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A classification of subsets with $w + w^* = d$ in polynomial association schemes

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1 Introduction

An association scheme with d classes is a pair $(X, \mathbf{R})$ of a finite set X and a set of $d+1$ relations $\mathbf{R} = \{R_0, R_1, \ldots, R_d\}$ on X satisfying certain regularity properties. We refer the reader to [1, Chapter 2] for terminologies and background materials.

Brouwer, Godsil, Koolen and Martin [2] introduced two new parameters, width and dual width, for subsets in association schemes. The *width* w of a non-empty subset C in a metric association scheme $(X, \mathbf{R})$ with respect to the ordering $R_0, R_1, \ldots, R_d$ of the relations (thus $\Gamma = (X, R_1)$ is a distance-regular graph and each R_i is the distance-i relation for Γ) is the maximum distance which occurs between members of C:

$$w = \max\{i : a_i \neq 0\}$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ is the *inner distribution* of C, namely

$$a_i = \frac{1}{|C|}|(C \times C) \cap R_i|.$$

Dually, the *dual width* w^* of a non-empty subset C in a cometric association scheme $(X, \mathbf{R})$ with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of the Bose-Mesner algebra $\mathscr{A}$ is defined by

$$w^* = \max\{i : (\mathbf{a}Q)_i \neq 0\}$$

where Q is the second eigenmatrix of the scheme. Obviously we have

$$w \geqslant s, \quad w^* \geqslant s^*$$

where $s = |\{i \neq 0 : a_i \neq 0\}|$, $s^* = |\{i \neq 0 : (\mathbf{a}Q)_i \neq 0\}|$ denote the *degree* and the *dual degree* of C, respectively [3, 1]. They showed that

$$w \geqslant d - s^*$$

for a non-empty subset C in a metric d-class association scheme, and that if equality holds then C is completely regular [2, Theorem 1]. (Suzuki [19] also obtained this result in a more general setting.) Moreover, they showed that

$$w^* \geqslant d - s$$

for a non-empty subset C in a cometric d-class association scheme, and that if equality holds then C induces a cometric s-class association scheme inside the original [2, Theorem 2].

In particular, we have $w + w^* \geqslant d$ for subsets in a metric and cometric d-class association scheme and if $w + w^* = d$ then equality is achieved in each of the above four inequalities as well. In fact, subsets with $w + w^* = d$ arise quite naturally in association schemes associated with regular semilattices [2, Theorem 5]. In this article, we give a classification of such subsets in (1) Grassmann graphs, (2) bilinear forms graphs and (3) dual polar graphs.

Throughout we shall use the following notation and description for each of the above graphs $(X, \mathbf{R})$. For (1), X is the set of d-dimensional subspaces of a vector space V of dimension n over the finite field $GF(q)$, where $n \geqslant 2d$. For (2), let V be a vector space of dimension $d + e$ over $GF(q)$ where $e \geqslant d$. Fix a subspace W of dimension e and let X be the set of d-dimensional subspaces γ of V with $\gamma \cap W = 0$. See [1, §9.5A]. For (3), we assume that V is a vector space over $GF(q)$ equipped with a specified nondegenerate form (alternating, Hermitian or quadratic) with Witt index d, and X is the set of maximal isotropic subspaces in this case.

The following is our main result:

Theorem 1. *Let $(X, \mathbf{R})$ be one of the above graphs and C a non-empty subset of X with $w + w^* = d$.*

(1) *If $(X, \mathbf{R})$ is a Grassmann graph, then either (a) C consists of all elements of X which contain a fixed subspace of dimension w^*, or (b) $n = 2d$ and C is the set of elements of X contained in a fixed subspace of dimension $d + w$.*

(2) *If $(X, \mathbf{R})$ is a bilinear forms graph, then either (a) C consists of all elements of X which contain a fixed subspace U of dimension w^* with $U \cap W = 0$, or (b) $e = d$ and C is the set of elements of X contained in a fixed subspace U' of dimension $d + w$ with $\dim U' \cap W = w$.*

(3) *If $(X, \mathbf{R})$ is a dual polar graph, then C consists of the set of all elements of X which contain a fixed isotropic subspace U of dimension w^*.*

A proof of Theorem 1 is given in Section 3. We remark that Brouwer et al. [2, Theorem 8] obtained a complete classification of subsets with $w + w^* = d$ for Johnson graphs and Hamming graphs as a consequence of a result of Meyerowitz on the completely regular codes of strength zero in these graphs. Thus, the classification of such subsets is complete for all classical distance-regular graphs associated with regular semilattices. Our proof of Theorem 1 is based on an observation that the parameters of the subscheme induced on a subset with $w + w^* = d$ are uniquely determined by w and $w^* = d - w$ (see Section 2), and in fact works for Johnson graphs and Hamming graphs as well.

