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WEAK AND STRONG CONVERGENCE THEOREMS FOR NONLINEAR OPERATORS AND THEIR APPLICATIONS

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ABSTRACT. In this article, we introduce two nonlinear operators of monotone type and nonexpansive type, i.e., inverse-strongly-monotone operators and relatively nonexpansive operators. Then, we obtain weak and strong convergence theorems for the nonlinear operators in a Hilbert space or a Banach space. Using these results, we consider some applications.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. An operator A of C into H is said to be *monotone* if

$$\langle x-y, Ax-Ay\rangle \geq 0$$

for all $x, y \in C$. An operator A of C into H is said to be inverse-strongly-monotone if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$$

for all $x, y \in C$. Such an operator A is said to be α-inverse-strongly-monotone. An operator A of C into H is said to be strongly monotone if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|x-y\|^2$$

for all $x, y \in C$. Such an operator A is said to be α-strongly monotone. An operator A of C into H is said to be Lipschitz continuous if there exists a positive real number β such that

$$\|Ax-Ay\| \leq \beta\|x-y\|$$

for all $x, y \in C$. Such an operator A is said to be β-Lipschitz continuous. If A is an α-strongly monotone and β-Lipschitz continuous operator of C into H, then A is α/β^2-inverse-strongly-monotone. The variational inequality problem is to find a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0$$

for all $v \in C$. Variational inequalities were initially studied by Stampacchia [20,23]. The set of solutions of the variational inequality is denoted by *VI(C, A)*. A mapping S of C into itself is said to be nonexpansive if

$$\|Sx-Sy\| \leq \|x-y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. Yamada [50] proved the following strong convergence theorem for strongly monotone and Lipschitz continuous operators in a Hilbert space.

Theorem 1 (Yamada [50]). *Let H be a real Hilbert space. Let S be a nonexpansive mapping of H into itself such that $F(S) \neq \emptyset$ and let A be an α-strongly monotone and β-Lipschitz continous operator of H into itself. Suppose $x_1 = x \in H$ and $\{x_n\}$ is given by*

$$x_{n+1} = Sx_n - \alpha_{n+1}\lambda ASx_n$$

for every $n = 1, 2, \dots,$ where $\{\alpha_n\}$ is a sequence in $[0,1]$ and λ is a positive real number. If $\{\alpha_n\}$ and λ are chosen so that $\lambda \in (0, 2\alpha/\beta^2)$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0,$$

then $\{x_n\}$ converges strongly to the uniquely existing solution of $VI(F(S), A)$.

On the other hand, Nakajo and Takahashi [30] proved the following strong convergence theorem by using the hybrid method in mathematical programming.

Theorem 2 (Nakajo and Takahashi [30]). *Let C be a closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n Sx_n, \\ C_n = \{z \in C : \ \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \ \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \dots,$ where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a, 1]$ for some a with $0 < a \leq 1$. Then $\{x_n\}$ converges strongly to $P_{F(S)}x$, where $P_{F(S)}$ is the metric projection from C onto $F(S)$.

In this article, motivated by Yamada [50], we first introduce four iterative schemes for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone operator in a Hilbert space. Then we obtain weak and strong convergence theorems for the iterative schemes. As in the above paragraph, if an operator is strongly monotone and Lipschitz continuous, then it is inverse-strongly-monotone. Further, we know important examples of inverse-strongly-monotone operators. So, using these results, we consider some applications; see Section 2. In Section 3, we define the notion of relatively nonexpansive mappings in a Banach space which generalizes nonexpansive mappings in a Hilbert space. Then we obtain two convergence theorems for relatively nonexpansive mappings in a Banach space. One of them solves a problem posed at the Symposium on Mathematical Economics sponsored by the Research Institute for Mathematical Science, Kyoto University, which was held during November 29 $\sim$ December 1, 2002; see [46].

2. Preliminaries

Let H be a real Hilbert space and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection form H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0$$

for all $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda Au)$$

for all $\lambda > 0$, where A is a monotone operator of C into H. It is also known that H satisfies Opial's condition [32], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. A set-valued monotone operator $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone operator. It is known that a monotone operator T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. An operator A of C into H is said to be hemicontinous if for all $x, y \in C$, the mapping $[0, 1] \ni t \mapsto A(tx + (1-t)y) \in H$ is continuous, where H has the weak topology. We denote by $N_C v$ the normal cone to C at a point $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}.$$

We know the following theorem [36]:

Theorem 3 (Rockafellar [36]). *Let C be a closed convex subset of a Hilbert space H and let A be a monotone and hemicontinuous operator of C into H. Let $T : H \to 2^H$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. We denote by $\langle \cdot, \cdot\rangle$ the duality product. The normalized duality mapping J form E to E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that

$\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach spase E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [7, 43, 44]. A Banach space E is said to have the Kadec-Klee property if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [7, 43, 44] for more details. Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2$$

for all $x, y \in E$. If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. If $\phi(x, y) = 0$, we have $\|x\| = \|y\|$. This implies $\langle y, Jx\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$; see [7, 43, 44] for more details. Recently, Kamimura and Takahashi [18] proved the following result. This plays an important role in the proof of Theorem 16.

