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The Simple Graphs Associated with Rings and Semigroups

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Let R be a commutative ring, and let $Z(R)$ denote its set of zero-divisors. We associate a simple graph $\Gamma(R)$ to R with vertices $Z(R)^{*}=Z(R)-\{0\}$, the set of nonzero zero-divisor of R. Two distinct vertices x and y are adjacent if $xy=0$. This graphs are called the *zero-divisor graphs* of rings R.

We also associate a simple graph $\Delta(\mathbf{Z}_n)$ to $\mathbf{Z}_n$ with vertices $\mathbf{Z}_n$ and for distinct elements $x,\ y\in \mathbf{Z}_n$, the vertices x and y are adjacent if and only if $y=x^{2}$ $(x\neq y)$. This graphs are called the *parabola graphs.*

For a commutative multiplicative semigroup S with 0 $(0x=0$ for all $x\in S)$, we can defined the zero-divisor graph $\Gamma(S)$ as above ([DMS]).

We denote an edge such that a and b are adjacent by $a-b$. We also denote a path by $a-b-c-d$ etc. Also, let $\chi(G)$ denote the chromatic number of the graph G and let $\chi^{'}(G)$ denote the edge chromatic number of the graph G.

The notion of a zero-divisor graph was first introduced by I. Beck in [B1] and further investigated in [A1], though their vertices set included the zero element. Let G be a graph.

The *diameter* of G is

$$diam(G)=sup\{d(x,y)\mid x \text{ and } y \text{ are distinct vertices of } G\},$$

where $d(x,y)$ denotes the length of the shortest path from x to y. The *grith* of G, denoted by $g(G)$, is defined as the length of the shortest cycle in G.

A complete subgraph of G is called *clique.* $\omega(G)$, the *clique number* of G, is the greatest integer $r\geq 1$ such that $K^{r}\subset G$. Also, $c(G)$, the *circumference* of G, is the length of the longest cycle in G. Let $n=p_1^{2n_1}\cdots p_k^{2n_k}q_1^{2m_1+1}\cdots q_r^{2m_r+1}$ for distinct primes $p_i,\ q_j$ and integers $n_i,m_j\geq 0$. Then

$$\omega(\Gamma(\mathbf{Z}_n))=p_1^{n_1}\cdots p_k^{n_k}q_1^{m_k}\cdots q_r^{m_r}+r-1$$

[1] This is a part of an abstract and details will be published elsewhere.

by [B, Proposition 2.3].

For a graph G and an integer $n \geq 1$, we define $\lambda(G,n)$ to be the number of complete subgraphs (cliques) of G of order n. Note that $\lambda(G,1)$ is the number of vertices of G, $\lambda(G,2)$ is the number of edges of G, $\lambda(G,3)$ is the number of triangles in G and $\lambda(G,n) = 0$ for all $n \geq \omega(G) + 1$ ([AFLL]).

A graph G is *planar* if it can be drawn in such a way that no two edges intersect.

We examine the zero-divisor graph $\Gamma(Z_n)$, of the ring Z_n where the residue class ring modulo n where n is a positive integer. Many parts of this note are contained in [DS] and [AF].

We also investigate the parabora graphs.

For any commutative semigroup S, let $C(S)$ = the *core* of $\Gamma(S)$ be the union of the cycles in $\Gamma(S)$. A vertex x of $\Gamma(S)$ is called an *end* point in case there is at most one edge in $\Gamma(S)$ with vertex x.

§1. Some exapmles of zero-divisor graphs and parabora graphs.

Example 1. Let S be the commutative nilsemigroup

$$S =< a, b \mid a^3 = a^2b = ab^2 = b^3 = 0 >= \{a, b, a^2, ab, b^2, 0\}.$$

For this zero-divisor graph $\Gamma(S) = (V(\Gamma(S)), E(\Gamma(S)))$, we have that

$$V(\Gamma(S)) = \{a, b, a^2, ab, b^2\}$$

is the set of all vertices and $E(\Gamma(S)) = \{a - a^2, a - b^2, a - ab, b^2 - ab, b^2 - a^2, b^2 - b, \}$. This graph is connected, $d(a, b^2) = 1, d(a, b) = 2$ and $a - a^2 - b^2 - a$ is a cycle of length 3. And so $diam(\Gamma(S)) = 2$ and $g(\Gamma(S)) = 3$. There is not an end poitnt in $\Gamma(S)$ and $C(S) = \Gamma(S)$. Also, we have that $\omega(\Gamma(S)) = 4$. Also we have that $\lambda(S, 1) = 5, \lambda(S, 2) = 9$ and $\lambda(S, 2) = 6$.