As an application of Theorem 1, we establish the *Erdős-Ko-Rado theorem* for Grassmann graphs and bilinear forms graphs in Section 4. This theorem was previously obtained by Hsieh [10], Frankl-Wilson [8] and Fu [9] for Grassmann graphs, and by Huang [11, 12] and Fu [9] for bilinear forms graphs. However their characterization for optimal intersecting families requires the assumption $\dim V \geqslant 2d+1$. We provide a proof which is valid for all $\dim V \geqslant 2d$.

2 Uniqueness of the parameters

Let $(X, \mathbf{R})$ be a metric and cometric association scheme with respect to the orderings $R_0, R_1, \ldots, R_d$ and $E_0, E_1, \ldots, E_d$ of the relations and the primitive idempotents of the Bose-Mesner algebra $\mathscr{A}$, respectively. Let Q denote the second eigenmatrix of $(X, \mathbf{R})$.

Let C be a non-empty subset of X such that $w + w^* = d$. Then C, together with the set of non-empty relations $\mathbf{R}|_{C\times C} = \{(C \times C) \cap R_i : 0 \leqslant i \leqslant w\}$, forms a cometric association scheme [2, Theorem 2]. In this section, we show that the parameters of the subscheme $(C, \mathbf{R}|_{C\times C})$ depend only on w and $w^* = d - w$, which is in fact implicit in the proof of [2, Theorem 2].

Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of $(X, \mathbf{R})$. For each matrix M in $\mathscr{A}$, let $\overline{M}$ denote the principal submatrix of M corresponding to the elements of C. Then $\overline{\mathscr{A}} = \{\overline{M} : M \in \mathscr{A}\}$ is the Bose-Mesner algebra of $(C, \mathbf{R}|_{C\times C})$.

Proposition 2. *With the above notation, the parameters of $(C, \mathbf{R}|_{C\times C})$ are uniquely determined by w and $w^* = d - w$.*

Proof. The set $\{\overline{A}_0, \overline{A}_1, \ldots, \overline{A}_w\}$ gives the basis of the adjacency matrices of $(C, \mathbf{R}|_{C\times C})$. Brouwer et al. has shown in the proof of [2, Theorem 2] that (i) $\{\overline{E}_0, \overline{E}_1, \ldots, \overline{E}_w\}$ is a basis, (ii) $\{\overline{E}_0, \ldots, \overline{E}_{j-1}, \overline{I}, \overline{E}_{w^*+j+1}, \ldots, \overline{E}_d\}$ is a basis for $0 \leqslant j \leqslant w$, and (iii) $\overline{E}_k \overline{E}_l = 0$ whenever $|k - l| > w^*$. Since $\overline{E}_j = |X|^{-1} \sum_{i=0}^{w} Q_{i,j} \overline{A}_i$, the base change matrices among these three types of bases do not depend on C. Thus, if we write

$$\overline{E}_i \overline{E}_j = \sum_{k=0}^{w} \tau_{i,j}^k(C) \overline{E}_k$$

for $i, j \in \{0, 1, \ldots, w\}$, then it suffices to verify that the $\tau_{i,j}^k(C)$ are independent of C. We use induction on i. By (ii) above, $\{\overline{E}_0, \ldots, \overline{E}_{i-1}, \overline{I}, \overline{E}_{w^*+i+1}, \ldots, \overline{E}_d\}$ is a basis for $\overline{\mathscr{A}}$. We have $\overline{E}_i \overline{I} = \overline{E}_i$, $\overline{E}_i \overline{E}_{w^*+i+1} = \cdots = \overline{E}_i \overline{E}_d = 0$ by (iii), and if $i > 0$ then the $\tau_{i,j}^k(C) = \tau_{j,i}^k(C)$ $(0 \leqslant j \leqslant i-1, 0 \leqslant k \leqslant w)$ are independent of C by the induction hypothesis. Since the base change matrix between $\{\overline{E}_0, \overline{E}_1, \ldots, \overline{E}_w\}$ and $\{\overline{E}_0, \ldots, \overline{E}_{i-1}, \overline{I}, \overline{E}_{w^*+i+1}, \ldots, \overline{E}_d\}$ does not depend on C, this shows that the assertion is true for i. $\square$

