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On Strongly Closed Subgraphs with Diameter Two and Q-Polynomial Property

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1 Introduction

Let $\Gamma=(X,R)$ be a distance-regular graph (DRG) of diameter D with vertex set X and edge set R. For vertices x and y, $\partial(x,y)$ denotes the distance between x and y, i.e., the length of a shortest path connecting x and y. For a vertex $u\in X$ and $j\in\{0,1,\ldots,D\}$, let

$$\Gamma_j(u)=\{x\in X\mid\partial(u,x)=j\}\text{ and }\Gamma(u)=\Gamma_1(u).$$

For two vertices u and $v\in X$ with $\partial(u,v)=j$ let

$$\begin{aligned}C(u,v)&=\Gamma_{j-1}(u)\cap\Gamma(v),\\ A(u,v)&=\Gamma_j(u)\cap\Gamma(v),\text{ and}\\ B(u,v)&=\Gamma_{j+1}(u)\cap\Gamma(v).\end{aligned}$$

The cardinalities $c_j=|C(u,v)|$, $a_j=|A(u,v)|$ and $b_j=|B(u,v)|$ depend only on $j=\partial(u,v)$, and they are called the *intersection numbers* of Γ. The number $k=b_0=|\Gamma(u)|$ is called the *valency* of Γ.

A subset Y of the vertex set X is said to be *strongly closed* if

$$C(u,v)\cup A(u,v)\subset Y\quad\text{for all }u,v\in Y.$$

We often identify a subset of X with the induced subgraph on it. In particular, when Y is strongly closed, Y is referred to as a strongly closed subgraph of Γ.

A *parallelogram* of length $j \geq 2$ is a four-vertex configuration (w, x, y, z) such that

$$\begin{aligned}\partial(w,x) &= \partial(y,z) = j-1 = \partial(x,z),\\ \partial(x,y) &= \partial(z,w) = 1 \text{ and } \partial(w,y) = j.\end{aligned}$$

A distance-regular graph Γ of diameter D is called a *regular near polygon* if there is no parallelogram of length 2 and that

$$a_i = c_i a_1 \text{ for } i = 1, 2, \ldots, D-1.$$

In addition, if $a_D = c_D a_1$, then Γ is called a regular near $2D$-gon.

Recently, in [7] P. Terwilliger and C. Weng showed that if θ_1 is the second largest eigenvalue of a regular near polygon with diameter $D \geq 3$, valency k and intersection numbers $a_1 > 0$, $c_2 > 1$, then

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{1.1}$$

Equality is attained above if and only if Γ is Q-polynomial with classical parameters with respect to θ_1.

Every regular near polygon contains a strongly closed subset Y such that the induced subgraph on Y is strongly regular, i.e., distance-regular of diameter 2. We noticed that the inequality in (1.1) and its equality condition are closely related to the existence of tight vectors that we defined in [4]. In this exposition, we shall explain the relation, apply the theory to parallelogram-free distance-regular graphs, and give a generalization of the results of Terwilliger and Weng above.

2 Terwilliger Algebra and Tight Vectors

Let $\Gamma = (X, R)$ be a distance-regular graph of diameter D. For $i \in \{0, 1, \ldots, D\}$ let A_i denote the *i-th adjacency matrix* in $\mathrm{Mat}_X(\boldsymbol{C})$ whose (x, y)-entry is defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i,\\ 0 & \text{otherwise.}\end{cases}$$

Let $E_0, E_1, \ldots, E_D$ be primitive idempotents corresponding to the eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of A.

Let Y be a nonempty subset of X. $E_i^* = E_i^*(Y) \in \mathrm{Mat}_X(\boldsymbol{C})$ $(i = 0, 1, \ldots, D)$ is defined by

$$(E_i^*)_{x,y} = \begin{cases} 1 & \text{if } x = y \text{ and } \partial(x, Y) = i,\\ 0 & \text{otherwise,}\end{cases}$$

and $E^* = E_0^*$. Then the Terwilliger algebra with respect to Y is a semisimple subalgebra of $\mathrm{Mat}_X(\boldsymbol{C})$ defined by:

$$\mathcal{T} = \mathcal{T}(Y) = \langle A, E_0^*, E_1^*, \ldots, E_D^* \rangle.$$

Let $V = \boldsymbol{C}^X$, and $W = E^*V$. For $x \in X$, let $\hat{x}$ denote the element of V with a 1 in the x-coordinate and 0 in all other coordinates. Then W is the vector subspace of V spanned by the set $\{\hat{y} \mid y \in Y\}$.

Let $w(Y) = \max\{\partial(y, y') \mid y, y' \in Y\}$ denote the width of Y. Then we have the following.