Example 2. Let S be the commutative nilsemigroup

$$S =< a, b \mid a^3 = a^2b = ab^2 = b^3 = 0,\ a^2 = b^2 >= \{a, b, a^2, ab, 0\}.$$

$\Gamma(S)$ is connected, $d(a, a^2) = 1, d(a, b) = 2$ and $a - a^2 - ab - a$ is a cycle of length 3. And so $diam(\Gamma(S)) = 2$ and $g(\Gamma(S)) = 3$. There is not an end poitnt in $\Gamma(S)$ and $C(S) = \Gamma(S)$. Also, we ahve that $\omega(\Gamma(S)) = 3$ and $c(\Gamma(S)) = 3$. Also we have that $\lambda(S, 1) = 4, \lambda(S, 2) = 5$ and $\lambda(S, 2) = 2$.

This graph $\Gamma(S)$ is planar.

Example 3. $V(\Gamma(\mathbf{Z}_{12}))$ of the zero-divisor graph $\Gamma(\mathbf{Z}_{12})$ is the set

$$\{2, 3, 4, 6, 8, 9, 10\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{12})) = \{2-6, 3-4, 3-8, 4-6, 4-9, 6-8, 6-10, 8-9\}.$$

The elements 4, 9 are idempotent elements and 6 is a nilpotent element. Also, $\{0, 6\}$ is an ideal of $\mathbf{Z}_{12}$, $\omega(\Gamma\mathbf{Z}_{12}) = 4$. Also we have that $\lambda(\mathbf{Z}_{12}, 1) = 7, \lambda(\mathbf{Z}_{12}, 2) = 8$ and $\lambda(\mathbf{Z}_{12}, 3) = 0$. Also $diam(\Gamma(\mathbf{Z}_{12})) = 3, g(\Gamma(\mathbf{Z}_{12})) = 4$ and $c(\Gamma(\mathbf{Z}_{12})) = 4$. We have that $\chi(\Gamma(\mathbf{Z}_{12})) = 2$ abd $\chi'(\Gamma(\mathbf{Z}_{12})) = 4$. $\Gamma(\mathbf{Z}_{12})$ is planar.

Example 4. $V(\Gamma(\mathbf{Z}_{15}))$ of the zero-divisor graph $\Gamma(\mathbf{Z}_{15})$ is the set

$$\{3, 5, 6, 9, 10, 12\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{15})) = \{3-5, 3-10, 5-6, 5-9, 5-12, 6-10, 9-10, 10-12\}.$$

The elements 6, 10 are idempotent elements. The ring $\mathbf{Z}_{15}$ has no ideals contained only two elements. We have that $\omega(\mathbf{Z}_{15}) = 4, c(\mathbf{Z}_{15}) = 4$ and also we have that $\lambda(\mathbf{Z}_{15}, 1) = 6, \lambda(\mathbf{Z}_{15}, 2) = 8$ and $\lambda(\mathbf{Z}_{15}, 2) = 0$. Also $diam(\Gamma(\mathbf{Z}_{15})) = 2$ and $g(\Gamma(\mathbf{Z}_{15})) = 4$. We have that $\chi(\Gamma(\mathbf{Z}_{15})) = 2$ abd $\chi'(\Gamma(\mathbf{Z}_{15})) = 4$. $\Gamma(\mathbf{Z}_{15})$ is planar.

