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The Burnside dimension of projective Mackey functors

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Abstract : In this note, I will apply methods exposed by J. P. May ([7]) to the special case of Mackey functors for a finite group G over a commutative ring R. In particular, any finitely generated projective Mackey functor has a *Burnside dimension*, which is an element of the Burnside algebra $RB(G)$ of G over R.

1. Mackey functors

There are several equivalent possible definitions of Mackey functors. In this note, I will use two of them. In both of them R is a commutative ring (with identity element), and G is a finite group :

1.1. Definition in terms of G-sets. The first definition of Mackey functors is due to A. Dress ([5]) :

A *Mackey functor* M for G over R is a bivariant functor $M = (M_*, M^*)$, from the category G-set of finite G-sets to the category R-Mod of R-modules, satisfying the following two conditions :

1. The functor M maps disjoint unions to direct sums : if X and Y are finite G-sets, if i_X and i_Y are the canonical inclusions from X and Y to the disjoint union $X \sqcup Y$, then the maps $(M_*(i_X), M_*(i_Y))$ and $\begin{pmatrix} M^*(i_X) \\ M^*(i_Y) \end{pmatrix}$ are mutual inverse isomorphisms of R-modules between $M(X) \oplus M(Y)$ and $M(X \sqcup Y)$.
2. If
$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] & Y \arrow[d, "c"] \\ Z \arrow[r, "d"'] & T \end{tikzcd}$$
is a cartesian (pullback) square of finite G-sets, then $M_*(b)M^*(a) = M^*(d)M_*(c)$.

A *morphism of Mackey functors* is a natural transformation of bivariant functors. The Mackey functors for G over R form a category, denoted by $\mathsf{Mack}_R(G)$.

1.2. The Mackey algebra. The second definition of Mackey functors is due to J. Thévenaz and P. Webb ([9]), who defined the Mackey algebra. The present exposition follows Chapter 4 of [2].

Let
$$\Omega_G = \bigsqcup_{H \subseteq G} G/H = \{xH \mid x \in G,\ H \subseteq G\}$$
denote the disjoint union of all transitive left G-sets G/H, where H runs through the set of subgroups of G.

If X is a finite (left) G-set, denote by $B(X)$ the Burnside group of X, i.e. the Grothendieck group of the category G-$\mathsf{set}\downarrow_X$ of G-sets *over* X. Similarly denote by $RB(X)$ the tensor product $R \otimes_{\mathbb{Z}} B(X)$.

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The *Mackey algebra* $\mu_R(G)$ of the group G over R is defined by

$$\mu_R(G) = RB(\Omega_G^2) \quad ,$$

where Ω_G^2 denotes the G-set $\Omega_G \times \Omega_G$ (for diagonal G-action). The multiplication on $\mu_R(G)$ is defined by R-linear extension from the pullback product

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "a"'] \arrow[dr, "b"] & & & & Y \arrow[dl, "c"'] \arrow[dr, "d"] & & & & X\times_{\Omega_G} Y \arrow[dl, "e"'] \arrow[dr, "f"] & \\ \Omega_G & & \Omega_G & \times & \Omega_G & & \Omega_G & \mapsto & \Omega_G & & \Omega_G \end{tikzcd} \quad ,$$

where $X\times_{\Omega_G} Y = \{(x,y) \in X\times Y \mid b(x) = c(y)\}$, and $e(x,y) = a(x)$, and $f(x,y) = d(y)$.

The identity element for this multiplication is the G-set

$$\begin{tikzcd} & \Omega_G \arrow[dl] \arrow[dr] & \\ \Omega_G & & \Omega_G \end{tikzcd}$$

where both maps are the identity map of Ω_G.

Now a *Mackey functor* for G over R is a left $\mu_R(G)$-module.

1.3. Equivalence. Saying that the above two definitions are equivalent means that the categories $\mathsf{Mack}_R(G)$ and $\mu_R(G)$-Mod are equivalent, and this equivalence can be seen as follows : suppose that V is a $\mu_R(G)$-module, and that X is a finite G-set. Then the R-module $RB_X = RB(\Omega_G \times X)$ has a natural structure of left $\mu_R(G)$-module, obtained by R-linear extension from the obvious pullback product. Thus setting

$$F_V(X) = \mathrm{Hom}_{\mu_R(G)}(RB_X, V)$$

defines a Mackey functor F_V, in the sense of Dress.

Conversely, if M is a Mackey functor in this sense, then $M(\Omega_G)$ has a natural structure of $\mu_R(G)$-module (see Section 4.3 of [2] for details).

1.4. Tensor product of Mackey functors. If M, N, and P are Mackey functors for G over R, a *bilinear morphism* $\varphi : M, N \to P$ is a collection of R-bilinear $\varphi_{X,Y}$: $M(X)\times N(Y) \to P(X\times Y)$, for any finite G-sets X and Y, which are moreover bivariant with respect to X and Y. The tensor product $M\hat{\otimes}N$ can be defined as the solution to the universal problem of bilinear morphisms : this means that the set of bilinear morphisms from M, N to P is in one to one correspondence with the set of morphisms of Mackey functors from $M\hat{\otimes}N$ to P (Proposition 1.8.2 of [2]).