Proposition 1 **([4, Proposition 9.2])** *For* $\boldsymbol{0} \neq \boldsymbol{v} \in W$,

$$|\{i \mid i \in \{0, 1, \ldots, D\},\ E_i\boldsymbol{v} = \boldsymbol{0}\}| \leq w(Y). \tag{2.2}$$

Now a nonzero vector $\boldsymbol{v} \in W$ is said to be *tight* (with respect to Y), if equality is attained in (2.2), i.e.,

$$|\{i \mid i \in \{0, 1, \ldots, D\},\ E_i\boldsymbol{v} = \boldsymbol{0}\}| = w(Y).$$

3 Strongly Closed, Strongly Regular Case

In this section, we review a result to guarantee the existence of strongly closed strongly regular subgraph Y, and inequalities related to the existence of tight vectors with respect to Y.

Proposition 2 **([10, Theorem 1], [3, Theorem 1.1])** *Let* $\Gamma = (X, R)$ *be a distance-regular graph of diameter* $D \geq 3$. *Suppose* $b_1 > b_2$ *and* $a_2 \neq 0$. *Then the following are equivalent.*

(i) *For every pair of vertices* x *and* y *with* $\partial(x, y) = 2$, *there is a strongly closed subgraph containing* x *and* y *of diameter* 2.

(ii) *There is no parallelogram of length 2 or 3.*

Moreover, if the conditions are satisfied, then strongly closed subgraphs guaranteed to exist are strongly regular.

Let Y be a strongly closed subset of X. Suppose the induced subgraph on Y is strongly regular, i.e., $w(Y) = 2$.

Set $\widetilde{A} = E^*AE^*$. Then there are three distinct eigenvalues η_0, η_1, η_2 of $\widetilde{A}$ on W, and they satisfy

$$\eta_0 = c_2 + a_2 > \eta_1 > -1 > \eta_2.$$

Let $\mathbf{1}_Y$ denote the characteristic vector of Y defined by

$$\mathbf{1}_Y = \sum_{y \in Y} \hat{y} \in W.$$

Let W_0, W_1 and W_2 be the eigenspaces of $\widetilde{A}$ in W corresponding to eigenvalues η_0, η_1 and η_2, respectively.

Then $W_0 = \langle \mathbf{1}_Y \rangle$, and

$$W = W_0 \oplus W_1 \oplus W_2.$$

Note that if $\boldsymbol{v} \in W_1 \oplus W_2$, then $E_0 \boldsymbol{v} = \mathbf{0}$. Hence an eigenvector $\boldsymbol{v}$ of $\widetilde{A}$ in $W_1 \oplus W_2$ is tight if $E_i \boldsymbol{v} = \mathbf{0}$ for some $i > 0$ as $w(Y) = 2$.

Proposition 3 ([4, Proposition 11.7]) *Let $\boldsymbol{v} \in W_j$ ($j = 1$ or 2) be an eigenvector of $\widetilde{A}$,*

(1) *For $i \in \{0, 1, \ldots, D\}$,*

$$\frac{\|E_i \boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = \frac{m_i(k - \theta_i)((1 + \eta_j)(1 + \theta_i) + b_1)}{k b_1 |X|} \geq 0.$$

(2) *The following hold.*

$$\theta_1 \leq -1 - \frac{b_1}{1 + \eta_2}, \text{ and } \theta_D \geq -1 - \frac{b_1}{1 + \eta_1}.$$

(3) *The following are equivalent.*

(a) *$\boldsymbol{v}$ is tight.*

(b) *One of the following holds.*

(i) $\theta_1 = -1 - \dfrac{b_1}{1 + \eta_2}$, *or*

(ii) $\theta_D = -1 - \dfrac{b_1}{1 + \eta_1}$.

Proof. The inequality in Proposition 3 (1) can be obtained by simple computation, and both (2) and (3) follow from (1) as $\theta_1 \geq \eta_1 > -1$ and $\theta_D \leq \eta_2 < -1$. ■

Suppose $\Gamma = (X, R)$ is a regular near polygon of diameter $D \geq 3$. Then it is known that Γ does not contain parallelograms of any length. In addition, assume that $a_1 > 0$ and $c_2 > 1$. Then by Proposition 2 there is a strongly closed subset Y such that the induced subgraph on Y is strongly regular. It is called a quad, and it has the following intersection array.

$$\left\{ \begin{array}{c} c_i \\ a_i \\ b_i \end{array} \right\} = \left\{ \begin{array}{ccc} * & 1 & c_2 \\ 0 & a_1 & c_2 a_1 \\ c_2(a_1+1) & (c_2-1)(a_1+1) & * \end{array} \right\}.$$

Hence in this case the eigenvalues can be expressed in a very simple form.