Proposition 4 (Kamimura and Takahashi [18]). *Let E be a uniformly convex and smooth Banach space, and let $\{y_n\}$ and $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex and smooth. Then for any $x \in E$, there exists a point $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y\in C} \phi(y, x).$$

The mapping P_C defined by $P_C x = x_0$ is called the generalized projection [1, 18]. The following are well-known results. For example, see [1, 18].

Proposition 5 (Alber [1], Kamimura and Takahashi [18]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = P_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0$$

for $y \in C$.

Proposition 6 (Alber [1], Kamimura and Takahashi [18]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, P_C x) + \phi(P_C x, x) \le \phi(y, x)$$

for all $y \in C$.

3. Inverse-Strongly-Monotone Operators

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. An operator A of C into H is said to be inverse-strongly-monotone [5] if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha \|Ax-Ay\|^2$$

for all $x, y \in C$. Such an A is said to be α-inverse-strongly-monotone. There are many examples of inverse-strongly-monotone operators. If $A = I - T$, where T is a nonexpansive mapping of C into itself, then A is 1/2-inverse-strongly-monotone; see [16]. Let f be a continuously Fréchet differentiable convex functional on H and let ∇f be the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-monotone; see [3]. We first establish a strong convergence theorem for inverse-strongly-monotone operators and nonexpansive mappings in a Hilbert space.

Theorem 7 (Iiduka and Takahashi [13]). *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n) \end{cases}$$

for every $n = 1, 2, \ldots,$ where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

Using Theorem 7, we obtain Wittmann's theorem [49].

Theorem 8 ([49]). *Let C be a closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping C into itself such that $F(S) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) S x_n$$

for every $n = 1, 2, \ldots,$ where $\{\alpha_n\}$ is a sequence in $[0,1)$. If $\{\alpha_n\}$ is chosen so that

$$\lim_{n\to\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)}x$, where $P_{F(S)}$ is the metric projection from C onto $F(S)$.

Proof. In Theorem 7, put $Ax = 0$ for all $x \in C$. Then A is inverse-strongly-monotone. We have $C = VI(C,A)$ and

$$\begin{aligned} x_{n+1} &= \alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n) \\ &= \alpha_n x + (1-\alpha_n) S P_C x_n \\ &= \alpha_n x + (1-\alpha_n) S x_n. \end{aligned}$$

Using Theorem 7, $\{x_n\}$ converges strongly to $P_{F(S)}x$. □

A mapping $T: C \to C$ is called strictly pseudocontractive if there exists k with $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2$$

for all $x, y \in C$. Such a mapping T is said to be k-strictly pseudocontractive. If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T: C \to C$ is a k-strictly pseudocontractive mapping. Then, A is $\frac{1-k}{2}$-inverse-strongly-monotone; see [5]. Actually, by the definition of T, we have, for all $x, y \in C$,

$$\|(I-A)x - (I-A)y\|^2 \le \|x-y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I-A)x - (I-A)y\|^2 = \|x-y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \ge \frac{1-k}{2}\|Ax - Ay\|^2.$$

Using Theorem 7, we can also prove a strong convergence theorem for a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Theorem 9. *Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself and let T be a k-strictly pseudocontractive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) S((1-\lambda_n)x_n + \lambda_n T x_n) \end{cases}$$

for every $n = 1, 2, \dots,$ where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [a,b] \subset (0, 1-k)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}x$.

We obtain another strong convergence theorem by using the hybrid method in mathematical programming.

Theorem 10 (Iiduka and Takahashi [11]). *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone operator of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \ \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \ \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \dots,$ where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\alpha_n \in [c,1]$ for some c with $0 < c \le 1$ and $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$, then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$, where $P_{F(S)\cap VI(C,A)}$ is the metric projection from C onto $F(S) \cap VI(C,A)$.

Using Theorem 10, we can prove the following strong convergence theorem in a Hilbert space.