If X is a finite G-set, then $M\hat{\otimes}N(X)$ can be computed as follows :

$$M\hat{\otimes}N(X) = \Big(\bigoplus_{Y\xrightarrow{f}X} M(Y)\otimes N(Y)\Big)/\mathcal{J} \quad ,$$

where $\mathcal{J}$ is the R-submodule generated by expressions $[M_*(a)(u)\otimes v]_{Z,g}-[u\otimes N^*(a)(v)]_{Y,f}$ and $[M^*(a)(u')\otimes v']_{Y,f}-[u'\otimes N_*(a)(v')]_{Z,g}$, for every commutative triangle of finite G-sets

$$\begin{tikzcd} Y \arrow[rr, "a"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

for every $u\in M(Y)$, $v\in N(Z)$, $u'\in M(Z)$, $v'\in N(Y)$, where e.g. $[M_*(a)(u)\otimes v]_{Z,g}$ denotes the element $M_*(a)(u)\otimes v$ of the component $M(Z)\otimes N(Z)$ indexed by $g:Z\to X$ in the direct sum.

The tensor product of Mackey functors is commutative (or symmetric), and associative. The Burnside functor RB is an identity for this tensor product, which means that the functors $RB\hat{\otimes}-$ and $-\hat{\otimes}RB$ are both isomorphic to the identity functor of $\mathsf{Mack}_R(G)$ (see Section 2.4 of [2] for details).

1.5. The Dress construction. Let M be a Mackey functor for G over R, and let X be a finite G-set. The bivariant functor M_X obtained by composition of M with the endofunctor $Y\mapsto Y\times X$ of G-set is a Mackey functor for G over R. This construction $\mathrm{Id}_X : M\mapsto M_X$ is an endofunctor of the category $\mathsf{Mack}_R(G)$, called the *Dress construction associated to* X. This functor Id_X is self adjoint (Lemma 3.1.1 of [2]).

1.6. Internal Hom. If M and N are Mackey functors for G over R, the functor $\mathcal{H}(M,N)$ was defined in Section 1.3 of [2]. It is another Mackey functor for G over R, whose value at the G-set X is

$$\mathcal{H}(M,N)(X)=\mathrm{Hom}_{\mathsf{Mack}_R(G)}(M,N_X)\quad .$$

The construction $(M,N)\mapsto\mathcal{H}(M,N)$ is an internal Hom in the category $\mathsf{Mack}_R(G)$. It is right adjoint to the tensor product of Mackey functors, in the following sense : if M, N, and P are Mackey functors for G over R, then there are isomorphisms of Mackey functors

$$\mathcal{H}(M\hat{\otimes}N,P)\cong\mathcal{H}\big(N,\mathcal{H}(M,P)\big)$$

which are natural in M, N, and P.

In the same situation, there is also *a composition morphism*

$$\gamma:\mathcal{H}(M,N)\hat{\otimes}\mathcal{H}(N,P)\to\mathcal{H}(M,P)$$

defined as follows : let X be a finite G-set. Then $\mathcal{H}(M,P)(X)=\mathrm{Hom}_{\mathsf{Mack}_R(G)}(M,P_X)$, whereas $\mathcal{H}(M,N)\hat{\otimes}\mathcal{H}(N,P)(X)$ is a quotient of the direct sum

$$\Sigma=\bigoplus_{Y\xrightarrow{f}X}\mathrm{Hom}_{\mathsf{Mack}_R(G)}(M,N_Y)\otimes\mathrm{Hom}_{\mathsf{Mack}_R(G)}(N,P_Y)\quad .$$

Fix some G-set (Y,f) over X (where $f:Y\to X$), and let $a:M\to N_Y$ and $b:N\to P_Y$ be some morphisms of Mackey functors. Then the image by γ_X of the element $a\otimes b$ of the component of Σ indexed by (Y,f) is the morphism $M\to P_X$ whose evaluation at a G-set Z is the map $M(Z)\to P(Z\times X)$ obtained by the composition

$$M(Z)\xrightarrow{a_Z}N(ZY)\xrightarrow{b_{ZY}}P(ZY^2)\xrightarrow{P^*\left(\substack{zy\\ \downarrow\\ zyy}\right)}P(ZY)\xrightarrow{P_*\left(\substack{zy\\ \downarrow\\ zf(y)}\right)}P(ZX)\quad ,$$

where for short $\begin{pmatrix} zy \\ \downarrow \\ zyy \end{pmatrix}$ denote the map $(z,y) \in Z \times Y \mapsto (z,y,y) \in Z \times Y \times Y$.

Finally, the Burnside functor is a left unit for $\mathcal{H}$: for any Mackey functor, there is an isomorphism $\mathcal{H}(RB, M) \cong M$.

2. Burnside trace and dimension

The previous section recalls various constructions in $\mathsf{Mack}_R(G)$, and shows that this category is a *closed symmetric monoidal category.* In this general framework, J.P May has developed a theory of Euler characteristics and Burnside rings (see [7]), and one can try to see how this theory applies in this particular example.

2.1. Dualizable objects. The dual DM of a Mackey functor M for G over R is defined by

$$DM = \mathcal{H}(M, RB) \quad .$$

(this notion is *different* from the notion of *dual over R* defined in Section 6.2.2 of [2])

The isomorphism $\mathcal{H}(RB, M) \cong M$ gives a composition morphism

$$j_M : DM \hat{\otimes} M \to \mathcal{H}(M, M) \quad ,$$

and M is called *dualizable* if j_M is an isomorphism.

