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Numerical verification by infinite dimensional Newton's method for stationary solutions of the Navier-Stokes problems

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概要

We present a numerical method to enclose stationary solutions of the Navier-Stokes equations, especially 2-D driven cavity problem with regularized boundary condition. Our method is based on the infinite dimensional Newton's method by estimating the inverse of the corresponding linearized operator. The method can be applied to the case for high Reynolds numbers and we show some numerical examples which confirm us the actual effectiveness.

Key words: Numerical enclosure method, driven cavity flows, infinite dimensional Newton's method.

1 Introduction

We consider the following Navier-Stokes equations

$$\begin{cases} -\Delta u + R\cdot(u\cdot\nabla)u + \nabla p &= f \ \text{ in } \Omega, \\ \operatorname{div} u &= 0 \ \text{ in } \Omega, \\ u &= g \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where u, p and R are the velocity vector, pressure and the Reynolds number, respectively and the flow region Ω is a convex polygonal domain in $\mathbf{R}^2$. In what follows, for each rational number m, let $H^m(\Omega)$ denote the L^2-Sobolev space of order m on Ω. The function $f = (f_1, f_2)$ means a density of body forces with $f \in (H^1(\Omega))$ and $g = (g_1, g_2) \in H^{1/2}(\partial\Omega)$, where we assume that there exists a function $\varphi \in H^2(\Omega)$ satisfying $(\varphi_y, -\varphi_x) = g$ on $\partial\Omega$.

The above problem was disccused by Wieners [7] for low Reynolds numbers. The method proposed in it is based on Newton-Kantorovich theorem but it would not be able to apply to high Reynolds numbers, because the estimation for the inverse of the linearized operator directly depends on the Reynolds number. We also uses Newton type verification condition, but the method which verifies the invertibility of linearized operator is different from the Wieners' formulation. Our method has an advantage which enables us to verify the invertibility of the linearlized operator, even for high Reynolds numbers, provided that the approximation space is sufficiently accurate and that the inverse operator acually exists in the rigorous sense.

In Section 2 we introduce a stream function formulation of our problem and consider the linearized operator which is needed in the infinite dimensional Newton method. In Section 3, we formulate the verification method by computer to verify the invertibility of the linearized operator. We then derive the infinite dimensional Newton method to enclose the solution in Section 4 and we show some enclosure results in Section 5.

2 Stream function and the linearized operator

We first introduce a stream function ψ satisfying $u = (\psi_y,\ -\psi_x)$ by the incompressibility condition in (1.1), where subscripts x and y denote the partial derivative for x and y respectively. Using this function we can rewrite the equations (1.1) as

$$\left\{\begin{array}{rcl} \Delta^2\psi + R\cdot J(\psi,\Delta\psi) &=& (f_2)_x - (f_1)_y \ \ \text{in}\Omega, \\ \psi &=& \varphi \ \ \text{on } \partial\Omega, \\ \frac{\partial\psi}{\partial n} &=& \frac{\partial\varphi}{\partial n} \ \ \text{on } \partial\Omega, \end{array}\right. \tag{2.1}$$

where J is a bilinear form defined by $J(u,v) = u_x v_y - u_y v_x$ and $\frac{\partial}{\partial n}$ stands for the normal derivative. Newly denoting u as $\psi - \varphi$ we have

$$\left\{\begin{array}{rcll} \Delta^2 u + \Delta^2\varphi + R\cdot J(u+\varphi,\Delta(u+\varphi)) &=& (f_2)_x - (f_1)_y & \text{in}\Omega, \\ u = \frac{\partial u}{\partial n} &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2.2}$$

Our aim is to verify the existence of a weak solution $u \in H_0^2(\Omega)$ of (2.2), where $H_0^2(\Omega) \equiv \{v \in H^2(\Omega) \mid v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}$ with the inner product $< u,v >_{H_0^2} \equiv (\Delta u, \Delta v)_{L^2}$ for $u,v \in H_0^2(\Omega)$.