Theorem 11. *Let C be a closed convex subset of a real Hilbert space H. Let T be a k-strictly pseudocontractive mapping of C into itself and let S be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Supporse $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n S((1-\lambda_n)x_n + \lambda_n TAx_n), \\ C_n = \{z \in C : \ \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \ \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 1-k]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\alpha_n \in [c,1]$ for some c with $0 < c \leq 1$ and $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 1-k$, then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is the metric projection from C onto $F(S) \cap F(T)$.

We can also prove the following weak convegence theorem for inverse-strongly-monotone operators and nonexpansive mappings in a Hilbert sapce.

Theorem 12 (Takahashi and Toyoda [48]). *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone operator of C into H and let S be a nonexpansive mapping C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\alpha_n \in [a,b]$ for some a, b with $0 < a < b < 1$ and $\lambda_n \in [c,d]$ for some c, d with $0 < c < d < 2\alpha$, then $\{x_n\}$ converges weakly to some element z of $F(S) \cap VI(C,A)$. Further, $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)}x_n$, where $P_{F(S)\cap VI(C,A)}$ is the metric projection from C onto $F(S) \cap VI(C,A)$.

In this section, we finally establish a weak convergence theorem which generalizes Baillon's nonlinear ergodic theorem [2].

Theorem 13 (Iiduka and Takahashi [14]). *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone operator of C into H and let S be a nonexpansive mapping C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{z_n\}$ is given by*

$$\begin{cases} x_{n+1} = SP_C(x_n - \lambda_n Ax_n), \\ z_n = \frac{1}{n}\sum_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$. Then $\{z_n\}$ converges weakly to some element z of $F(S) \cap VI(C,A)$. Further, $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)}x_n$, where $P_{F(S)\cap VI(C,A)}$ is the metric projection from C onto $F(S) \cap VI(C,A)$.

Baillon's nonlinear ergodic theorem [2] is as follows:

Theorem 14 ([2]). *Let C be a closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping C into itself such that $F(S) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{z_n\}$ is given by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} S^{k-1}x$$

for every $n = 1, 2, \ldots$. Then $\{z_n\}$ converges weakly to some element z of $F(S)$. Further, $z = \lim_{n\to\infty} P_{F(S)}x_n$, where $P_{F(S)}$ is the metric projection from C onto $F(S)$.

Proof. In Theorem 13, put $Ax = 0$ for all $x \in C$. Then A is inverse-strongly-monotone. We have $C = VI(C, A)$ and

$$\begin{aligned} x_{n+1} &= SP_C(x_n - \lambda_n Ax_n) \\ &= SP_C x_n = Sx_n \\ &= S^n x. \end{aligned}$$

So, by Theorem 13, $\{z_n\}$ converges weakly to some element z of $F(S)$. □

Using Theorem 13, we can also obtain the following theorem.

Theorem 15. *Let H be a real Hilbert space. Let A be an α-inverse-strongly-monotone operator of H into itself and let S be a nonexpansive mapping H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Suppose $x_1 = x \in H$ and $\{z_n\}$ is given by*

$$\begin{cases} x_{n+1} = S(x_n - \lambda_n Ax_n), \\ z_n = \frac{1}{n}\sum_{k=1}^{n} x_k \end{cases}$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$. Then $\{z_n\}$ converges weakly to some element z of $F(S) \cap A^{-1}0$. Further, $z = \lim_{n\to\infty} P_{F(S)\cap A^{-1}0}x_n$, where $P_{F(S)\cap A^{-1}0}$ is the metric projection from H onto $F(S) \cap A^{-1}0$.

Proof. We have $A^{-1}0 = VI(H, A)$. So, putting $P_H = I$, by Theorem 13, we have that $\{z_n\}$ converges weakly to some element z of $F(S) \cap A^{-1}0$. □

4. Relatively Nonexpansive Mappings

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [35] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

The following is a strong convergence theorem for relatively nonexpansive mappings in a Banach space which generalizes Nakajo and Takahashi's theorem [30] in a Hilbert space.

Theorem 16 (Matsushita and Takahashi [28]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself with $F(T) \neq \phi$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_1 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x \end{cases}$$

for all $n = 1, 2, \ldots$, *where* J *is the duality mapping on* E. *Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x$, *where* $P_{F(T)}$ *is the generalized projection from* C *onto* $F(T)$.

Using Theorem 16, we can prove Nakajo and Takahashi's theorem (Theorem 2) as follows: To show Nakajo and Takahashi's theorem, it is sufficient to prove that if T is nonexpansive, then T is relatively nonexpansive. It is obvious that $F(T) \subset \hat{F}(T)$. If $u \in \hat{F}(T)$, then there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since T is nonexpansive, T is demiclosed. So, we have $u = Tu$. This implies $F(T) = \hat{F}(T)$. Further, in a Hilbert space H, we know that

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. So, $\|Tx - Ty\| \le \|x - y\|$ is equivalent to $\phi(Tx, Ty) \le \phi(x, y)$. Therefore, T is relatively nonexpansive. Using Theorem 16, we obtain the desired result.

