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FIXED-WIDTH CONFIDENCE INTERVAL ESTIMATION OF A FUNCTION OF TWO EXPONENTIAL SCALE PARAMETERS

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1 Introduction

Many researchers are working in the area of sequential estimation in the two-sample exponential case. To cite some recent works, Mukhopadhyay and Chattopadhyay [4] considered the sequential estimation of the difference between means. Sen [5] treated a sequential comparison of two exponential distributions. Uno [6] provided second-order approximations of the expected sample size and the risk of the sequential procedure for the ratio parameter $\theta = \sigma_1/\sigma_2$. Isogai and Futschik [2] dealt with the same parameter θ, using bounded risk estimation. Lim, et al. [3], investigated the construction of sequential confidence intervals for positive functions of the scale parameters. In this paper, we will use the results of Lim, et al. [3] for the function $h(\sigma_1, \sigma_2) = (\sigma_1/\sigma_2)^r, r \neq 0$. More specifically for the cases when $r = 1$ and $r = 2$.

Let $h(x, y)$ be a positive, real-valued and three-times continuously differentiable function defined on $\mathbb{R}_+^2 = (0, +\infty) \times (0, +\infty)$ with $h_x = \frac{\partial}{\partial x}h$, $h_y = \frac{\partial}{\partial y}h$ and $h_x^2(x, y) + h_y^2(x, y) > 0$ on $\mathbb{R}_+^2$.

Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be independent observations from two exponential populations Π_1 and Π_2, respectively, with their corresponding densities given as follows:

$$f_1(x) = \sigma_1^{-1} \exp(-x/\sigma_1)\, I(x > 0) \quad \text{and} \quad f_2(y) = \sigma_2^{-1} \exp(-y/\sigma_2)\, I(y > 0),$$

where the scale parameters $\sigma_1 > 0$ and $\sigma_2 > 0$ are both unknown and $I(\cdot)$ stands for the indicator function of $(\cdot)$. Taking samples of size n from Π_1 and Π_2, we estimate $\theta = h(\sigma_1, \sigma_2)$ by

$$\hat{\theta}_n = h(\bar{X}_n, \bar{Y}_n),$$

where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i$.

Given $d > 0$ and $\alpha \in (0,1)$, we want to construct a confidence interval I_n for $\theta = h(\sigma_1, \sigma_2)$ with length $2d$ and coverage probability $1-\alpha$, based on samples of size n, $\{X_1, \cdots, X_n\}$ and $\{Y_1, \cdots, Y_n\}$, from Π_1 and Π_2, respectively. Throughout the paper, we shall assume that '$\xrightarrow{d}$', '$\xrightarrow{p}$' and '$\xrightarrow{a.s.}$' stand for convergence in distribution, convergence in probability and almost sure convergence, respectively.

Let us look at the succeeding result which gives the asymptotic distribution of $\hat{\theta}_n = h(\bar{X}_n, \bar{Y}_n)$. This result provides the asymptotic normality of $\sqrt{n}(\hat{\theta}_n - \theta)$.

Proposition 1. ([3]) *Let a function g on $\mathbb{R}_+^2$ be defined by*

$$g(x,y) = h_x^2(x,y)x^2 + h_y^2(x,y)y^2.$$

Then

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, g(\sigma_1, \sigma_2)) \quad \text{as } n \to \infty.$$

For a given $d > 0$ and $0 < \alpha < 1$, let $I_n = [\,\hat{\theta}_n - d,\, \hat{\theta}_n + d\,]$ be a confidence interval for θ with length $2d$. This interval I_n must satisfy

$$P\{\theta \in I_n\} = P\{|\hat{\theta}_n - \theta| \le d\} \ge 1 - \alpha\,. \tag{1}$$

Choose $a = a_\alpha > 0$ such that $\Phi(a) = 1 - \alpha/2$, where Φ is the standard normal distribution function. Set

$$n^* = \frac{a^2}{d^2}\, g(\sigma_1, \sigma_2)\,. \tag{2}$$

Then it follows from Proposition 1 that for all $n \ge n^*$,

$$\begin{aligned} P\{\theta \in I_n\} &= P\left\{\left|\sqrt{n}(\hat{\theta}_n - \theta)/\sqrt{g(\sigma_1,\sigma_2)}\right| \le d\sqrt{n}/\sqrt{g(\sigma_1,\sigma_2)}\right\} \\ &\ge P\left\{\left|\sqrt{n}(\hat{\theta}_n - \theta)/\sqrt{g(\sigma_1,\sigma_2)}\right| \le a\right\} \approx 1 - \alpha \end{aligned}$$

if n^* is sufficiently large. For simplicity, assume n^* to be an integer. Then n^* is the asymptotically smallest sample size which approximately satisfies equation (1).