Let S_h be a finite dimensional subspace of $H_0^2(\Omega)$ that depends on h $(0 < h < 1)$. Usually S_h is taken to be a finite element subspace with mesh size h.

We calculate an approximate solution $u_h \in C^1(\Omega)$ of (2.2) in the finite dimensional space, satisfying for all $v_h \in S_h$

$$(\Delta u_h + \Delta\varphi, \Delta v_h)_{L^2} + (R \cdot J(u_h + \varphi, \Delta(u_h + \varphi)), v_h)_{L^2} = ((f_2)_x - (f_1)_y, v_h)_{L^2}, \quad (2.3)$$

and calculate $u_s \in C^2(\Omega)$ by smoothing of u_h. Then the linearized operator at u_s is represented as

$$\mathcal{L}u \equiv \Delta^2 u + R \cdot \{J(u_s + \varphi, \Delta u) + J(u, \Delta(u_s + \varphi))\},$$

and $\mathcal{L}$ is considerd as the operator from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$ in weak sense. We will verify the existence of the inverse $\mathcal{L}^{-1} : H^{-2}(\Omega) \to H_0^2(\Omega)$ and formulate the infinite dimensional Newton's method.

3 Invertibility of the linearized operator

By direct computations, we find that for any $q \in H^{-2}(\Omega)$ there exists a unique solution $v \in H_0^2(\Omega)$ satisfying

$$\begin{cases} \Delta^2 v &= q \quad \text{in } \Omega, \\ v = \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

For $q \in H^{-2}(\Omega)$, let Kq be the unique solution $v \in H_0^2(\Omega)$ of the equation (3.1) then K is a compact operator from $H^{-1}(\Omega)$ to $H_0^2(\Omega)$. Using the following compact operator on $H_0^2(\Omega)$

$$F_1(u) \equiv -R \cdot K\{J(u_s + \varphi, \Delta u) + J(u, \Delta(u_s + \varphi))\},$$

the equation $\mathcal{L}u = 0$ is equivalent to the fixed point equation

$$u = F_1(u). \quad (3.2)$$

In order to show the invertibility of the linearized operator $\mathcal{L}$, by the Fredholm alternative, we only have to show the uniqueness of the solution of the equation $\mathcal{L}u = 0$.

Now let $P_h : H_0^2(\Omega) \longrightarrow S_h$ denote the H_0^2-projection defined by

$$(\Delta(u - P_h u), \Delta v_h)_{L^2} = 0 \qquad \text{for all } v_h \in S_h,$$

and we derive some error estimations for P_h. In what follows, we restrict ourselves to that the domain Ω is a unit square $(0,1) \times (0,1)$, and that S_h is the set of piecewise bicubic Hermite functions with uniform mesh on Ω (e.g., [5]). However, our verification principle can also be applied to more general domains and approximation subspaces, when the appropriate a priori error estimates are obtained.

At first we derive the following interpolation error estimation.

Lemma 1. *Let $\mathcal{I}_\Omega$ denote the cubic Hermite interpolation on $\Omega = (0,1)^2$. For $u \in H^4(\Omega) \cap H_0^3(\Omega)$ we have*

$$\|u - \mathcal{I}_\Omega u\|_{H_0^2} \leq \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2}. \tag{3.3}$$

Proof. At first, we have

$$|u|_{H^4(\Omega)} = \|\Delta^2 u\|_{L^2} \text{ for } u \in H^4(\Omega) \cap H_0^3(\Omega)$$

where $|\cdot|_{H^4(\Omega)}$ denotes the H^4 seminorm on Ω defined by

$$|u|_{H^4}^2 = \sum_{n_1+n_2=4,\ n_1,n_2 \in \mathbf{N}\cup\{0\}} \left\| \frac{\partial^4 u}{\partial x^{n_1} \partial y^{n_2}} \right\|_{L^2}^2.$$