Conversely, there is a morphism

$$e_M : M \hat{\otimes} DM \to \mathcal{H}(RB, RB) \cong RB \quad .$$

2.2. Lemma : *The Mackey functor M is dualizable if and only if M is finitely generated and projective.*

Proof: (Sketch) Suppose that M is dualizable. Evaluating j_M at the trivial G-set $\bullet$ gives an isomorphism

$$(DM \hat{\otimes} M)(\bullet) \to \mathrm{End}_{\mathsf{Mack}_R(G)}(M) \quad .$$

Choosing an element in the left hand side mapping to the identity of M shows that there exist a positive integer n, finite G-sets Y_i, and morphisms $a^{(i)} : M \to RB_{Y_i}$ and $b^{(i)} : RB \to M_{Y_i}$, for $i \in \{1, \ldots, n\}$ such that for any G-set Z

$$\mathrm{Id}_{M(Z)} = \sum_{i=1}^{n} M_* \begin{pmatrix} zy_i \\ \downarrow \\ z \end{pmatrix} M^* \begin{pmatrix} zy_i \\ \downarrow \\ zy_iy_i \end{pmatrix} \circ b^{(i)}_{ZY_i} \circ a^{(i)}_Z \quad . \tag{2.3}$$

Using the adjunction $\mathrm{Hom}_{\mathsf{Mack}_R(G)}(RB, M_{Y_i}) \cong \mathrm{Hom}_{\mathsf{Mack}_R(G)}(RB_{Y_i}, M)$, the morphisms $b^{(i)}$ give morphisms $\tilde{b}^{(i)} : RB_{Y_i} \to M$, and one can check that equality 2.3 is equivalent to

$$\mathrm{Id}_M = \sum_{i=1}^{n} \tilde{b}^{(i)} \circ a^{(i)} \quad .$$

Setting $X = \bigsqcup_{i=1}^{n} Y_i$, this shows that M is a direct summand of RB_X. Since RB_X is finitely generated and projective, so is M.

Conversely, if M is finitely generated and projective, then M is a direct summand of some functor RB_X, for a finite G-set X. Since any direct summand of a dualizable object is a dualizable object, it suffices to show that RB_X is dualizable. And this is easy, because

$$\mathcal{H}(RB_X, RB)\hat{\otimes}RB_X \cong \mathcal{H}(RB_X, RB)_X \cong \mathcal{H}(RB_X, RB_X) \quad .$$

Here the first isomorphism is a consequence of Lemma 7.2.3 of [2], which implies that for any Mackey functors M and N, and any finite G-set X, one has that $(M\hat{\otimes}N)_X \cong M_X\hat{\otimes}N \cong M\hat{\otimes}N_X$, and from the fact that RB is a unit for the tensor product $\hat{\otimes}$. The second isomorphism follows easily from the definitions of $\mathcal{H}$. □

2.4. Burnside trace and dimension. Let M be a dualizable (i.e. finitely generated and projective) Mackey functor, and let $f \in \mathrm{End}_{\mathsf{Mack}_R(G)}(M)$. There is a sequence of morphisms

$$RB \overset{i_M}{\to} \mathcal{H}(M,M) \overset{j_M^{-1}}{\to} DM\hat{\otimes}M \overset{\sigma_M}{\to} M\hat{\otimes}DM \overset{f\otimes \mathrm{Id}}{\to} M\hat{\otimes}DM \overset{e_M}{\to} RB \quad .$$

Here the morphism i_M is the unique morphism of Green functors from RB to $\mathcal{H}(M,M)$: there is a unique such morphism, because $\mathcal{H}(M,M)$ is a Green functor (Proposition 2.1.1 of [2]), and RB is an initial object in the category of Green functors for G over R (Proposition 2.4.4 of [2]). The morphism σ_M comes from the commutativity of $\hat{\otimes}$.

The composition of these morphisms of Mackey functors is an endomorphism of RB. Since $\mathrm{End}_{\mathsf{Mack}_R(G)}(RB)$ is isomorphic to the evaluation $RB(\bullet)$ at the trivial G-set, i.e. to the Burnside algebra $RB(G)$ of G over R, this gives an element denoted by $\mathrm{Btr}(f)$ of $RB(G)$.

2.5. Definition and Notation : *This element* $\mathrm{Btr}(f)$ *of* $RB(G)$ *will be called* the Burnside trace *of the endomorphism* f. *When* $f = \mathrm{Id}$, *it will be called* the Burnside dimension *of* M, *and denoted by* $\mathrm{Bdim}(M)$.

2.6. Proposition : *Let M be a dualizable Mackey functor for G over R, let X be a finite G set, and let $p : RB_X \to M$ and $s : M \to RB_X$ be morphisms of Mackey functors such that $p \circ s = \mathrm{Id}_M$. Let ϵ_X the element of $RB(X^2)$ corresponding to the diagonal inclusion $x \mapsto (x,x)$ of X into X^2. If $f \in \mathrm{End}_{\mathsf{Mack}_R(G)}$, then :*

$$\mathrm{Btr}(f) = RB_*\begin{pmatrix} x \\ \downarrow \\ \bullet \end{pmatrix} RB^*\begin{pmatrix} x \\ \downarrow \\ xx \end{pmatrix} s_X f_X p_X(\epsilon_X) \quad .$$

Proof: First of all, the isomorphism $\mathrm{End}_{\mathsf{Mack}_R(G)}(RB) \cong RB(\bullet)$ is the map sending the endomorphism f of RB to $f_\bullet(\bullet) \in RB(\bullet)$. Now the image of $\bullet \in RB(\bullet)$ in $\mathcal{H}(M,M)(\bullet) = \mathrm{End}_{\mathsf{Mack}_R(G)}(M)$ is the identity map of M. The hypotheses imply that the image by $(j_M^{-1})_\bullet$ of the identity map of M is the element $s \otimes \tilde{p}$ of the component $X \to \bullet$ in the direct sum

$$\bigoplus_{X \to \bullet} \mathrm{Hom}_{\mathsf{Mack}_R(G)}(M, RB_X) \otimes \mathrm{Hom}_{\mathsf{Mack}_R(G)}(RB, M_X)$$