Using Theorem 16, we can also consider a proximal-type algorithm for finding zero points of maximal monotone operators in a Banach space. Let A be a multivalued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \phi\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. An operator A is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \ge 0$ for each $x_i \in D(A)$ and $y_i \in Tx_i, i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well-known.

Theorem 17 (Rockafellar [36]). *Let* E *be a reflexive, strictly convex and smooth Banach space and let* A *be a monotone operator from* E *to* E^*. *Then* A *is maximal if and only if* $R(J + rA) = E^*$ *for all* $r > 0$.

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorem 17 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

If $J_r x = x_r$, then we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such a J_r is called the resolvent of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$; see [43, 44] for more details. Using Theorem 16, we can prove a strong convergence threorem for maximal monotone operators in a Banach space. Such a problem has been also studied in [18, 22, 31, 33, 35, 39].

Theorem 18. *Let* E *be a uniformly convex and uniformly smooth Banach space, let* A *be a maximal monotone operator from* E *to* E^*, *let* J_r *be the resolvent of* A, *where* $r > 0$ *and let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \le \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$. *Suppose that* $\{x_n\}$ *is given by*

$$\begin{cases} x_1 = x \in E, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JJ_r x_n), \\ H_n = \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x \end{cases}$$

for all $n = 1, 2, \dots$, where J is the duality mapping on E. If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$, where $P_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

Proof. We first show that $\hat{F}(J_r) \subset A^{-1}0$. Let $p \in \hat{F}(J_r)$. Then, there exists $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty}(z_n - J_r z_n) = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r}(Jz_n - JJ_r z_n) \to 0.$$

It follows from $\frac{1}{r}(Jz_n - JJ_r z_n) \in AJ_r z_n$ and the monotonicity of A that

$$\langle w - J_r z_n, w^* - \tfrac{1}{r}(Jz_n - JJ_r z_n)\rangle \geq 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have

$$\langle w - p, w^*\rangle \geq 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Therefore from the maximality of A, we obtain $p \in A^{-1}0$. On the other hand, we know that $F(J_r) = A^{-1}0$ and $F(J_r) \subset \hat{F}(J_r)$. Therefore $A^{-1}0 = F(J_r) = \hat{F}(J_r)$. Next we show that J_r is a relatively nonexpansive mapping with respect to $A^{-1}0$. Let $w \in E$ and $p \in A^{-1}0$. From the monotonicity of A, we have

$$\begin{aligned}
\phi(p, J_r w) &= \|p\|^2 - 2\langle p, JJ_r w\rangle + \|J_r w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_r w - Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2\langle J_r w - p - J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2\langle J_r w - p, Jw - JJ_r w\rangle \\
&\quad + 2\langle J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2r\langle J_r w - p, \tfrac{1}{r}(Jw - JJ_r w)\rangle \\
&\quad + 2\langle J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&\leq \|p\|^2 + 2\langle J_r w, Jw - JJ_r w\rangle - 2\langle p, Jw\rangle + \|J_r w\|^2 \\
&= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_r w\|^2 + 2\langle J_r w, Jw\rangle - \|w\|^2 \\
&= \phi(p, w) - \phi(J_r w, w) \\
&\leq \phi(p, w).
\end{aligned}$$

This implies that J_r is a relatively nonexpansive mapping. Using Theorem 16, we can conclude that $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$. □

Next, we obtain a weak convergence theorem for relatively nonexpansive mappings in a Banach space which is connected with Browder and Petryshyn's theorem [5] and Rockafellar's theorem [37]. Before proving it, we need the following proposition.

Proposition 19 (Matsushita and Takahashi [27]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself such that*

$F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$. Let $x_1 \in C$ and let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = P_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n)$$

for $n = 1, 2, \ldots$. Then $\{P_{F(T)} x_n\}$ converges strongly to a fixed point of T, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

Using Proposition 19, we can prove the following weak convergence theorem.

Theorem 20 (Matsushita and Takahashi [27]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself such that $F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that*

$$0 \leq \alpha_n \leq 1 \text{ and } \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0.$$

Let $x_1 \in C$ and let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = P_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n)$$

for $n = 1, 2, \ldots$. *If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to u, where $u = \lim_{n\to\infty} P_{F(T)} x_n$ and $P_{F(T)}$ is the generalized projection from C onto $F(T)$*

Using Theorem 20, we can prove the following two weak convergence theorems.