Actually, by expanding as $u = \sum_{m,n=1}^{\infty} a_{mn} \sin m\pi x \sin n\pi y$, noting the convergence of the Fourier series, we have

$$|u|_{H^4}^2 = \frac{1}{4} \sum_{m,n=1}^{\infty} \{(m\pi)^2 + (n\pi)^2\}^4 a_{mn}^2 = \|\Delta^2 u\|_{L^2}^2.$$

Denoting $\mathcal{I}_H$ the Hermite interpolation on $I = (0,1)$, we have by [5]

$$\begin{aligned} \|u - \mathcal{I}_H u\|_{H_0^2} &\leq \|u''\|_{L^2} \text{ for } u \in H_0^2(I), \\ \|u - \mathcal{I}_H u\|_{H_0^2} &\leq \frac{h^2}{\pi^2} \|u^{(4)}\|_{L^2} \text{ for } u \in H^4(I) \cap H_0^2(I). \end{aligned}$$

Representing $\mathcal{I}_H$ as $\mathcal{I}_{H_x}$ and $\mathcal{I}_{H_y}$ for x and y direction respectively, we have

$$
\begin{aligned}
\|u-\mathcal{I}_\Omega u\|_{H_0^2} &= \|u-\mathcal{I}_{I_x}u+\mathcal{I}_{I_x}u-\mathcal{I}_{H_x}\mathcal{I}_{H_y}u\|_{H_0^2} \\
&\le \|u-\mathcal{I}_{H_x}u\|_{H_0^2}+\|\mathcal{I}_{H_x}(u-\mathcal{I}_{H_y}u)\|_{H_0^2} \\
&\le \|u-\mathcal{I}_{H_x}u\|_{H_0^2}+\|\mathcal{I}_{H_x}(u-\mathcal{I}_{H_y}u)-(u-\mathcal{I}_{H_y}u)\|_{H_0^2}+\|u-\mathcal{I}_{H_y}u\|_{H_0^2} \\
&\le \frac{h^2}{\pi^2}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{L^2}+\left\|\frac{\partial^2}{\partial x^2}(u-\mathcal{I}_{H_y}u)\right\|_{L^2}+\frac{h^2}{\pi^2}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_{L^2} \\
&\le \frac{h^2}{\pi^2}\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{L^2}+\frac{h^2}{\pi^2}\left\|\frac{\partial^4}{\partial x^2\partial y^2}u\right\|_{L^2}+\frac{h^2}{\pi^2}\left\|\frac{\partial^4 u}{\partial y^4}\right\|_{L^2} \\
&\le \frac{h^2}{\pi^2}|u|_{H^4}=\frac{h^2}{\pi^2}\|\Delta^2 u\|_{L^2}
\end{aligned}
$$

∎

Using the estimation (3.3) and the relation

$$\|u-P_h u\|_{H_0^2}=\inf_{\xi\in S_h}\|u-\xi\|_{H_0^2}\le\|u-\mathcal{I}_\Omega u\|_{H_0^2},$$

we have the following error estimatinos for P_h.

In what follows, we will discuss under the assumption that the error estimation (3.3) is valid for all $u\in H^4(\Omega)\cap H_0^2(\Omega)$. Of course, as this assumption is not yet assured, we have to validate it in another paper or to decide a correct constant, which might be greater than $\frac{h^2}{\pi^2}$, for further study.

Lemma 2. *For $u\in H^4(\Omega)\cap H_0^2(\Omega)$ we have*

$$\|u-P_h u\|_{H_0^2} \le \frac{h^2}{\pi^2}\|\Delta^2 u\|_{L^2}, \tag{3.4}$$

$$\|u-P_h u\|_{H_0^1} \le \frac{h^3}{\pi^3}\|\Delta^2 u\|_{L^2}, \tag{3.5}$$

$$\|u-P_h u\|_{L^2} \le \frac{h^4}{\pi^4}\|\Delta^2 u\|_{L^2}. \tag{3.6}$$

Proof. The first estimation (3.4) is trivial by Lemma 1.