3 Proof of Theorem 1

In this section, we prove Theorem 1. We retain the notation in the previous section.

Let $(X, \mathbf{R})$ be one of the graphs in Theorem 1. Then $(X, \mathbf{R})$ is naturally associated with a regular semilattice (see [4, 18]) and each object in the semilattice gives rise to a subset satisfying $w + w^* = d$. Namely, for $0 \leqslant t \leqslant d$, let U be a subspace of V of dimension t. For (2) we assume $U \cap W = 0$, and for (3) we assume that U is isotropic. It is a standard fact that the set

$$C_U = \{\gamma \in X : U \subseteq \gamma\}$$

has width $d-t$ and dual width t (cf. [2, Theorem 5]). Moreover, $(C_U, \mathbf{R}|_{C_U \times C_U})$ preserves all classical parameters [1] except the diameter. In particular, C_U is convex (i.e., geodetically closed).

Let C be a non-empty subset of X with width $w = d - t$ and dual width $w^* = t$. Then since $(C, \mathbf{R}|_{C \times C})$ has the same parameters as $(C_U, \mathbf{R}|_{C_U \times C_U})$ by Proposition 2, C is also convex. Lambeck [13, Chapter 5] classified the convex subgraphs in all classical distance-regular graphs except those in the quadratic forms graphs over the finite fields of characteristic two (see [15] for this case). Thus Theorem 1 immediately follows from his result. $\square$

Remark. In fact, it is possible to give a direct and quite simple proof of Theorem 1 without using Lambeck's result. See [20] for details.

Remark. Our proof of Theorem 1 clearly works for Johnson graphs and Hamming graphs as well, but relies heavily on the existence of specific examples of subsets with $w + w^* = d$. It is an interesting problem whether it is possible to derive the convexity without reference to the existence of such examples or not. There are also certain nice posets naturally associated with the other classical distance-regular graphs, namely alternating forms graphs, Hermitian forms graphs and quadratic forms graphs (see e.g. [17]). However, in general we do not obtain subsets satisfying $w + w^* = d$ from these poset structures.

4 The Erdős-Ko-Rado theorem

The *Erdős-Ko-Rado theorem* [7, 21] is a classical result in extremal set theory which asserts that the largest possible families $\mathscr{F}$ of d-subsets of an n-set such that $|\gamma \cap \delta| \geqslant t$ for all $\gamma, \delta \in \mathscr{F}$ where $n > (t+1)(d-t+1)$ are the families of all d-subsets containing some fixed t-subset.

In this section, we prove the following:

Theorem 3. (1) *Let $\mathscr{F}$ be a collection of elements of the vertex set X of a Grassmann graph with the property that* $\dim \gamma \cap \delta \geqslant t$ *for all γ, δ in $\mathscr{F}$, where $0 \leqslant t \leqslant d$. Then we have $|\mathscr{F}| \leqslant \left[{n-t \atop d-t}\right]$, and equality holds if and only if either (a) $\mathscr{F}$ consists of all elements of X which contain a fixed t-dimensional subspace of V, or (b) $n = 2d$ and $\mathscr{F}$ is the set of all elements of X contained in a fixed*

$(n-t)$-dimensional subspace of V.
(2) Let $\mathscr{F}$ be a collection of elements of the vertex set X of a bilinear forms graph with the property that $\dim \gamma \cap \delta \geqslant t$ *for all γ, δ in $\mathscr{F}$, where* $0 \leqslant t \leqslant d$. *Then we have $|\mathscr{F}| \leqslant q^{(d-t)e}$, and equality holds if and only if either (a) $\mathscr{F}$ consists of all elements of X which contain a fixed t-dimensional subspace U with $U \cap W = 0$, or (b) $e = d$ and $\mathscr{F}$ is the set of all elements of X contained in a fixed $(2d-t)$-dimensional subspace U' with* $\dim U' \cap W = d - t$.