Example 5. $V(\Gamma(\mathbf{Z}_{16}))$ of the zero-divisor graph $\Gamma(\mathbf{Z}_{16})$ is the set

$$\{2, 4, 6, 8, 10, 12, 14\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{16})) = \{2-8, 4-8, 4-12, 6-8, 8-10, 8-12, 8-14\}.$$

The element 8 is an idempotent element and 4, 12 are nilpotent elements. The set $\{0, 8\}$ is an ideal of $\mathbf{Z}_{16}$. We have that $\omega(\mathbf{Z}_{16}) = 3$. Also we have that $\lambda(\mathbf{Z}_{16}, 1) = 7, \lambda(\mathbf{Z}_{16}, 2) = 7$ and $\lambda(\mathbf{Z}_{16}, 3) = 1$. Also $diam(\Gamma(\mathbf{Z}_{16})) =$

$2, g(\Gamma(\mathbf{Z}_{16})) = 3$ and $c(\Gamma(\mathbf{Z}_{16})) = 3$. We have that $\chi(\Gamma(\mathbf{Z}_{16})) = 3$ and $\chi'(\Gamma(\mathbf{Z}_{16})) = 6$. $\Gamma(\mathbf{Z}_{16})$ is planar.

Example 6. Let $\Gamma(\mathbf{Z}_4[X]/(X^2)) = \Gamma(\mathbf{Z}_4[x])$ $(x^2 = 0)$ be a zero-divisor graph associated with the ring $\mathbf{Z}_4[X]/(X^2)$. Set $R_4[x] = \mathbf{Z}_4[x]$. Then $V(\Gamma(R_4[x])) = \{2, x, 2+x, 2x, 2+2x, 3x, 2+3x\}$ and $E(R_4[x]) = \{2-2x, 2-2+2x, x-2x, x-3x, 2+x-2x, 2x-2+2x, 2+3x-2x\}$.

We have that $\omega(R_4[x]) = 3, c(R_4[x]) = 3$ and also we have that $\lambda(R_4[x], 1) = 7, \lambda(R_4[x], 2) = 8$, $\lambda(R_4[x], 3) = 2$ and $\lambda(R_4[x], 4) = 0$. Also $diam(\Gamma(R_4[x])) = 2$ and $g(\Gamma(R_4[x])) = 3$. We have that $\chi(\Gamma(R_4[x])) = 3$ and $\chi'(\Gamma(R_4[x])) = 6$.

$\Gamma(R_4[x])$ is planar.

Lemma 1. *$\Gamma(\mathbf{Z}_n)$ is connected and $diam(\mathbf{Z}_n) \leq 3$.*

Lemma 2. *If $\Gamma(\mathbf{Z}_n)$ contains a cycle, then $g(\mathbf{Z}_n) \leq 4$.*

Conjecture. The number of end points of $\Gamma(\mathbf{Z}_n)$ is an even number.

This conjecture is valid for $n \leq 30$.

Lemma 3. *If $n \geq 9$ and $\Gamma(\mathbf{Z}_n)$ is a zero-divisor graph of $\mathbf{Z}_n$, then there exists an element x such that $\{0, x\}$ is an ideal of $\mathbf{Z}_n$, $Ann(x)$ is a maximal ideal of $\mathbf{Z}_n$ and $\mathbf{Z}_n/Ann(x) \cong \mathbf{Z}_2$.*

If p is a prime number $(\neq 2)$, then $\{0, p\}$is an ideal of $\mathbf{Z}_{2p}$. So $\Gamma(\mathbf{Z}_{2p})$ is a star graph.

Lemma 4. *If any vertex in $\Gamma(\mathbf{Z}_n)$ is either a vertex of the core $C(\Gamma(\mathbf{Z}_n))$ or else is an end point of $\Gamma(\mathbf{Z}_n)$.*

Lemma 5 ([B] and [AN]). *Let p, q and r be all distinct prime numbers. The following satements are satisfied.*

(1) $\omega(\Gamma(\mathbf{Z}_n)) = 1$ *if and only if* $n = 4$.

(2) $\omega(\Gamma(\mathbf{Z}_n)) = 2$ *if and only if* $n = 8,\ 9,\ pq,\ 4p\ (p \neq 2)$.

(3) $\omega(\Gamma(\mathbf{Z}_n)) = 3$ *if and only if* $n = prq,\ 4pq\ (p \neq 2, q \neq 2),\ 8p\ (p \neq 2),\ 9p\ (p \neq 3),\ 16,\ 27$.