For $e \equiv u - P_h u$ we denote $\phi \in H^4(\Omega) \cap H_0^2(\Omega)$ as the solution of the following equations:

$$\begin{cases} \Delta^2 \phi &= e \quad \text{in } \Omega \\ \phi = \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Then we have

$$\begin{aligned} \|u - P_h u\|_{L^2}^2 &= (u - P_h u, u - P_h u)_{L^2} \\ &= (e, \Delta^2 \phi)_{L^2} \\ &= (\Delta e, \Delta \phi)_{L^2} \\ &= (\Delta e, \Delta(\phi - P_h \phi)_{L^2} \\ &\le \|\Delta e\|_{L^2} \|\Delta(\phi - p_h \phi)\|_{L^2} \\ &\le \|\Delta e\|_{L^2} \cdot \frac{h^2}{\pi^2} \|\Delta^2 \phi\|_{L^2} \\ &\le \|\Delta e\|_{L^2} \cdot \frac{h^2}{\pi^2} \|e\|_{L^2}. \end{aligned}$$

Therefore $\|u - P_h u\|_{L^2} \le \frac{h^2}{\pi^2} \|u - P_h u\|_{H_0^2}$ holds and the third estimation (3.6) is proved. Finally using (3.4) and (3.6) we can prove the second estimation (3.5) as follows:

$$\begin{aligned} \|u - P_h u\|_{H_0^1}^2 &= (\nabla(u - P_h u), \nabla(u - P_h u))_{L^2} \\ &= (-\Delta(u - P_h u), u - P_h u)_{L^2} \\ &\le \|\Delta(u - P_h u)\|_{L^2} \|u - P_h u\|_{L^2} \\ &\le \frac{h^2}{\pi^2} \|\Delta^2 u\|_{L^2} \cdot \frac{h^4}{\pi^4} \|\Delta^2 u\|_{L^2}. \end{aligned}$$

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Now, as in [1] or [3], we decompose (3.2) into the finite and infinite dimensional parts:

$$\begin{cases} P_h u &= P_h F_1(u), \\ (I - P_h) u &= (I - P_h) F_1(u). \end{cases} \tag{3.8}$$

Since we apply a Newton-like method only for the former part of (3.8), we define the following operator:

$$\mathcal{N}_h^1(u) \equiv P_h u - [I - F_1]_h^{-1} (P_h u - P_h F_1(u)),$$

where I is the identity map on $H_0^2(\Omega)$. And we assume that the restriction to S_h of the operator $P_h[I-F_1]: S_h \to S_h$ has the inverse $[I-F_1]_h^{-1}$. The validity of this assumption can be numerically confirmed in actual computations.

We next define the operator $T_1 : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$ by

$$T_1(u) \equiv \mathcal{N}_h^1(u) + (I - P_h)F_1(u).$$

Then T_1 becomes a compact map on $H_0^2(\Omega)$ and we have the following equivalence relation

$$u = T_1(u) \Longleftrightarrow u = F_1(u).$$

Our purpose is to find a unique fixed point of T_1 in a certain set $U \subset H_0^2(\Omega)$, which is called a 'candidate set'. Given positive real numbers γ and α we define the corresponding candidate set U by

$$U \equiv U_h \oplus [\alpha], \tag{3.9}$$

where $U_h \equiv \{\phi_h \in S_h \mid \|\phi_h\|_{H_0^2} \leq \gamma\}$, $[\alpha] \equiv \{\phi_\perp \in S_\perp \mid \|\phi_\perp\|_{H_0^2} \leq \alpha\}$ and $S_\perp$ means the orthogonal complement of S_h in $H_0^2(\Omega)$. If the relation

$$\overline{T_1(U)} \subset \mathrm{int}(U) \tag{3.10}$$

holds, by Schauder's fixed point theorem and the linearity of T_1, there exists a fixed point u of T_1 in U and the fixed point is unique, i.e., $u = 0$, which implies that the operator $\mathcal{L}$ is invertible. Decomposing (3.10) into finite and infinite dimensional parts we have a sufficient condition for (3.10) as follows:

$$\begin{cases} \sup_{u\in U} \|\mathcal{N}_h^1(u)\|_{H_0^2} < \gamma \\ \sup_{u\in U} \|(I-P_h)F_1(u)\|_{H_0^2} < \alpha. \end{cases} \tag{3.11}$$

We now derive the following theorem in which the verification condition (3.11) is numerically and simply described.