defining $(DM\hat{\otimes}M)(\bullet)$, where $\tilde{p}$ is the morphism $RB \to M_X$ obtained by adjunction from the morphism $p : RB_X \to M$. The image of this element $s \otimes \tilde{p}$ by $(\sigma_M)_\bullet$ is the element $\tilde{p} \otimes s$ of the component $X \to \bullet$ in the direct sum

$$\bigoplus_{X \to \bullet} \mathrm{Hom}_{\mathsf{Mack}_R(G)}(RB, M_X) \otimes \mathrm{Hom}_{\mathsf{Mack}_R(G)}(M, RB_X)$$

defining $(M\hat{\otimes}DM)(\bullet)$. By the map $(f\otimes \mathrm{Id})_\bullet$, this element $\tilde{p}\otimes s$ is sent to $f_X\tilde{p}\otimes s$ in the same component of the direct sum. And finally, by the map $(e_M)_\bullet$, this is sent to the endomorphism $\tilde{s}f_X\tilde{p}$ of RB, where $\tilde{s}: M_X \to RB$ is the morphism deduced by adjunction from $s: M\to RB_X$.

It follows that

$$\mathrm{Btr}(f) = (\tilde{s}f_X\tilde{p})_\bullet(\bullet) \quad .$$

So it is the image of $\bullet \in RB(\bullet)$ by the map

$$RB(\bullet) \xrightarrow{\tilde{p}_\bullet} M(X) \xrightarrow{f_X} M(X) \xrightarrow{\tilde{s}_\bullet} RB(\bullet) \quad .$$

Now the map $\tilde{p}_\bullet$ is the map

$$RB(\bullet) \xrightarrow{RB^*\left(\begin{smallmatrix} x\\ \downarrow\\ \bullet\end{smallmatrix}\right)} RB(X) \xrightarrow{RB_*\left(\begin{smallmatrix} x\\ \downarrow\\ xx\end{smallmatrix}\right)} RB(X^2) \xrightarrow{p_X} M(X) \quad ,$$

and $RB_*\left(\begin{smallmatrix} x\\ \downarrow\\ xx\end{smallmatrix}\right) RB^*\left(\begin{smallmatrix} x\\ \downarrow\\ \bullet\end{smallmatrix}\right)(\bullet) = \epsilon_X$. Moreover the map $\tilde{s}_\bullet$ is the map

$$M(X) \xrightarrow{s_X} RB(X^2) \xrightarrow{RB^*\left(\begin{smallmatrix} x\\ \downarrow\\ xx\end{smallmatrix}\right)} RB(X) \xrightarrow{RB_*\left(\begin{smallmatrix} x\\ \downarrow\\ \bullet\end{smallmatrix}\right)} RB(\bullet) \quad .$$

It follows that $\mathrm{Btr}(f) = RB_*\left(\begin{smallmatrix} x\\ \downarrow\\ \bullet\end{smallmatrix}\right) RB^*\left(\begin{smallmatrix} x\\ \downarrow\\ xx\end{smallmatrix}\right) s_X f_X p_X(\epsilon_X)$, as was to be shown. □

The following is the special case $M = RB_X$: then $\mathrm{End}_{\mathsf{Mack}_R(G)}(M) \cong RB(X^2)$:

2.7. Corollary : *Let X and Z be finite G-sets, and let a and b be maps of G-sets from Z to X. Let*

$$f = \begin{tikzcd} & Z \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X & & X \end{tikzcd}$$

be the corresponding element of $RB(X^2)$, viewed as an endomorphism of RB_X. Then

$$\mathrm{Btr}(f) = \{z\in Z \mid a(z) = b(z)\} \quad .$$

In particular $\mathrm{Bdim}(RB_X) = X$.

Proof: In this case, one can suppose that the maps p and s are the identity map. The result for $\mathrm{Btr}(f)$ follows from a straightforward computation, and the result for $\mathrm{Bdim}(RB_X)$ is the special case $Z = X$ and $a = b = \mathrm{Id}_X$. □

2.8. Example : Suppose that $X = G/\mathbf{1}$. Then $\mathrm{End}_{\mathsf{Mack}_R(G)}(RB_X)$ is isomorphic to the group algebra RG : this isomorphism $RG \to RB\big((G/\mathbf{1})^2\big)$, denoted by $x\mapsto \hat{x}$, maps the element $g\in G$ of the canonical basis of RG to the element

$$\hat{g} = \begin{tikzcd} & G/\mathbf{1} \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "g"] & \\ G/\mathbf{1} & & G/\mathbf{1} \end{tikzcd}$$

of $RB\big((G/\mathbf{1})^2\big)$, where the right hand side arrow is right multiplication by g on the G-set $G/\mathbf{1}$. In this case

$$\mathrm{Btr}(\hat{g}) = \delta_{g,1} \cdot G/\mathbf{1} \quad ,$$

where $\delta_{g,1}$ is a Kronecker symbol, so in general $\mathrm{Btr}(\hat{x}) = \mathrm{tr}_{RG}(x) \cdot G/\mathbf{1}$, where $\mathrm{tr}_{RG}(x)$ is the usual trace map on the group algebra RG.