$$\eta_0 = c_2(a_1+1) > \eta_1 = a_1 > \eta_2 = -c_2.$$

Now the inequalities of Proposition 3 (2) yield

$$\theta_1 \leq -1 - \frac{b_1}{1-c_2}, \text{ and } \theta_D \geq -1 - \frac{b_1}{1+a_1}.$$

The first inequality can also be expressed as

$$\theta_1 \leq -1 - \frac{b_1}{1-c_2} = \frac{k-a_1-c_2}{c_2-1}. \tag{3.3}$$

4 A Theorem of Terwilliger and Weng

Theorem 4 (Terwilliger–Weng [7]) *Let Γ denote a regular near polygon with diameter $D \geq 3$, valency k and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ. Then*

$$\theta_1 \leq \frac{k-a_1-c_2}{c_2-1}. \tag{4.4}$$

Moreover, the following (i) - (iii) *are equivalent.*

(i) *Equality is attained in* (4.4).

(ii) Γ *is Q-polynomial with respect to θ_1.*

(iii) Γ *is a dual polar graph or a Hamming graph.*

The inequality in (4.4) is nothing but the one in (3.3). Terwilliger and Weng obtained it using a so-called balanced condition and showed that Γ satisfies the Q-polynomial property if equality is attained.

In view of Proposition 3, the theorem above asserts under the same assumption that the following are equivalent.

(i) There is a tight vector in W_2.

(ii) Γ is Q-polynomial with respect to θ_1.

The following theorem identifies typical tight vectors in W_1 and W_2.

Theorem 5 *Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$, and an intersection number $a_2 > 0$. Let Y be a strongly closed subset of X of width 2. Then the induced subgraph on Y is strongly regular with eigenvalues $\eta_0 = c_2 + a_2 > \eta_1 > -1 > \eta_2$, and the following are equivalent.*

(i) *There is a nonzero vector $\boldsymbol{v} \in E^*V$ such that $E_0\boldsymbol{v} = E_i\boldsymbol{v} = 0$ for some $i \in \{1, 2, \ldots, D\}$.*

(ii) *Either one of the following holds.*

(a) *For every $x, y \in Y$ with $\partial(x, y) = 2$, $E_1\boldsymbol{u} = 0$ and $\theta_1 = -1 - b_1/(1 + \eta_2)$, where*

$$\boldsymbol{u} = \sum_{z \in A(y,x)} \hat{z} - \sum_{w \in A(x,y)} \hat{w} - \eta_2(\hat{x} - \hat{y}), \text{ or}$$

(b) *For every $x, y \in Y$ with $\partial(x, y) = 2$, $E_D\boldsymbol{u} = 0$ and $\theta_D = -1 - b_1/(1 + \eta_1)$, where*

$$\boldsymbol{u} = \sum_{z \in A(y,x)} \hat{z} - \sum_{w \in A(x,y)} \hat{w} - \eta_1(\hat{x} - \hat{y}).$$

The conditions in (ii) are related to a balanced condition in the following theorem.

Theorem 6 (Terwilliger [5]) *Let $\Gamma = (V, R)$ be a distance-regular graph of diameter $D \geq 3$. Let*

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j$$

be a primitive idempotent such that $q_i(j) \neq q_i(0)$ for every $j = 1, \ldots, D$. Then the following are equivalent.

(i) *Γ is Q-polynomial with respect to E_i.*

(ii) *The following two 'balanced' conditions are satisfied.*

(a) *For all $x, y \in X$ with $\partial(x, y) = 2$,*

$$\sum_{z \in A(y,x)} E_i\hat{z} - \sum_{w \in A(x,y)} E_i\hat{w} \in \langle E_i(\hat{x} - \hat{y}) \rangle.$$

(b) *For all* $x, y \in X$ *with* $\partial(x, y) = 3$,

$$\sum_{z \in C(y,x)} E_i \hat{z} - \sum_{w \in C(x,y)} E_i \hat{w} \in \langle E_i(\hat{x} - \hat{y}) \rangle.$$

In view of Theorem 6, there is a tight vector in W_2 if and only if Γ satisfies (ii)(a), the first half of the condition for Γ to be Q-polynomial.

5 Parallelogram Free DRGs

Recall that every regular near polygon is parallelogram-free. If we assume that Γ is of parallelogram free, we can prove a bit more. Before we state our result, we review the definition of a distance-regular graph with classical parameters. Such graph is always Q-polynomial. See [1].

Definition 1 Let Γ denote a distance-regular graph with diameter $D \geq 3$. We say Γ has *classical parameters* (D, q, α, β) whenever the intersection numbers are given by

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad (0 \leq i \leq D),$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (0 \leq i \leq D),$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}.$$

Now we assume the following.

Hypothesis 1 Let $\Gamma = (X, R)$ be a parallelogram-free distance-regular graph with diameter $D \geq 3$. Suppose $a_2 > 0$ and $b_1 > b_2$.