2 Main Results

In this section, we will propose a sequential procedure and give its asymptotic properties. We have seen from the previous section that n^* in (2) is the asymptotically smallest sample size. Now, since σ_1 and σ_2 are unknown, then n^* is also unknown. It is known that fixed sample size procedures are not available for scale families. Thus, we propose the following stopping rule:

$$N = N_d = \inf\left\{ n \ge m : n \ge \frac{a^2}{d^2} g(\bar{X}_n, \bar{Y}_n) \right\}, \tag{3}$$

where $m \geq 2$ is the initial sample size. Then in view of the SLLN and the definition of N_d, we can show the lemma below.

Lemma 1. ([3])

(i) $\quad P\{N_d < +\infty\} = 1$ *for each* $d > 0$.

(ii) $\quad N_d \xrightarrow{a.s.} +\infty$ *as* $d \to 0$.

(iii) $\quad N_d/n^* \xrightarrow{a.s.} 1$ *as* $d \to 0$.

The following proposition gives the asymptotic normality of $\sqrt{N}(\hat{\theta}_N - \theta)$ which will play the important role in showing the asymptotic consistency of the sequential confidence intervals $\{I_N\}$.

Proposition 2. ([3]) *As* $d \to 0$,

$$\sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, g(\sigma_1, \sigma_2)),$$

where

$$g(\sigma_1, \sigma_2) = h_x^2(\sigma_1, \sigma_2)\sigma_1^2 + h_y^2(\sigma_1, \sigma_2)\sigma_2^2.$$

Once sampling is stopped after taking N observations from populations Π_1 and Π_2, respectively, we use the confidence interval $I_N = [\,\hat{\theta}_N - d,\ \hat{\theta}_N + d\,]$ for θ. The next result shows the asymptotic consistency of the sequential confidence intervals $\{I_N\}$.

Theorem 1. ([3]) [Asymptotic Consistency]

$$\lim_{d \to 0} P\{\theta \in I_N\} = 1 - \alpha.$$

Throughout the remainder of this section, we let

$$U_i = (X_i - \sigma_1)/\sigma_1, \quad V_i = (Y_i - \sigma_2)/\sigma_2 \quad \text{and} \quad \boldsymbol{X}_i = (U_i, V_i) \quad \text{for } i = 1, 2, \cdots.$$

Consider also the following notations:

$$Z_{1n} = \sqrt{n}(\bar{X}_n - \sigma_1)/\sigma_1, \quad Z_{2n} = \sqrt{n}(\bar{Y}_n - \sigma_2)/\sigma_2,$$

$$D_n = n\bar{U}_n = \sum_{i=1}^{n} U_i = n(\bar{X}_n - \sigma_1)/\sigma_1 = \sqrt{n}Z_{1n},$$

$$Q_n = n\bar{V}_n = \sum_{i=1}^{n} V_i = n(\bar{Y}_n - \sigma_2)/\sigma_2 = \sqrt{n}Z_{2n},$$

$$\boldsymbol{S}_n = (D_n, Q_n) \quad \text{and} \quad \boldsymbol{c} = \left(-\sigma_1 \frac{g_x(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)}, -\sigma_2 \frac{g_y(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)}\right).$$