Theorem 1. *Let $\{\phi_i\}$ be the basis of S_h and define the following constatns:*

$$C_0 = \frac{h}{\pi},\ C_1^s = \|\nabla(u_s+\varphi)\|_\infty,\ C_2^s = \left\|\nabla\frac{\partial(u_s+\varphi)}{\partial x}\right\|_\infty + \left\|\nabla\frac{\partial(u_s+\varphi)}{\partial y}\right\|_\infty$$

$$
\begin{aligned}
C_3^s &= \|\nabla\Delta(u_s+\varphi)\|_\infty, \ C_p = \frac{1}{\pi\sqrt{2}}, \ M_1 = \|L^T G^{-1} L\|_E, \\
K_1 &= C_1^s + C_0^2 C_3^s, \\
K_2 &= C_1^s + C_0 C_3^s C_p, \\
K_3 &= \sqrt{2} C_1^s + C_p(C_2^s + C_0 C_3^s),
\end{aligned}
$$

where $\|\nabla v\|_\infty \equiv (\|\nabla v_x\|_\infty^2 + \|\nabla v_y\|_\infty^2)^{\frac{1}{2}}$, $\|\cdot\|_E$ *denotes the matrix norm corresponding to the Euclidian vector norm,* C_p *is the Poincaré constant, the matrix* $G = (G_{ij})$ *is defined by*

$$G_{ji} \equiv R(J(u_s+\varphi, \Delta\phi_i) + J(\phi_i, \Delta(u_s+\varphi)), \phi_j)_{L^2} + (\Delta\phi_i, \Delta\phi_j)_{L^2},$$

and $D = LL^T$ *is a Cholesky decomposition for the matrix* $D = (D_{ij})$ *defined by*

$$D_{ij} \equiv (\Delta\phi_i, \Delta\phi_j)_{L^2}.$$

For these constants, if the inequality

$$RC_0(K_1 + K_2 K_3 M_1 R C_0) < 1 \tag{3.12}$$

holds then the operator $\mathcal{L}$ *is invertible.*

Proof. We show sufficient conditions for (3.11). Denoting $u = u_1 + u_2$, $u_1 \in U_h$, $u_2 \in [\alpha]$, by some simple calculations we have $\mathcal{N}_h^1(u) = [I - F_1]_h^{-1} P_h F_1(u_2)$, and thus

$$\|\mathcal{N}_h^1(u)\|_{H_0^2} \le M_1 \|P_h F_1(u_2)\|_{H_0^2} \tag{3.13}$$

holds. (See [1] or [3] for details to such estimation.) Using error estimation in Lemma 2, we can estimate $\|P_h F_1(u_2)\|_{H_0^2}$ as follows:

$$\|P_h F_1(u_2)\|_{H_0^2} \le RC_0 K_3 \alpha. \tag{3.14}$$

Thus we derive a sufficient condition for the first inequality in (3.11) as

$$M_1 R C_0 K_3 \alpha < \gamma. \tag{3.15}$$

Now we estimate the left hand side of the second inequality in (3.11). Noting that

$$\begin{aligned} \|(I-P_h)F_1(u)\|_{H_0^2} &\le R\|(I-P_h)KJ(u_s+\varphi,\Delta u)\|_{H_0^2} \\ &\qquad + R\|(I-P_h)KJ(u,\Delta(u_s+\varphi))\|_{H_0^2} \\ &\le RC_0K_2\gamma + RC_0K_1\alpha, \end{aligned}$$

we obtain the sufficient condition for the second inequality in (3.11) as

$$RC_0(K_1\alpha + K_2\gamma) < \alpha. \tag{3.16}$$

Combining the conditions (3.15) and (3.16) we finally obtain the sufficient condition for (3.11) as $RC_0(K_1 + K_2K_3M_1RC_0) < 1$.