2.9. Remark : Through the equivalence 1.3, the category of Mackey functors is equivalent to the category of $\mu_R(G)$-modules. For $\mu_R(G)$-modules, there is *the Hattori-Stallings trace map* $\mathrm{Tr}_{\mu_R(G)}$ (see [6] and [8]), which associates to any endomorphism f of a finitely generated projective $\mu_R(G)$-module, an element $\mathrm{Tr}_{\mu_R(G)}(f)$ in the zero-th Hochschild homology group of $\mu_R(G)$, i.e.

$$\mathrm{Tr}_{\mu_R(G)}(f) \in HH_0(\mu_R(G)) = \mu_R(G)/[\mu_R(G), \mu_R(G)] \quad .$$

One can show easily that with this equivalence, the Burnside trace $\mathrm{Btr}(f)$ is the image of $\mathrm{Tr}_{\mu_R(G)}(f)$ by the map

$$HH_0(\mu_R(G)) \to RB(G)$$

induced by the "equalizer map" from $\mu_R(G) = RB(\Omega_G^2)$ to $RB(G)$, sending the element

$$\begin{tikzcd} & Z \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \Omega_G & & \Omega_G \end{tikzcd}$$

to the equalizer $\{z \in Z \mid a(z) = b(z)\}$, viewed as an element of $RB(G)$.

3. Functorial properties

3.1. Composition with a biset. Let G and H be finite groups, and let U be a finite (H, G)-biset. If X is a finite G-set, define

$$U \circ X = \{(u, x) \in U \times X \mid \forall g \in G,\ u \cdot g = u \Rightarrow g \cdot x = x\} \quad ,$$

and denote by $U \circ_G X$ the quotient of $U \circ X$ by the right action of G given by

$$(u, x) \cdot g = (u \cdot g, g^{-1} \cdot x),\ \forall (u, x, g) \in U \times X \times G \quad .$$

This construction extends to a map $X \mapsto U \circ_G X$ from $B(G)$ to $B(H)$.

The construction $X \mapsto U \circ_G X$ is a functor γ_U from G-set to H-set, which preserves disjoint unions and pullback squares. Conversely, any functor G-set $\to$ H-set with these two properties is isomorphic to a functor γ_U, for some finite (H, G)-biset U (see [1] for details).

By composition, the functor γ_U induces a functor

$$\Gamma_U : \mathsf{Mack}_R(H) \to \mathsf{Mack}_R(G) \quad ,$$

defined by $\Gamma_U(N) = N \circ \gamma_U$, for any Mackey functor N for H over R.

3.2. Adjoint functors. These functors Γ_U have left and right adjoints, respectively denoted by $\mathcal{L}_U$ and $\mathcal{R}_U$: an explicit, but rather complicated description of the functors $\mathcal{L}_U$ was given in Chapter 9 of [2]. A simpler description ([4]) can be obtained as follows, using the equivalence 1.3 : consider

$$R\mathcal{B}_U = R\mathcal{B}\Big(\Omega_H \times (U \circ_G \Omega_G)\Big) \quad .$$

This is a $\Big(\mu_R(H), \mu_R(G)\Big)$-bimodule, for the actions extending linearly the following products : suppose that $\Big(X, (a, b)\Big)$ is an H-set over $\Omega_H \times \Omega_H$, that $\Big(Y, (c, d)\Big)$ is a G-set over $\Omega_G \times \Omega_G$, and that $\Big(Z, (e, f)\Big)$ is an H-set over $\Omega_H \times (U \circ_G \Omega_G)$. Build the following diagram

$$\begin{tikzcd}[column sep=small]
 & & & E \arrow[dl, "k"'] \arrow[dr, "l"] & & & \\
 & & C \arrow[dl, "g"'] \arrow[dr, "h"] & & D \arrow[dl, "i"'] \arrow[dr, "j"] & & \\
 & X \arrow[dl, "a"'] \arrow[dr, "b"] & & Z \arrow[dl, "e"'] \arrow[dr, "f"] & & U \circ_G Y \arrow[dl, "U \circ_H c"'] \arrow[dr, "U \circ_H d"] & \\
\Omega_H & & \Omega_H & & U \circ_G \Omega_G & & U \circ_G \Omega_G
\end{tikzcd}$$

where all the squares are pull-back squares. Then the left and right actions on $R\mathcal{B}_U$ are defined by

$$\Big(X, (a, b)\Big).\Big(Z, (e, f)\Big).\Big(Y, (c, d)\Big) = \Big(E, (agk, (U \circ_H d)jl)\Big) \quad .$$

It is easy to this from this definition that there is an isomorphism of left $\mu_R(H)$-modules

$$R\mathcal{B}_U \cong R\mathcal{B}_{U \circ_G \Omega_G}(\Omega_H) \quad .$$

In particular $R\mathcal{B}_U$ is projective and finitely generated as $\mu_R(H)$-module. Moreover, one can show that if N is a Mackey functor for H over R, then the natural isomorphism of R-modules

$$(N \circ U)(\Omega_G) = N(U \circ_G \Omega_G) \cong \mathrm{Hom}_{\mu_R(H)}\Big(R\mathcal{B}_U, N(\Omega_H)\Big)$$

is an isomorphism of $\mu_R(G)$-modules. Thus if M is a Mackey functor for G over R, this gives by standard arguments

$$\mathcal{L}_U(M)(\Omega_H) \cong R\mathcal{B}_U \otimes_{\mu_R(G)} M(\Omega_G) \quad .$$

More generally, if Z is a finite H-set, the same argument shows that

$$\textbf{(3.3)} \qquad \mathcal{L}_U(M)(Z) \cong R\mathcal{B}(Z \times U \circ_G \Omega_G) \otimes_{\mu_R(G)} M(\Omega_G) \quad ,$$

where the right $\mu_R(G)$-module structure on $R\mathcal{B}(Z \times U \circ_G \Omega_G)$ is given by pullback as above.