Define the function f on $\mathbb{R}_+^2$ as $f(x,y) = g(\sigma_1,\sigma_2)/g(x,y)$. Since g is positive and continuous on $\mathbb{R}_+^2$, so is f. Then the stopping time N in (3) can be written as

$$N = \inf\{n \geq m : Z_n \geq n^*\}, \tag{4}$$

where

$$Z_n = nf(\bar{X}_n, \bar{Y}_n) = n - \sigma_1 \frac{g_x(\sigma_1,\sigma_2)}{g(\sigma_1,\sigma_2)} D_n - \sigma_2 \frac{g_y(\sigma_1,\sigma_2)}{g(\sigma_1,\sigma_2)} Q_n + \xi_n, \tag{5}$$

$$\xi_n = \frac{1}{2}\left\{{\sigma_1}^2 f_{xx}(\eta_1,\eta_2)Z_{1n}^2 + 2\sigma_1\sigma_2 f_{xy}(\eta_1,\eta_2)Z_{1n}Z_{2n} + {\sigma_2}^2 f_{yy}(\eta_1,\eta_2)Z_{2n}^2\right\},$$

η_1 and η_2 are random variables satisfying $|\eta_1 - \sigma_1| < |\bar{X}_n - \sigma_1|$ and $|\eta_2 - \sigma_2| < |\bar{Y}_n - \sigma_2|$. In the notations of Aras and Woodroofe [1], we can rewrite (5) as

$$Z_n = n + \langle \boldsymbol{c}, \boldsymbol{S}_n \rangle + \xi_n,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Let

$$T = \inf\{n \geq 1 : n + \langle \boldsymbol{c}, \boldsymbol{S}_n\rangle > 0\} \quad \text{and} \quad \rho = \frac{E\{(T + \langle \boldsymbol{c}, \boldsymbol{S}_T\rangle)^2\}}{2E\{T + \langle \boldsymbol{c}, \boldsymbol{S}_T\rangle\}}. \tag{6}$$

Consider the following assumptions:

(A1) $\left\{\left[\left(Z_n - \frac{n}{\epsilon_0}\right)^+\right]^3, n \geq m\right\}$ is uniformly integrable for some $0 < \epsilon_0 < 1$, where $x^+ = \max(x,0)$.

(A2) $\sum_{n=m}^{\infty} nP\{\xi_n < -\epsilon_1 n\} < \infty$ for some $0 < \epsilon_1 < 1$.

The following theorem gives the second-order approximation of the expected sample size $E(N)$.

Theorem 2. ([3]) *If (A1) and (A2) hold, then*

$$E(N) = n^* + \rho - \nu + o(1) \quad \text{as } d \to 0,$$

where

$$\nu = \left\{\sigma_1^2 f_{xx}(\sigma_1,\sigma_2) + \sigma_2^2 f_{yy}(\sigma_1,\sigma_2)\right\}/2$$

and ρ in (6) satisfies

$$0 < \rho < \{1 + \langle \boldsymbol{c}, \boldsymbol{c}\rangle\}/2.$$

3 Example

We consider the estimation of the rth power of the ratio of two scale parameters, namely, $\theta = h(\sigma_1, \sigma_2) = (\sigma_1/\sigma_2)^r$ for $r \neq 0$. Theorem 3 that follows, gives the expected sample size of the sequential procedure for the given function θ.

Theorem 3. *If $m > \max\{1, 6|r|\}$, then*

$$E(N) = n^* + \rho - 4r^2 + o(1) \quad \text{as } d \to 0,$$

where ρ in (6) satisfies

$$0 < \rho < \frac{1+8r^2}{2}.$$

Proof. For this function, the stopping random variable N in (4) can be written as

$$N = \inf\{n \geq m : Z_n \geq n^*\},$$

where

$$Z_n = n - 2r(D_n - Q_n) + \xi_n \tag{7}$$

and

$$\xi_n = r\theta^2 \left(\frac{\eta_2}{\eta_1}\right)^{2r} \left\{(2r+1)\frac{\sigma_1{}^2}{\eta_1{}^2} Z_{1n}^2 - 4r\frac{\sigma_1\sigma_2}{\eta_1\eta_2} Z_{1n}Z_{2n} + (2r-1)\frac{\sigma_2{}^2}{\eta_2{}^2} Z_{2n}^2\right\},$$

η_1 and η_2 are random variables satisfying $|\eta_1 - \sigma_1| < |\bar{X}_n - \sigma_1|$ and $|\eta_2 - \sigma_2| < |\bar{Y}_n - \sigma_2|$. In the notations of Aras and Woodroofe [1], we can rewrite (7) as

$$Z_n = n + \langle \boldsymbol{c}, \boldsymbol{S}_n \rangle + \xi_n,$$

where $\boldsymbol{c} = (-2r, 2r)$. In order to use Theorem 2 to determine the expected sample size, we need to satisfy conditions (A1) and (A2) of the theorem. Let $u > 1$ and $v > 1$ be such that $u^{-1} + v^{-1} = 1$ and M a generic positive constant.