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4 Verification procedure for nonlinear problem

In what follows we assume that the invertibility of the linearized operator $\mathcal{L}$ is confirmed by the method described in the previous section. We will verify the existence of solutions for (2.2) in the neighborhood of $u_X \in C^1(\Omega)$ satisfying for all $v_h \in S_h$

$$(\Delta u_X + \Delta\varphi, \Delta v_h)_{L^2} + (R\cdot J(u_s+\varphi,\Delta(u_s+\varphi)), v_h)_{L^2} = ((f_2)_x - (f_1)_y, v_h)_{L^2}. \tag{4.1}$$

Considering the function $\bar{u}$ satisfying

$$\begin{cases} \Delta^2\bar{u} = -\Delta^2\varphi - R\cdot J(u_s+\varphi,\Delta(u_s+\varphi)) + (f_2)_x - (f_1)_y & \text{in } \Omega, \\ \bar{u} = \frac{\partial \bar{u}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and writing $w \equiv u - \bar{u}$, $v_0 \equiv \bar{u} - u_X$, $u - u_X$ can be represented as $w + v_0$.

Noting that $u_X = P_h\bar{u}$, we see that $v_0 \in S_\perp$ and, by Lemma 2 and its proof, the error estimates for v_0 can be derived:

$$\|v_0\|_{H_0^2} \le \frac{h^2}{\pi^2}\|-\Delta^2\varphi - R\cdot J(u_s+\varphi,\Delta(u_s+\varphi)) + (f_2)_x - (f_1)_y\|_{L^2},$$

$$\begin{aligned}
\|v_0\|_{H_0^1} &\le \frac{h}{\pi}\|v_0\|_{H_0^2}, \\
\|v_0\|_{L^2} &\le \frac{h^2}{\pi^2}\|v_0\|_{H_0^2}.
\end{aligned}$$

Now we can rewrite (2.2) as

$$\left\{\begin{aligned}
\Delta^2 w &= -R\cdot J(w+u_X+v_0+\varphi, \Delta(w+u_X+v_0+\varphi)) \\
&\quad +R\cdot J(u_s+\varphi, \Delta(u_s+\varphi)) \ \text{ in } \Omega, \\
w=\tfrac{\partial w}{\partial n} &= 0 \ \text{ on } \partial\Omega.
\end{aligned}\right. \tag{4.3}$$

Thus defining the following compact map on $H_0^2(\Omega)$

$$F_2(w) \equiv RK\{J(u_s+\varphi, \Delta(u_s+\varphi)) - J(w+u_X+v_0+\varphi, \Delta(w+u_X+v_0+\varphi))\}, \tag{4.4}$$

we have the fixed point equation

$$w = F_2(w). \tag{4.5}$$

Now we formulate the infinite dimensional Newton method for the equation (4.5). Note that $w-[I-F_2'(-v_0-u_X+u_s)]^{-1}(I-F_2)(w)$ can be equivalently represented as $\mathcal{L}^{-1}q(w)$, where $F_2'(-v_0-u_X+u_s)$ stands for Fréchet derivative of F_2 at $-v_0-u_X+u_s$ and

$$\begin{aligned}
q(w) \equiv R\{&J(u_s+\varphi, \Delta(u_s+\varphi)) - J(w+u_X+v_0+\varphi, \Delta(w+u_X+v_0+\varphi)) \\
&+ J(u_s+\varphi, \Delta w) + J(w, \Delta(u_s+\varphi))\}.
\end{aligned}$$

Then we have the relation

$$w = F_2(w) \Longleftrightarrow w = T_2(w), \tag{4.6}$$

where $T_2(w) \equiv \mathcal{L}^{-1}q(w)$ is a compact map on $H_0^2(\Omega)$.