3.4. Remark : A similar argument (see [4]), considering the $\big(\mu_R(G),\mu_R(H)\big)$-bimodule

$$RB_U^\sharp = \mathrm{Hom}_{\mu_R(H)}\Big(R\mathcal{B}_U,\mu_R(H)\Big)$$

gives the description of the right adjoint $\mathcal{R}_U$, by

$$\mathcal{R}_U(M)(\Omega_H) \cong \mathrm{Hom}_{\mu_R(G)}\Big(RB_U^\sharp, M(\Omega_G)\Big) \quad .$$

3.5. Proposition : *Let G be a finite group. If M and N are finitely generated projective Mackey functors for G over R, then DM is finitely generated and projective, and*

$$\begin{aligned} \mathrm{Bdim}(M\oplus N) &= \mathrm{Bdim}M + \mathrm{Bdim}N \\ \mathrm{Bdim}(M\hat{\otimes}N) &= \mathrm{Bdim}M\cdot\mathrm{Bdim}N \\ \mathrm{Bdim}(DM) &= \mathrm{Bdim}M \quad . \end{aligned}$$

Proof: This is Proposition 2.7 and Proposition 4.3 of [7]. □

3.6. Proposition : *Let G and H be finite groups, and let U be a finite (H,G)-biset. Let M be a finitely generated projective Mackey functor for G over R, and let $f\in\mathrm{End}_{\mathsf{Mack}_R(G)}(M)$. Then*

$$\mathrm{Btr}\big(\mathcal{L}_U(f)\big) = U\circ_G\mathrm{Btr}(f) \quad .$$

Proof: Since M is finitely generated and projective, there exist a finite G-set X and morphisms $p: RB_X\to M$ and $s: M\to RB_X$ such that $p\circ s=\mathrm{Id}_M$. In this case, with the notation of Proposition 2.6

$$\mathrm{Btr}(f) = RB_*\left(\begin{smallmatrix} x\\ \downarrow\\ \bullet\end{smallmatrix}\right) RB^*\left(\begin{smallmatrix} x\\ \downarrow\\ xx\end{smallmatrix}\right) s_Xf_Xp_X(\epsilon_X) \quad .$$

Applying the functor $\mathcal{L}_U$ gives morphisms

$$\mathcal{L}_U(p):\mathcal{L}_U(RB_X)\to\mathcal{L}_U(M) \qquad \mathcal{L}_U(s):\mathcal{L}_U(M)\to\mathcal{L}_U(RB_X)$$

such that $\mathcal{L}_U(p)\circ\mathcal{L}_U(s)=\mathrm{Id}_{\mathcal{L}_U(M)}$. Now there is an isomorphism of Mackey functors (see Lemma 5.4 of [3])

$$\mathcal{L}_U(RB_X)\cong RB_{U\circ_G X} \quad ,$$

which can be seen as follows : if Z is a finite H-set, then there is a map

$$\pi_{Z,X}:\mathcal{L}_U(RB_X)(Z)\cong RB(Z\times U\circ_G\Omega_G)\otimes_{\mu_R(G)} RB(\Omega_G\times X)\to RB(Z\times U\circ_G X)$$

sending $S\otimes T$, where S is some H-set over $Z\times U\circ_G\Omega_G$ and T is some G-set over $\Omega_G\times X$, to the pullback product of S and $\gamma_U(T)$ over $U\circ_G\Omega_G$. To check that this map $\pi_{Z,X}$ is an isomorphism, first consider the case $X=\Omega_G$, where it is trivial, and then the case where X is a disjoint union of copies of Ω_G. The general case follows, because any G-set is a subset of such a disjoint union, and the corresponding map $\pi_{Z,X}$ is then the retract of an isomorphism.

Denote by T the H-set $U\circ_G X$. By definition now

$$\mathrm{Btr}\big(\mathcal{L}_U(f)\big)=RB_*\begin{pmatrix} t\\ \downarrow\\ \bullet\end{pmatrix}RB^*\begin{pmatrix} t\\ \downarrow\\ tt\end{pmatrix}\mathcal{L}_U(s)_T\mathcal{L}_U(f)_T\mathcal{L}_U(p)_T(\epsilon_T)\quad .$$

To compute this, the first thing to do is to find a preimage of ϵ_T by the above isomorphism $\pi_{T,X}$: suppose that $\omega : X\to \bigsqcup_{j=1}^{n}\Omega_G$ is an inclusion. For $j\in\{1,\ldots,n\}$, denote by X_j the inverse image by ω of the j-th component of $\bigsqcup_{j=1}^{n}\Omega_G$. Denote by ω_j the restriction of ω to X_j, and by i_j the inclusion of X_j into X. Then

$$\pi_{T,X}^{-1}(\epsilon_T)=\sum_{j=1}^{n}\begin{array}{ccc} & U\circ_G X_j & \\ {\scriptstyle U\circ_G i_j}\swarrow & & \searrow{\scriptstyle U\circ_G\omega_j}\\ U\circ_G X & & U\circ_G\Omega_G\end{array}\otimes\begin{array}{ccc} & X_j & \\ {\scriptstyle \omega_j}\swarrow & & \searrow{\scriptstyle i_j}\\ \Omega_G & & X\end{array}\quad .$$