For Grassmann graphs (1), Hsieh [10] proved Theorem 3 for $n \geqslant 2d+1$ and $(n,q) \neq (2d+1, 2)$. Frankl and Wilson [8] obtained the bound $|\mathscr{F}| \leqslant \left[{n-t \atop d-t}\right]$ for $n \geqslant 2d$ and $q \geq 2$. They asserted [8, p.229] that the uniqueness of the optimal families for $n \geqslant 2d+1$ can also be obtained using the methods of [6]. They also stated that for $n = 2d$ it appears likely that there are only two non-isomorphic optimal families. Thus our result verifies the validity of their observation. In fact, Theorem 3 (1) is an immediate consequence of Theorem 1 (1) and their result.

For bilinear forms graphs (2), Huang [11] proved Theorem 3 for $e \geqslant d+1$ and $(e,q) \neq (d+1, 2)$ (see also [12]). As pointed out in [11, p.192, Remark], the bound $|\mathscr{F}| \leqslant q^{(d-t)e}$ for $e \geqslant d$ and $q \geq 2$ follows from a result of Delsarte [3, Theorem 3.9] and his construction [5] of (d,e,t,q)-Singleton systems for all values of the parameters d, e, t and q. A slightly more detailed analysis of this argument yields Theorem 3 (2).

Fu [9] proved the results of [10, 11] in a unified way using the notion of quantum matroids. For the Erdős-Ko-Rado theorems for other graphs, see [14] for Hamming graphs and [16] for dual polar graphs.

Proof. (1) The proof of the bound by Frankl and Wilson [8] is an application of Delsarte's linear programming bound [3]. Let χ be the (column) characteristic vector of $\mathscr{F}$. They constructed a matrix A in the Bose-Mesner algebra $\mathscr{A}$ such that (i) the (γ, δ)-entry of A is 0 whenever $\dim \gamma \cap \delta \geqslant t$, and (ii) the matrix $A + I - \left[{n-t \atop d-t}\right]^{-1} J$ is positive semidefinite and the i-th eigenvalue of $A + I - \left[{n-t \atop d-t}\right]^{-1} J$ is positive precisely when $t+1 \leqslant i \leqslant d$. (See [8, §5] for the latter half of (ii). There is a minor error in the middle of page 235 in that paper: '$\lambda_e < -1$' must be '$\lambda_e > -1$'.) Then $\chi^{\mathsf{T}} A \chi = 0$ since $\mathscr{F}$ is t-intersecting, and moreover

$$0 \leqslant \chi^{\mathsf{T}} \left(A + I - \begin{bmatrix} n-t \\ d-t \end{bmatrix}^{-1} J \right) \chi = |\mathscr{F}| - \begin{bmatrix} n-t \\ d-t \end{bmatrix}^{-1} |\mathscr{F}|^2,$$

or equivalently $|\mathscr{F}| \leqslant \left[{n-t \atop d-t}\right]$. In the case of equality χ is in the null space of $A + I - \left[{n-t \atop d-t}\right]^{-1} J$ which is exactly $V_0 + V_1 + \cdots + V_t$, where V_i is the i-th eigenspace of $\mathscr{A}$. Thus if equality holds then $w^* \leqslant t$. Together with $w \leqslant d-t$ and the general inequality $w + w^* \geqslant d$, we conclude $w = d - t$ and $w^* = t$. Now the result immediately follows from Theorem 1 (1).

(2) A (d,e,t,q)-*Singleton system* is a t-design in X of index 1. Equivalently, a subset $Y \subseteq X$ with inner distribution $\mathbf{b} = (b_0, b_1, \ldots, b_d)$ is a (d,e,t,q)-*Singleton system* if $(\mathbf{b}Q)_1 = \cdots = (\mathbf{b}Q)_t = 0$ and $|Y| = q^{te}$. In this case it

turns out that Y is also a $(d-t)$-codesign, i.e., $b_1 = \cdots = b_{d-t} = 0$ [5, Theorem 5.4]. The inner distribution $\mathbf{b}$ is uniquely determined by d, e, t and q, and for $0 \leqslant i \leqslant t-1$, $b_{d-i} = b(d, e, t, q; i)$ is given by the formula

$$b_{d-i} = b(d,e,t,q;i) = \begin{bmatrix} d \\ i \end{bmatrix} \sum_{j=0}^{t-i-1} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} d-i \\ j \end{bmatrix} (q^{(t-i-j)e} - 1)$$

[5, Theorem 5.6].