§2. The parabola graph $y = x^2$

We associate a simple graph $\Delta(\mathbf{Z}_n)$ to $\mathbf{Z}_n$ with vertices $\mathbf{Z}_n$ and for distinct elements $x,\ y \in \mathbf{Z}_n$, the vertics x and y are adjacent if and only if $y = x^2$ $(x \neq y)$.

Example 1. The vertices set $V(\Delta(\mathbf{Z}_7))$ of the simple graph $\Delta(\mathbf{Z}_7)$ is the set $\mathbf{Z}_7$ and $E(\Delta(\mathbf{Z}_7)) = \{1-6, 2-3, 2-4, 4-5\}$. $\lambda(\Delta(\mathbf{Z}_7), 1) = 7, \lambda(\Delta(\mathbf{Z}_7), 2) = 4$ and $\lambda(\Delta(\mathbf{Z}_7), 3) = 0$. We have that $diam(\Delta(\mathbf{Z}_7)) = 3, \omega(\Delta(\mathbf{Z}_7)) = 2, c(\Delta(\mathbf{Z}_7)) = g(\Delta(\mathbf{Z}_7)) = 0$.

Example 2. Let $\Delta(\mathbf{Z}_{11}) = (V(\Delta(\mathbf{Z}_{11})), E(\Delta(\mathbf{Z}_{11})))$ be a simple graph associated with $\mathbf{Z}_{11}$. We have that $V(\Delta(\mathbf{Z}_{11}) = \mathbf{Z}_{11}$ and $E(\Delta(\mathbf{Z}_{11})) = \{2-10, 3-5, 3-6, 3-9, 4-5, 4-9, 5-7, 8-9\}$.
$\lambda(\Delta(\mathbf{Z}_{11}), 1) = 11, \lambda(\Delta(\mathbf{Z}_{11}), 2) = 8$ and $\lambda(\Delta(\mathbf{Z}_{11}), 3) = 0, \lambda(\Delta(\mathbf{Z}_{11}), 4) = 1$. We have that $diam(\Delta(\mathbf{Z}_{11})) = 4, \omega(\Delta(\mathbf{Z}_{11})) = 2, c(\Delta(\mathbf{Z}_{11})) = 4, g(\Delta(\mathbf{Z}_{11})) = 4$. This graph is not a forest.

Theorem. (a) *Let $\Delta(\mathbf{Z}_7)$ be a parabola graph and let A_7 be an adjacent matrix of a parabola graph $\Delta(\mathbf{Z}_7)$. Also, $f_n(X) = f_{A^n}(X)$ be a minimal polynomial of A_7. Then the following statements hold.*

(1) *If n is an even natural number, then $f_n(X)$ has a divisor $X^2 - L_nX + 1$.*

(2) *If n is an odd natural number, then $f_n(X)$ has a divisor $X^2 - L_nX - 1$, where L_n is a Lucas number.*

(b) (1) *$\Delta(\mathbf{Z}_p)$ has no triangles, that is, $\lambda(\Delta(\mathbf{Z}_p), 3) = 0$ for a prime number p.*

(2) *$\lambda(\Delta(\mathbf{Z}_n)) = 0$ for $1 \leq n \leq 10$ and $12 \leq n \leq 20$.*

(3) *For parabora graphs $\Delta(\mathbf{Z}_n)$ $(2 \leq n \leq 20)$, their graphs are forest except $n = 11$.*

References

[AF] D.F.Anderson, A.A.Frazier, A.Lauve and P.S.Livingston, The zero-divisor graph of a commutative ring II, Lecture Notes in Pure and Appli. Math., Marcel Dekker, New York, 220 (2001), 61-72.

[A1] D.D.Anderson and M.Naseer, Beck's coloring of a commutative ring, J. Algebra 1993, 159, 500-514.

[B1] I.Beck, Coloring of commutative rings. J. Algebra 1988, 116, 208-226.

[DS] F.DeMeyer and K.Schneider, Automorphisms and zero divisor graphs of commutative rings, International J. of commutative rings, to appear.

[DMS] F.R. DeMeyer, T. McKenzie and K. Schneider, The zero-divisor graph of a commutative semigroup, Semigroup Forums, 65 (2002) 206-214.

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