The image of this element by $\mathcal{L}_U(s)_T\mathcal{L}_U(f)_T\mathcal{L}_U(p)_T$ is equal to

$$\sum_{j=1}^{n}\begin{array}{ccc} & U\circ_G X_j & \\ {\scriptstyle U\circ_G i_j}\swarrow & & \searrow{\scriptstyle U\circ_G\omega_j}\\ U\circ_G X & & U\circ_G\Omega_G\end{array}\otimes s_{\Omega_G}f_{\Omega_G}p_{\Omega_G}\left(\begin{array}{ccc} & X_j & \\ {\scriptstyle \omega_j}\swarrow & & \searrow{\scriptstyle i_j}\\ \Omega_G & & X\end{array}\right)\quad ,$$

and the image of this by the isomorphism $\pi_{T,X}$ is equal to the element

$$S=\sum_{j=1}^{n}\begin{array}{ccc} & U\circ_G X_j & \\ {\scriptstyle U\circ_G i_j}\swarrow & & \searrow{\scriptstyle U\circ_G\omega_j}\\ U\circ_G X & & U\circ_G\Omega_G\end{array}\times_{U\circ_G\Omega_G}U\circ_G\left[s_{\Omega_G}f_{\Omega_G}p_{\Omega_G}\left(\begin{array}{ccc} & X_j & \\ {\scriptstyle \omega_j}\swarrow & & \searrow{\scriptstyle i_j}\\ \Omega_G & & X\end{array}\right)\right]\quad ,$$

of $RB(T^2)$, where $\times_{U\circ_G\Omega_G}$ denotes the pullback product over $U\circ_G\Omega_G$. Then

$$\mathrm{Btr}\big(\mathcal{L}_U(f)\big)=RB_*\begin{pmatrix} t\\ \downarrow\\ \bullet\end{pmatrix}RB^*\begin{pmatrix} t\\ \downarrow\\ tt\end{pmatrix}(S)\quad .$$

Since the functor γ_U preserves pullbacks, this is equal to

$$U\circ_G RB_*\begin{pmatrix} x\\ \downarrow\\ \bullet\end{pmatrix}RB^*\begin{pmatrix} x\\ \downarrow\\ xx\end{pmatrix}\Big[\sum_{j=1}^{n}\begin{array}{ccc} & X_j & \\ {\scriptstyle i_j}\swarrow & & \searrow{\scriptstyle \omega_j}\\ X & & \Omega_G\end{array}\times_{\Omega_G}s_{\Omega_G}f_{\Omega_G}p_{\Omega_G}\left(\begin{array}{ccc} & X_j & \\ {\scriptstyle \omega_j}\swarrow & & \searrow{\scriptstyle i_j}\\ \Omega_G & & X\end{array}\right)\Big]\quad ,$$

where $\times_{\Omega_G}$ denotes the pullback product over Ω_G. Now

$$\begin{tikzcd}[column sep=small] & X_j \arrow[dl, "\omega_j"'] \arrow[dr, "i_j"] & \\ \Omega_G & & X \end{tikzcd} = RB_{X*}(\omega_j)RB^*_X(i_j)(\epsilon_X) \quad ,$$

and since s, f, and p are morphisms of Mackey functors, it follows that

$$s_{\Omega_G} f_{\Omega_G} p_{\Omega_G}\left(\begin{tikzcd}[column sep=small] & X_j \arrow[dl, "\omega_j"'] \arrow[dr, "i_j"] & \\ \Omega_G & & X \end{tikzcd} \right) = RB_{X*}(\omega_j)RB^*_X(i_j)s_X f_X p_X(\epsilon_X) \quad ,$$

and this is also equal to

$$\begin{tikzcd}[column sep=small] & X_j \arrow[dl, "\omega_j"'] \arrow[dr, "i_j"] & \\ \Omega_G & & X \end{tikzcd} \times_X s_X f_X p_X(\epsilon_X) \quad ,$$

where $\times_X$ is the pullback over X. Finally $\mathrm{Btr}\big(\mathcal{L}_U(f)\big)$ is equal to

$$U \circ_G RB_*\left(\begin{smallmatrix} x \\ \downarrow \\ \bullet \end{smallmatrix}\right) RB^*\left(\begin{smallmatrix} x \\ \downarrow \\ xx \end{smallmatrix}\right) \sum_{j=1}^{n} \begin{tikzcd}[column sep=small] & X_j \arrow[dl, "i_j"'] \arrow[dr, "\omega_j"] & \\ X & & \Omega_G \end{tikzcd} \times_{\Omega_G} \left(\begin{tikzcd}[column sep=small] & X_j \arrow[dl, "\omega_j"'] \arrow[dr, "i_j"] & \\ \Omega_G & & X \end{tikzcd} \times_X s_X f_X p_X(\epsilon_X) \right) \quad ,$$

and by associativity of pullback products, this is equal to

$$U \circ_G RB_*\left(\begin{smallmatrix} x \\ \downarrow \\ \bullet \end{smallmatrix}\right) RB^*\left(\begin{smallmatrix} x \\ \downarrow \\ xx \end{smallmatrix}\right) \sum_{j=1}^{n} \left(\begin{tikzcd}[column sep=small] & X_j \arrow[dl, "i_j"'] \arrow[dr, "\omega_j"] & \\ X & & \Omega_G \end{tikzcd} \times_{\Omega_G} \begin{tikzcd}[column sep=small] & X_j \arrow[dl, "\omega_j"'] \arrow[dr, "i_j"] & \\ \Omega_G & & X \end{tikzcd} \right) \times_X s_X f_X p_X(\epsilon_X) \quad .$$

Now

$$\sum_{j=1}^{n} \left(\begin{tikzcd}[column sep=small] & X_j \arrow[dl, "i_j"'] \arrow[dr, "\omega_j"] & \\ X & & \Omega_G \end{tikzcd} \times_{\Omega_G} \begin{tikzcd}[column sep=small] & X_j \arrow[dl, "\omega_j"'] \arrow[dr, "i_j"] & \\ \Omega_G & & X \end{tikzcd} \right) = \epsilon_X \quad ,$$

and ϵ_X is the identity element for $\times_X$. Hence

$$\mathrm{Btr}\big(\mathcal{L}_U(f)\big) = U \circ_G RB_*\left(\begin{smallmatrix} x \\ \downarrow \\ \bullet \end{smallmatrix}\right) RB^*\left(\begin{smallmatrix} x \\ \downarrow \\ xx \end{smallmatrix}\right) s_X f_X p_X(\epsilon_X) = U \circ_G \mathrm{Btr}(f) \quad ,$$

as was to be shown. □

3.7. p-permutation modules. Suppose that $R = k$ is a field of characteristic p. It was shown in Section 12 of [9], that evaluation at the trivial subgroup induces a one to one correspondence between the isomorphism classes of indecomposable projective Mackey functors for G over k, which are moreover projective relative to p-subgroups of G, and isomorphism classes of indecomposable trivial source kG-modules.