To prove (A1), it suffices to show that

$$\sup_{n\geq m} E\left\{\left[(Z_n - n/\epsilon_0)^+\right]^3\right\} < \infty.$$

Now

$$(Z_n - n/\epsilon_0)^+ = n\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{2r} - \epsilon_0^{-1}\right\} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1}\right\}}.$$

Thus,

$$\begin{aligned}
E\left\{\left[(Z_n - n/\epsilon_0)^+\right]^3\right\} &\leq n^3 E\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{6r} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1}\right\}}\right\} \\
&\leq n^3 E\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{6r} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1},\, \bar{U}_n+1 < 1-\epsilon_0\right\}}\right\} \\
&\quad + n^3 E\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{6r} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1},\, \bar{U}_n+1 \geq 1-\epsilon_0\right\}}\right\} \\
&\equiv K_1(n) + K_2(n), \text{ say.}
\end{aligned}$$

By the independence of $\bar{U}_n$ and $\bar{V}_n$ and by Hölder's Inequality, we have $K_1(n) \leq n^3 E(\bar{V}_n+1)^{6r}\left\{E(\bar{U}_n+1)^{-6ru}\right\}^{1/u}\left\{P(|\bar{U}_n|>\epsilon_0)\right\}^{1/v}$. By Lemma 1 of Uno [6], $E(\bar{V}_n+1)^{6r} \leq M$ and $E(\bar{U}_n+1)^{-6ru} \leq M$ for $n \geq m > 6|r|u$. By Markov's Inequality, $P(|\bar{U}_n|>\epsilon_0) \leq (n\epsilon_0)^{-q}E|D_n|^q$ for $q \geq 2$. But by Marcinkiewicz-Zygmund Inequality, $E|D_n|^q = O(n^{q/2})$ as $n \to \infty$. Thus, it follows that $K_1(n) \leq Mn^{3-q/2v}$ for $n \geq m > 6|r|u$. Since $m > 6|r|$, we can choose $u > 1$ such that $m > 6|r|u$. Then choose $q > \max\{2, \frac{6u}{u-1}\}$. Thus, $3 - q/2v \leq 0$ which shows that $\sup_{n \geq m} K_1(n) < \infty$. Let $\delta = \epsilon_0{}^{-1/2r}(1-\epsilon_0) > 1$ and $r > 0$ for small $0 < \epsilon_0 < 1$. Then

$$\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1},\ \bar{U}_n+1 \geq 1-\epsilon_0\right\} \subset \left\{\bar{V}_n+1 \geq \delta\right\}.$$

It follows that for $r > 0$,

$$\begin{aligned}
K_2(n) &\leq n^3(1-\epsilon_0)^{-6r}E\left\{(\bar{V}_n+1)^{6r}I_{\{\bar{V}_n+1\geq\delta\}}\right\} \\
&\leq n^3(1-\epsilon_0)^{-6r}\left\{E(\bar{V}_n+1)^{6ru}\right\}^{1/u}\left\{P(\bar{V}_n+1\geq\delta)\right\}^{1/v} \\
&\leq n^3(1-\epsilon_0)^{-6r}\left\{E(\bar{V}_n+1)^{6ru}\right\}^{1/u}\left\{P(|\bar{V}_n|\geq\delta-1)\right\}^{1/v},
\end{aligned}$$

where $\frac{1}{u}+\frac{1}{v}=1$ and $u > 1$. Thus, in the same way as $K_1(n)$, $\sup_{n\geq m} K_2(n) < \infty$ for $m > 6r$. For $r < 0$, by similar arguments as above, $\sup_{n\geq m} K_2(n) < \infty$ for $m > 6|r|$. This completes the proof of (A1).

