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Cospectral graphs of the Grassmann graphs

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Let q be a prime power, and V be a n-dimensional space over the $GF(q)$ the field with q elements. Let $1 \leq e \leq n-1$ be an integer.

The **Grassmann Graph** $G_q(n,e)$ has as vertices the e-dimensional subspaces and $S \sim T$ iff their intersection is $(e-1)$-dimensional.

To construct graphs with the same spectrum as $G_q(n,e)$ we first will look at a partial linear space.

Let n, e be positive integers such that $4 \leq 2e \leq n$.

Let V be a n-dimensional vector space over $GF(q)$ and

let H be a $2e$-dimensional subspace of V.
We first construct the partial linear space

$$\mathcal{LG}_q(n, e, e+1).$$

Its points are the e-dimensional subspaces of V.

There are two kinds of lines:

Lines of the first kind: $(e+1)$-dimensional subspaces L of V which are not a subspace of H. A line L as points the e-dimensional subspaces contained in L.

b Lines of the second kind: $(e-1)$-dimensional spaces M contained in H. A line M has as points the e-dimensional spaces contained in H which contain M as a subspace.

Now $\mathcal{LG}_q(n, e, e+1)$ has
$\left[{n \atop e}\right]$ points,
$\left[{n \atop e+1}\right]$ lines,
each point is incident with $\left[{n-e \atop 1}\right]$ lines
and each line is incident with $\left[{e+1 \atop 1}\right]$ points.

Through any pair of points there is at most one line.
If P and Q are points then they are on a line iff $P \cap Q$ is $(e-1)$-dimensional.

Define $P_q(n, e+1)$ as the line graph of $\mathcal{LG}_q(n, e, e+1)$, that is its vertices are the lines of $\mathcal{LG}_q(n, e, e+1)$ and two lines are adjacent iff they have exactly one point in common.

Theorem 1 *(i) $P_q(n, e+1)$ is cospectral with $G_q(n, e+1)$,*
(ii) $P_q(n, e+1)$ is distance-regular iff $n = 2e+1$.
(iii) $P_q(2e+1, e+1)$ is not isomorphic to the Grassmann graph $G_q(2e+1, e+1)$.

(i) Let N be the point-line incidence matrix. Then $NN^T - \left[{n-e \atop 1}\right]I$ is the adjacency matrix of the point graph. As the point graph is clearly $G_q(n,e)$, we know the spectrum of NN^T. Now except for the zero eigenvalue the spectrum of NN^T is the same as the N^TN. This implies that $P_q(n,e+1)$ is cospectral with $G_q(n,e+1)$ as $NN^T - \left[{e+1 \atop 1}\right]I$ is the adjacency matrix for $P_q(n,e+1)$.

(ii) If $n < 2e + 1$, then there is $e + 1$-dimensional space L which intersects H in a $(e - 1)$-dimensional space M Now in $P_q(n, e+1)$ the distance between L and M is 2 and it easy to see that they have $\left[{2 \atop 1}\right]\left[{e+1 \atop 1}\right]$ common neighbours where in the Grassmann graph $c_2 = \left[{2 \atop 1}\right]^2$.

If $n = 2e + 1$, then it is possible to check that it is distance-regular. An easy way to see this is true we use a result by Fiol and Garriga which states that if a graph has the same spectrum as a distance-regular graph Γ with diameter d is distance-regular iff for all vertices x we have $k_d(x) = k_d(\Gamma)$. And this is easily checked.

(iii) Let $n = 2e+1$. Let K be an $(e+2)$-dimensional space which intersects H in $e + 1$ dimensions. Now the $(e + 1)$-dimensional subsapces of K which are not contained in H forms a maximal clique of size $\left[{e+2 \atop 1}\right] - 1$ in $P_q(2e + 1, e + 1)$, whereas the Grassmann graph $G_q(2e + 1, e + 1)$ has maximal cliques of sizes $\left[{e+2 \atop 1}\right]$ and $\left[{e+1 \atop 1}\right]$.

This shows the theorem.

(i) By looking at maximal cliques in $P_q(2e+1, e+1)$, it is easy to see that it is not vertex-transitive. The group $P\Gamma L(2e+1)_{2e}$ is an automorphism group of the graph. It was shown by M. Tagami that this is the full automorphism group.

(ii) For large q and e we were able to show that the local graph of a line of type 1 is not cospectral to the local graph of a line of type 2. We suspect that this is always the case. This implies, for example, that the the Terwilliger Algebra depends on the base vertex for $P_q(2e+1, e+1)$.