Delsarte [5, §6] constructed a (d, e, t, q)-Singleton system $Y(d, e, t, q)$ for each $e \geqslant d \geqslant t \geqslant 0$ and $q \geqslant 2$. In fact, $Y(d, e, t, q)$ is a subgroup of the additive group $(X, +)$ (where we regard X as the set of $d \times e$ matrices over $GF(q)$). Thus the dual subgroup $Y(d, e, t, q)^{\perp}$ of $Y(d, e, t, q)$ with respect to a nondegenerate inner product on $(X, +)$ is a $(d, e, d-t, q)$-Singleton system and in particular $q^{-te}\mathbf{b}Q$ is the inner distribution of $Y(d, e, d-t, q)$.

Let $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ be the inner distribution of $\mathscr{F}$. Then $a_{d-t+1} = \cdots = a_d = 0$, and [3, Theorem 3.9] gives the inequality

$$|\mathscr{F}| \cdot |Y(d,e,t,q)| \leqslant |X|$$

or equivalently $|\mathscr{F}| \leqslant q^{(d-t)e}$. Moreover in the case of equality, $\mathbf{a}$ and the inner distribution $\mathbf{b} = (b_0, b_1, \ldots, b_d)$ of $Y(d, e, t, q)$ satisfy

$$(\mathbf{a}Q)_i(\mathbf{b}Q)_i = 0 \quad \text{for all } i \in \{1, 2, \ldots, d\}.$$

(See also [1, p.55, Proposition 2.5.3].) In order to apply Theorem 1 (2), we only have to show $b(d, e, t, q; i) \neq 0$ for all d, e, t, q and $0 \leqslant i \leqslant t-1$. Indeed, since $(\mathbf{b}Q)_{d-i} = q^{te} b(d, e, d-t, q; i)$ for $0 \leqslant i \leqslant d-t-1$, this implies $(\mathbf{a}Q)_{t+1} = \cdots = (\mathbf{a}Q)_d = 0$ whenever $|\mathscr{F}| = q^{(d-t)e}$, and it follows from $w \leqslant d-t$, $w^* \leqslant t$ and $w + w^* \geqslant d$ that in fact $w = d-t$ and $w^* = t$.

We follow [8, §5]. Namely, since the expression for $b(d, e, t, q; i)$ is an alternating sum, it is sufficient to prove that the terms decrease in absolute value. We need the following two inequalities:

$$\frac{b-1}{a-1} < \frac{b}{a} \qquad \text{for } a > b \geqslant 1,$$
$$\frac{q^b - 1}{q^a - 1} < q^{b-a+1} \quad \text{for } a \geqslant 1, q \geqslant 2.$$

Let $\mu_j = q^{\binom{j}{2}} \left[\begin{smallmatrix} d-i \\ j \end{smallmatrix}\right] (q^{(t-i-j)e} - 1)$. Then, for $0 \leqslant j \leqslant t-i-2$ we have

$$\begin{aligned} \frac{\mu_{j+1}}{\mu_j} &= \frac{q^{\binom{j+1}{2}} \left[\begin{smallmatrix} d-i \\ j+1 \end{smallmatrix}\right] (q^{(t-i-j-1)e} - 1)}{q^{\binom{j}{2}} \left[\begin{smallmatrix} d-i \\ j \end{smallmatrix}\right] (q^{(t-i-j)e} - 1)} \\ &= q^j \cdot \frac{q^{d-i-j} - 1}{q^{j+1} - 1} \cdot \frac{q^{(t-i-j-1)e} - 1}{q^{(t-i-j)e} - 1} \\ &< q^j \cdot q^{d-i-2j} \cdot q^{-e} = q^{d-i-j-e} \leqslant 1. \end{aligned}$$

This completes the proof of Theorem 3 (2). □

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