By Taylor's Theorem,

$$\begin{aligned}
&(\bar{V}_n+1)^{2r}(\bar{U}_n+1)^{-2r} \\
&= \left(1+2r\bar{V}_n+r(2r-1)\phi_2^{2(r-1)}\bar{V}_n^2\right)\left(1-2r\bar{U}_n+r(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\right),
\end{aligned}$$

where ϕ_1 and ϕ_2 are positive random variables between $(\bar{U}_n+1)$ and 1, and $(\bar{V}_n+1)$ and 1, respectively. Thus, it follows from (7) that

$$\begin{aligned}
\xi_n = Z_n - n + 2r(D_n - Q_n) &= n\left[(\bar{V}_n+1)^{2r}(\bar{U}_n+1)^{-2r} - 1 + 2r(\bar{U}_n - \bar{V}_n)\right] \\
&= n\left[-4r^2\bar{U}_n\bar{V}_n + r(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2 + 2r^2(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right] \\
&\quad + n\Big[r(2r-1)\phi_2^{2(r-1)}\bar{V}_n^2 - 2r^2(2r-1)\phi_2^{2(r-1)}\bar{U}_n\bar{V}_n^2 \\
&\quad + r^2(4r^2-1)\phi_1^{-2(r+1)}\phi_2^{2(r-1)}\bar{U}_n^2\bar{V}_n^2\Big].
\end{aligned}$$

Thus, setting $\epsilon_2 = \epsilon_1/6$ for $0 < \epsilon_1 < 1$, we have

$$
\begin{aligned}
&P\{\xi_n < -\epsilon_1 n\} \\
&\leq P\left\{\left|4r^2\bar{U}_n\bar{V}_n\right| > \epsilon_2\right\} + P\left\{\left|r(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\right| > \epsilon_2\right\} \\
&\quad + P\left\{\left|2r^2(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right| > \epsilon_2\right\} + P\left\{\left|r(2r-1)\phi_2^{2(r-1)}\bar{V}_n^2\right| > \epsilon_2\right\} \\
&\quad + P\left\{\left|2r^2(2r-1)\phi_2^{2(r-1)}\bar{U}_n\bar{V}_n^2\right| > \epsilon_2\right\} \\
&\quad + P\left\{\left|r^2(4r^2-1)\phi_1^{-2(r+1)}\phi_2^{2(r-1)}\bar{U}_n^2\bar{V}_n^2\right| > \epsilon_2\right\} \\
&\equiv \sum_{i=1}^{6} I_i(n), \text{ say.}
\end{aligned}
$$

By the independence of $\bar{U}_n$ and $\bar{V}_n$, and by Marcinkiewicz-Zygmund Inequality, $E\{|D_nQ_n|^q\} = E\{|D_n|^q\}E\{|Q_n|^q\} \leq Mn^q$, for $q \geq 2$. Thus, by Markov's Inequality,

$$I_1(n) = P\left\{4r^2|D_nQ_n| > n^2\epsilon_2\right\} \leq M\,n^{-2q}E\{|D_nQ_n|^q\} \leq Mn^{-q}.$$

Now, since ϕ_1 is a random variable between 1 and $\bar{U}_n + 1$, then $\phi_1 > 1/2$ on the set $\left\{|\bar{U}_n| \leq 1/4\right\}$. Thus, for $r+1 \geq 0$, we have

$$
\begin{aligned}
I_2(n) &\leq P\left\{M\left|\phi_1^{-2(r+1)}\bar{U}_n^2\right| > \epsilon_2, |\bar{U}_n| \leq 1/4\right\} + P\left\{|\bar{U}_n| > 1/4\right\} \\
&\leq P\left\{M(1/2)^{-2(r+1)}(1/2)^2|\bar{U}_n| > \epsilon_2\right\} + P\left\{|\bar{U}_n| > 1/4\right\} \\
&\leq P\left\{|\bar{U}_n| > M\right\} + Mn^{-q/2} \leq Mn^{-q/2}.
\end{aligned}
$$