Then by Proposition 2, Γ contains a strongly closed subset Y such that the induced subgraph on Y is strongly regular. Let

$$\eta_0 = c_2 + a_2 > \eta_1 > \eta_2$$

be its distinct eigenvalues.

Theorem 7 *Under Hypothesis 1, the following hold.*

(i) $\theta_1 \leq -1 - \dfrac{b_1}{1+\eta_2}$, *and* $\theta_D \geq -1 - \dfrac{b_1}{1+\eta_1}$.

(ii) *Suppose* $\theta \in \{\theta_1, \theta_D\}$ *attains one of the bounds above. Let* $q = b_1/(\theta + 1)$. *Then the following hold.*

(a) *The intersection numbers of* Γ *are such that*

$$qc_i - b_i - q(qc_{i-1} - b_{i-1})$$

is independent of i $(1 \leq i \leq D)$.

(b) $c_3 \geq (c_2 - q)(q^2 + q + 1)$.

(c) *If* $\theta = \theta_1$, *then* $q + 1 \geq c_2$ *and* $q^2 + q + 1 \geq c_3$, *and if* $\theta = \theta_D$, *then* $q + 1 \leq -a_1$.

(d) *The equality holds in (b) if and only if* Γ *is* Q*-polynomial with classical parameters* (D, q, α, β) *with suitable choices of real numbers* α *and* β.

If Γ is a regular near polygon, then $\eta_2 = -c_2$ and $q = c_2 - 1$. Hence by (b), $c_3 \geq q^2 + q + 1$ and by (c), $q^2 + q + 1 \geq c_3$. Therefore Γ is Q-polynomial with classical parameters by (d).

As a by-product, we obtained the following result as well.

Proposition 8 *Let* $\Gamma = (X, R)$ *be a parallelogram-free distance-regular graph with diameter* $D \geq 3$ *and intersection numbers* $a_2 = s - 1 > 0$, $b_1 = b_2$. *Suppose for all* $x, y \in X$ *with* $\partial(x, y) = 2$,

$$\sum_{z \in A(y,x)} E_i\hat{z} - \sum_{w \in A(x,y)} E_i\hat{w} \in \langle E_i(\hat{x} - \hat{y}) \rangle.$$

Then Γ *is a regular near* $2D$*-gon and* $c_3 \geq 1 - q^3$, *where* $q = -s = -(a_1 + 1)$. *If equality holds, then* Γ *is a classical distance-regular graph with parameters*

$$(D, q, \alpha, \beta) = (D, -s, \frac{s}{1-s}, \frac{k(1+s)}{1-(-s)^D}).$$

If $D = 3$, then Γ is a generalized hexagon. No examples are known if $D > 3$.

6 Examples

1. If Γ contains a strongly closed subgraph isomorphic to (the collinearity graph of) a generalized quadrangle, θ_D attains the bound if and only if $\theta_D = -k/(a_1+1)$.

2. Dual polar graphs and Hamming graphs are the only Q-polynomial regular near polygons of diameter $D \geq 4$ with intersection numbers $c_2 > 1$ and $a_1 > 0$ and these are distance-regular graphs having classical parameters with $\alpha = 0$ and $a_1 \neq 0$. These graphs are Q-polynomial with respect to θ_1 and attain both of the bounds.

3. Let Γ be a parallelogram-free Q-polynomial distance-regular graph of diameter $D \geq 4$ with $a_2 > 0$. Then Γ has classical parameters (D, q, α, β) and Γ is either a regular near polygon or $q < -1$. Distance-regular graphs having classical parameters (D, q, α, β) with $q < -1$ are said to be of negative type. These graphs satisfy the bound for θ_D.

Finally we include a table of the list of known parallelogram-free Q-polynomial distance-regular graphs taken from [1]. There is a series of excellent articles on parallelogram-free distance-regular graphs by C. Weng and others. See [2, 6, 8, 9, 10, 11]. We hope that our observations may shed light on the classification of this class of distance-regular graphs.

Known Parallelogram-Free Q-DRGs

Name	*Diam.*	b	$\alpha+1$	$\beta+1$
$H(D,q)$	D	1	1	q
$DP(D,q,e)$	D	q	1	q^e+1
$U(2D,r)$	D	$-r$	$\frac{1+r^2}{1-r}$	$\frac{1-(-r)^{D+1}}{1-r}$
$Her_D(r)$	D	$-r$	$-r$	$-(-r)^D$
$GH(q,q^3)$	3	$-q$	$\frac{1}{1-q}$	q^2+q+1
M_{24}	3	-2	-3	11
M_{23}	3	-2	-1	6
$ExtTGolay$	3	-2	-2	9

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The content of this exposition is included in the following.

[12] H. Suzuki, On strongly closed subgraphs with diameter two and Q-polynomial property, preprint.