We intend to find a fixed point of T_2 in a set W defined by

$$W = \{w \in H_0^2(\Omega) \mid \|w\|_{H_0^2} \le \alpha\}, \tag{4.7}$$

where α is a positive number. If the relation

$$T_2(W) \subset W \tag{4.8}$$

holds, by Schauder's fixed point theorem there exists a fixed point of T_2 in W. Since a sufficient condition for (4.8) is

$$\sup_{w \in W} \|T_2(w)\|_{H_0^2} \leq \alpha, \tag{4.9}$$

by estimating the left hand side of (4.8), we can derive the following theorem.

Theorem 2. *Assume that the invertibility condition (3.12) holds. Using the same constants in Theorem 1, we define the following constants:*

$$\begin{aligned}
\kappa &\equiv C_0R(K_1 + K_2K_3M_1C_0R),\\
\tau_1 &= \frac{C_0RM_1K_2}{1-\kappa},\ \tau_2 = \frac{1}{1-\kappa},\\
\tau_3 &= M_1(C_0RK_3\tau_1 + 1),\ \tau_4 = M_1C_0RK_3\tau_2,\\
b &= \|v_0\|_{H_0^2},\ C_4 = \frac{1}{\pi},
\end{aligned}$$

where C_4 is an embedding constant satisfying $\|\nabla u\|_{L^4} \leq C_4\|\Delta u\|_{L^2}$ for $u \in H_0^2(\Omega)$ and we have used the optimal embedding estimates $C_4 = \frac{1}{\pi}$ which can be derived by the result in [6]. Moreover for a matrix $S = \begin{pmatrix} \tau_1^2 + \tau_3^2 & \tau_1\tau_2 + \tau_3\tau_4 \\ \tau_1\tau_2 + \tau_3\tau_4 & \tau_2^2 + \tau_4^2 \end{pmatrix}$ and $M_2 \equiv \|S\|_E^{\frac{1}{2}}$, define the following constants:

$$\begin{aligned}
C_1^X &= \|\nabla(u_X + \varphi)\|_\infty,\ C_2^X = \left\|\nabla\frac{\partial(u_X+\varphi)}{\partial x}\right\|_\infty + \left\|\nabla\frac{\partial(u_X+\varphi)}{\partial y}\right\|_\infty,\\
C_3^X &= \|\nabla\Delta(u_X + \varphi)\|_\infty,\ D_1^\delta = \|\nabla(u_X - u_s)\|_{L^2},\\
D_2^\delta &= \|J(u_X - u_s, \Delta(u_s + \varphi))\|_{L^2},\ D_3^\delta = \|\Delta(u_X - u_s)\|_{L^2},.
\end{aligned}$$

If there exists a real number $\alpha > 0$ satisfying

$$\begin{aligned}
&M_2R\{C_4^2(\alpha + b)^2 + C_4^2\alpha D_3^\delta + C_3^XC_pC_0b + \alpha D_1^\delta C_p\\
&+ C_0b(\sqrt{2}C_1^X + C_pC_3^X) + C_p^2D_2^\delta + C_pC_1^XD_3^\delta\} \leq \alpha,
\end{aligned} \tag{4.10}$$

then there exists a fixed point of T_2 in W.