In a similar way, we get

$$
\begin{aligned}
I_3(n) &\leq P\left\{M\left|\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right| > \epsilon_2, |\bar{U}_n| \leq 1/4\right\} + P\left\{|\bar{U}_n| > 1/4\right\} \\
&\leq P\left\{|\bar{U}_n\bar{V}_n| > M\right\} + Mn^{-q/2} \\
&\leq Mn^{-2q}E\{|D_nQ_n|^q\} + Mn^{-q/2} \leq Mn^{-q/2}.
\end{aligned}
$$

Suppose that $r+1 < 0$. Then it follows by convexity and Lemma 1 of Uno [6] that for any $q \geq 2$

$$E\left\{\phi_1^{-4(r+1)q}\right\} \leq 1 + E\left[(\bar{U}_n + 1)^{-4(r+1)q}\right] \leq M.$$

Thus,

$$I_2(n) \leq ME\left\{\phi_1^{-4(r+1)q}\right\}^{1/2} E\left\{|\bar{U}_n|^{4q}\right\}^{1/2} \leq Mn^{-q}. \tag{8}$$

From (8), we obtain

$$
\begin{aligned}
I_3(n) &\leq ME\left\{\left|\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right|^q\right\} = ME\left\{\left|\phi_1^{-2(r+1)}\bar{U}_n^2\right|^q\right\} E\left\{|\bar{V}_n|^q\right\} \\
&\leq Mn^{-q}n^{-q/2} \leq Mn^{-q/2}.
\end{aligned}
$$

Thus, from the above relations, $I_i(n) \le Mn^{-q/2}$ for $i = 1, 2, 3$. Hence, taking $q = 6$, we have $\sum_{n=1}^{\infty} nI_i(n) < \infty$ for $i = 1, 2, 3$. By similar arguments, we can show that $\sum_{n=1}^{\infty} nI_i(n) < \infty$ for $i = 4, 5, 6$. Therefore, (A2) is satisfied. Now, $\nu = 4r^2$. Hence, it follows from Theorem 2 that for $m > \max\{1, 6|r|\}$,

$$E(N) = n^* + \rho - 4r^2 + o(1) \quad \text{as } d \to 0,$$

where $0 < \rho < (1 + 8r^2)/2$. This completes the proof. $\square$

To illustrate these results, let us consider two cases. For the case when $r = 1$, we consider two stopping rules; N in (3) and N^* given in Isogai and Futschik [2], and compare the coverage probabilities of the sequential confidence intervals, corresponding to N and N^*. The stopping rule N becomes

$$N = N_d = \inf\left\{ n \ge m : \ n \ge \frac{2a^2\overline{X}_n^2}{d^2\overline{Y}_n^2} \right\}.$$

Then, letting $L(n) \equiv 1$ and replacing w by d^2/a^2, N in (4) is the same as N_w in Isogai and Futschik [2] who also showed that (A1) and (A2) hold with $m > 6$ and $\boldsymbol{c} = (-2, 2)$. Thus, it follows from Theorem 2 that

$$E(N) = n^* + \rho - 4 + o(1) \quad \text{and} \quad 0 < \rho < 9/2.$$

By simulation, we can get $\rho = 2.03$. Thus, taking this ρ into account, we consider another stopping rule:

$$N^* = N_d^* = \inf\left\{ n \ge m : \ n \ge L(n)\frac{2a^2\overline{X}_n^2}{d^2\overline{Y}_n^2} \right\} \quad \text{where } L(n) = 1 + \frac{1.97}{n}.$$

From Theorem 2.1 of Isogai and Futschik [2], if $m > 6$ then $E(N^*) = n^* + o(1)$ as $d \to 0$.

Now, from Proposition 2.1 of Isogai and Futschik [2] if $m > 12$, then

$$E(\hat{\theta}_N) - \theta = -\frac{3d}{a\sqrt{2n^*}} + o(d^2) \quad \text{as } d \to 0.$$

From this result, we propose the following bias-corrected sequential confidence intervals:

$$I_N^\dagger = [\,\hat{\theta}_N^\dagger - d,\ \hat{\theta}_N^\dagger + d\,] \quad \text{and} \quad I_{N^*}^\dagger = [\,\hat{\theta}_{N^*}^\dagger - d,\ \hat{\theta}_{N^*}^\dagger + d\,],$$

where $\hat{\theta}_N^\dagger = \hat{\theta}_N + (3d)/(a\sqrt{2N})$ and $\hat{\theta}_{N^*}^\dagger = \hat{\theta}_{N^*} + (3d)/(a\sqrt{2N^*})$.