Thus if V is such an indecomposable trivial source module, denote by P_V the projective Mackey functor for G over k such that $P_V(1) = V$. It is natural to look at the Burnside dimension of P_V.

Proposition 3.6 involves the special case of restriction to a subgroup : if H is a subgroup of G, and if $U = G$, viewed as an (H, G)-biset by left and right multiplication, then the corresponding functor $\mathcal{L}_U$ is the restriction functor $\mathrm{Res}_H^G : \mathsf{Mack}_R(G) \to \mathsf{Mack}_R(H)$ (see Section 9.9.1 of [2]). The functor γ_U : G-set $\to$ H-set is also the restriction functor.

Suppose that Q is a p-subgroup of G. Then the module $\mathrm{Res}_Q^G V$ is a permutation kQ-module. So there is a finite Q-set X_Q, such that

$$\mathrm{Res}_Q^G V \cong kX_Q \quad ,$$

and up to isomorphism, the Q-set X_Q does not depend on the choice of such a Q-stable basis. In particular, this gives a well defined element $X_Q \in kB(Q)$. Then obviously, if $Q' \subseteq Q$

$$\mathrm{Res}_{Q'}^Q X_Q = X_{Q'} \quad ,$$

and if $x \in P$, then ${}^x X_Q = X_{{}^x Q}$.

Hence the sequence $\beta_V = (X_Q)_{Q \in \underline{s}_p(G)}$, indexed by the set $\underline{s}_p(G)$ of all p-subgroups of G, is an element of $\big(\varprojlim_{Q \in \underline{s}_p(G)} kB(Q)\big)^G$. The map

$$\lambda_G : (u_Q) \in \big(\varprojlim_{Q \in \underline{s}_p(G)} kB(Q)\big)^G \mapsto -\sum_{Q \in \underline{s}_p(G)/G} \frac{\tilde{\chi}]Q,\cdot[_{\underline{s}_p(G)}}{|N_G(Q) : Q|} \mathrm{Ind}_Q^G u_Q \in kB(G) \quad ,$$

where $\tilde{\chi}]Q,\cdot[_{\underline{s}_p(G)}$ is the reduced Euler-Poincaré characteristics of the poset of p-subgroups of G containing Q as a proper subgroup, is injective, and right inverse to the map

$$\rho_G : X \in kB(G) \mapsto (\mathrm{Res}_Q^G X)_{Q \in \underline{s}_p(G)} \quad .$$

Now back to the projective Mackey functor P_V : since Res_Q^G and evaluation at the trivial subgroup commute, the evaluation at the trivial subgroup of the Mackey functor $\mathrm{Res}_Q^G P_V$ is isomorphic to kX_Q. Thus

$$\mathrm{Res}_Q^G P_V \cong kB_{X_Q} \quad .$$

It follows that $\mathrm{Res}_Q^G \mathrm{Bdim}(P_V) = X_Q$, and this gives finally

$$\rho_G\big(\mathrm{Bdim}(P_V)\big) = \beta_V \quad .$$

3.8. Remark : It is natural to ask if the stronger result

$$\mathrm{Bdim}(P_V) = \lambda_G(\beta_V)$$

holds. One can show it is the case if k is the residue field of a discrete valuation ring of characteristic 0 (e.g. if k is perfect).

3.9. Tensor induction. In this section, the ground ring R is the ring $\mathbb{Z}$ of integers. If G and H are finite groups, if U is a finite (H, G)-biset, one can define another functor $T_U : \mathsf{Mack}_{\mathbb{Z}}(G) \to \mathsf{Mack}_{\mathbb{Z}}(H)$, called *tensor induction*, associated to U (see [3] for details). This functor is not additive, but rather multiplicative (i.e. it commutes with the tensor product of Mackey functors).

It is defined by extending the functor $B_X \mapsto B_{\mathrm{Hom}_G(U^{op}, X)}$, defined on the subcategory of *permutation Mackey functors*, to a right exact (non additive) functor defined on $\mathsf{Mack}_{\mathbb{Z}}(G)$. It follows in particular from this construction that

$$T_U(B_X) = B_{\mathrm{Hom}_G(U^{op}, X)} \quad ,$$

and that T_U maps finitely generated projective Mackey functors to finitely generated projective Mackey functors. So it is natural to look at the connection between this tensor induction and Burnside traces and dimensions. One can show the following :

3.10. Proposition : *Let G and H be finite groups, and let U be a finite (H, G)-biset. Let moreover M be a finitely generated projective Mackey functor for G over $\mathbb{Z}$. If $f \in \mathrm{End}_{\mathsf{Mack}_{\mathbb{Z}}(G)}(M)$, then*

$$\mathrm{Btr}\big(T_U(f)\big) = \mathrm{Hom}_G\big(U^{op}, \mathrm{Btr}(f)\big) \quad .$$

3.11. Remark : Here in the right hand side, the map $\mathrm{Hom}_G(U^{op}, -) : B(G) \to B(H)$ is the extension to the Burnside ring of the natural map defined on G-sets. This extension can be achieved by considering *polynomial maps*, or as in Section 3 of [3], by considering finite G-posets and associated Lefschetz invariants.

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