Proof. For $q(w) \in H^{-2}(\Omega)$ consider the solution $\phi \in H_0^2(\Omega)$ of the problem

$$\begin{cases} \mathcal{L}\phi &= q(w) \quad \text{in } \Omega, \\ \phi &= \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

Then writing $\phi = \phi_h + \phi_\perp$, $\phi_h \in S_h$, $\phi_\perp \in S_\perp$, we have

$$\begin{cases} \|\phi_h\|_{H_0^2} &\leq M_1 R C_0 K_3 \|\phi_\perp\|_{H_0^2} + M_1 \|P_h K q(w)\|_{H_0^2}, \\ \|\phi_\perp\|_{H_0^2} &\leq R C_0 (K_1 \|\phi_\perp\|_{H_0^2} + K_2 \|P_h K q(w)\|_{H_0^2}) + \|(I - P_h) K q(w)\|_{H_0^2}. \end{cases} \tag{4.12}$$

Noting that $\kappa < 1$ holds because of the invertibility of $\mathcal{L}$, we have

$$\begin{cases} \|\phi_h\|_{H_0^2} &\leq \tau_3 \|P_h K q(w)\|_{H_0^2} + \tau_4 \|(I - P_h) K q(w)\|_{H_0^2}, \\ \|\phi_\perp\|_{H_0^2} &\leq \tau_1 \|P_h K q(w)\|_{H_0^2} + \tau_2 \|(I - P_h) K q(w)\|_{H_0^2}. \end{cases} \tag{4.13}$$

Therefore we obtain

$$\|\phi\|_{H_0^2} \leq M_2 \|K q(w)\|_{H_0^2} \leq M_2 \|q(w)\|_{H^{-2}}. \tag{4.14}$$

Furthermore, we have the estimations

$$\|q(w)\|_{H^{-2}} = \sup_{\theta \in H_0^2,\ \|\theta\|_{H_0^2} = 1} \left| < q(w), \theta >_{H^{-2}, H_0^2} \right|$$

$$\leq R\{C_4^2(\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_p C_0 b + \alpha D_1^\delta C_p$$

$$+ C_0 b(\sqrt{2} C_1^X + C_p C_3^X) + C_p^2 D_2^\delta + C_p C_1^X D_3^\delta\}$$

Thus we obtain

$$\|\mathcal{L}^{-1} q(w)\|_{H_0^2} \leq$$

$$M_2 R\{\|C_4^2(\alpha + b)^2 + C_4^2 \alpha D_3^\delta + C_3^X C_p C_0 b + \alpha D_1^\delta C_p$$

$$+ C_0 b(\sqrt{2} C_1^X + C_p C_3^X) + C_p^2 D_2^\delta + C_p C_1^X D_3^\delta\}.$$

This means that the inequality (4.10) is a sufficient condition for $T_2(W) \subset W$ and the desired assertion is proved.

∎

5 Numerical examples

Particularly, we consider the two dimensional driven cavity problem with $f = 0$ and $g = (\varphi_y, -\varphi_x)$ in (1.1), where $\varphi(x, y) = x^2(1-x)^2 y^2(1-y)$.

In calculations, we used interval arithmetic in order to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on the DELL Precision WorkStation 650 (Intel Xeon 3.2GHz) using MATLAB (Ver. 6.5.1) and the interval arithmetic toolbox INTLAB (Ver. 4.2.1) coded by Prof. Rump in TU Hamburg-Harburg ([4]). The verification results are shown in Table 1, in which 'smallest α' means the smallest bound α satisfing the verification condition (4.9) and the solution u in (2.2) is enclosed as $\|u - u_X\|_{H_0^2(\Omega)} \leq \|v_0\|_{H_0^2(\Omega)} + \alpha$.

表 1: Verification Results for Driven Cavity Problem ($h = 1/22$)

R	M_1	M_2	$\|v_0\|_{H_0^2}$	D_3^δ	α
100	1.0183	1.4511	9.2855e-4	1.2940e-4	1.1199e-3
200	1.0355	2.4945	1.0094e-3	1.3666e-4	7.5815e-3

It seems that Wieners' method would not be able to apply to the Reynolds number higher than 20 in [7]. On the other hand, we enclosed the stationary solution for the Reynolds number over 100, and our method can be applied, in principle, to more higher Reynolds numbers by using more accurate approximation subspaces, i.e., smaller mesh sizes.

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