For the case when $r = 2$, the stopping rule in (3) becomes

$$N = N_d = \inf\left\{ n \ge m : \ n \ge \frac{8a^2\overline{X}_n^4}{d^2\overline{Y}_n^4} \right\},$$

and by Theorem 3, for $m > 12$, the expected sample size is

$$E(N) = n^* + \rho - 16 + o(1) \quad \text{and} \quad 0 < \rho < 33/2.$$

Now, by simulation using 100,000 repetitions, we can get $\rho = 4.02$. Considering this value for ρ, we propose another stopping rule as follows:

$$N^* = N_d^* = \inf\left\{ n \geq m : \ n \geq L(n) \frac{8a^2 \overline{X}_n^4}{d^2 \overline{Y}_n^4} \right\}, \quad L(n) = 1 + \frac{11.98}{n}.$$

Simulation Results. We shall give simulation results for the case when $(\sigma_1, \sigma_2) = (2, 1)$. The coverage probability is set at $1 - \alpha = 0.95$ and the pilot sample size at $m = 13$. The following results are based on 10,000 repetitions.

Table 1.1 Using N $(r = 1)$ $\theta = 2$

n^*	20	100	200	500	1000
d	1.239588	0.554360	0.391992	0.247918	0.175304
$E(N)$	21.4789	96.7799	197.5324	497.2630	995.8871
$E(\hat{\theta}_N)$	1.865092	1.917183	1.965773	1.986306	1.991834
$E(\hat{\theta}_N^\dagger)$	2.172344	1.981733	1.996558	1.998415	1.997865
$P(\theta \in I_N)$	0.9864	0.9079	0.9361	0.9477	0.9485
$P(\theta \in I_N^\dagger)$	0.9878	0.9241	0.9444	0.9501	0.9518

Table 1.2. Using N^* $(r = 1)$ $\theta = 2$

n^*	20	100	200	500	1000
d	1.239588	0.554360	0.391992	0.247918	0.175304
$E(N^*)$	22.7216	98.7350	199.7856	499.7327	1000.3485
$E(\hat{\theta}_{N^*})$	1.860984	1.920277	1.967711	1.987303	1.994298
$E(\hat{\theta}_{N^*}^\dagger)$	2.160043	1.983678	1.998279	1.999382	2.000316
$P(\theta \in I_{N^*})$	0.9881	0.9122	0.9360	0.9460	0.9478
$P(\theta \in I_{N^*}^\dagger)$	0.9883	0.9271	0.9437	0.9476	0.9509

Table 2.1. Using N $(r = 2)$ $\theta = 4$

n^*	20	100	200	500	1000
d	4.958350	2.217442	1.567968	0.991670	0.701217
$E(N)$	23.7237	83.0451	173.9840	475.8522	980.8059
$E(\hat{\theta}_N)$	3.489305	3.288380	3.492996	3.812448	3.924260
$P(\theta \in I_N)$	0.9992	0.8055	0.8122	0.8973	0.9298

Table 2.2 Using N^* $(r = 2)$ $\theta = 4$

n^*	20	100	200	500	1000
d	4.958350	2.217442	1.567968	0.991670	0.701217
$E(N^*)$	29.0472	98.0366	192.7931	493.2553	996.6286
$E(\hat{\theta}_{N^*})$	3.448231	3.438694	3.623196	3.853191	3.939215
$P(\theta \in I_{N^*})$	0.9994	0.8556	0.8657	0.9184	0.9378

The tables show that the rate of convergence of the coverage probability $P(\theta \in I_N)$ to $1 - \alpha$ seems to be slow. For the case when $r = 1$, the bias-corrected sequential confidence intervals, $I_N^\dagger$ and $I_{N^*}^\dagger$, are more effective than the ordinary ones, I_N and I_{N^*}. Furthermore, there seems to be no significant difference between the coverage probabilities of the intervals, I_N and I_{N^*}. An improvement on the stopping rule in (4) is needed.

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