Theorem 21 (Browder and Petryshyn [5]). *Let C be a nonempty closed convex subset of a Hilbert space H, let T be a nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$ and let λ be a real number such that $0 < \lambda < 1$. Let $x_1 \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \lambda x_n + (1-\lambda) T x_n$$

for $n = 1, 2, \ldots$. *Then $\{x_n\}$ converges weakly to u, where $u = \lim_{n\to\infty} P_{F(T)} x_n$ and $P_{F(T)}$ is the metric projection from C onto $F(T)$*

Proof. Let $\alpha_n = \lambda$ for each $n \in \mathbb{N}$. It is clear that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) = \lambda(1-\lambda) > 0$. We know that if T is nonexpansive, then T is relatively nonexpansive. Using Theorem 20, we obtain the desired result. □

Theorem 22. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^* such that $A^{-1}0 \neq \emptyset$, let J_r be the resolvent of A where $r > 0$, and let $\{\alpha_n\}$ be a sequence of real numbers such that*

$$0 \leq \alpha_n \leq 1 \quad \text{and} \quad \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0.$$

Let $x_1 \in E$ and let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J J_r x_n)$$

for $n = 1, 2, \ldots$. *If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to u in $A^{-1}0$, where $u = \lim_{n\to\infty} P_{A^{-1}0} x_n$ and $P_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.*

Proof. As in the proof of Theorem 18, we have that

$$\phi(p, J_r x) \leq \phi(p, x)$$

for all $x \in E$ and $p \in A^{-1}0$ and $F(J_r) \subset \hat{F}(J_r)$. Further, we know that $\hat{F}(J_r) \subset A^{-1}0$; see [17, 22, 28]. So, we obtain that J_r is a relatively nonexpansive mapping and $\hat{F}(J_r) = F(J_r) = A^{-1}0$. Applying Theorem 20, we get that $\{x_n\}$ converges weakly to $\lim_{n\to\infty} P_{A^{-1}0} x_n$, where $P_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$ □

REFERENCES

[1] Y. L. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996, pp. 15-50.

[2] J. B. Baillon, *Un thèoréme de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris A-B **280** (1975), 1511-1514

[3] J. B. Baillon and G. Haddad, *Quelques propriétés des operateurs angle-bornés et n-cycliquement monotones*, Israel J. Math. **26** (1977), 137–150.

[4] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283-301

[5] F. E. Browder and W. V. Petryshym, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[6] Y. Cencer and S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization **37** (1996), 323-339.

[7] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[8] J. Diestel, *Geometry of Banach Spaces - Selected Topics*, Lecture Notes in Mathematics 485, Springer-Verlag, Berlin-New York, 1975

[9] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics 83, Marcel Dekker, Inc., New York, 1984.

[10] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[11] H. Iiduka and W. Takahashi, *Strong convergence theorems by a hybrid method for nonexpansive mappings and inverse-strongly-monotone mappings*, in Fixed Point Theory and Applications (J. G. Falset, E. L. Fuster and B. Sims Eds.) , Yokohama Publishers, Yokohama, 2004, pp. 81-94.

[12] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive nonself-mappings and inverse-strongly-monotone mappings*, J. Convex Anal. **11** (2004), 69-79.

[13] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly-monotone mappings*, Nonlinear Anal. **61** (2005), 341-350.

[14] H. Iiduka and W. Takahashi, *Weak convergence theorems by Cesaro means for nonexpansive mappings and inverse-strongly-monotone mappings*, to appear.

[15] H. Iiduka and W. Takahashi, *Strong convergence theorem by a hybrid method for nonlinear mappings of nonexpansive and monotone type and applications*, to appear.

[16] H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, PanAmer. Math. J. **14** (2004), 49-61

[17] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.

[18] ———, *Strong convergence of a proximal-type algorithm in a Banach apace*, SIAM. J. Optim. **13** (2002), 938-945.

[19] S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417-429.

[20] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[21] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-1006.

[22] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. **2004** (2004), 239-249.

[23] P. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493-517.

[24] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. **6** (1998), 313-344.

[25] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[26] S. Matsushita and D. Kuroiwa, *Approximation of fixed points of nonexpansive nonself- mappings*, Sci. Math. Jpn. **57** (2003), 171-176.

[27] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2004** (2004), 37-47.

[28] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, to appear.

[29] K. Nakajo and W. Takahashi, *Strong and weak convergence theorems by an improved splitting method*, Comm. Appl. Nonlinear Anal. **9** (2002), 99-107.

[30] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372-378.

[31] S. Ohsawa and W. Takahashi, *Strong convergence theorems for resolvents of maximal monotone operators*, Arch. Math. **81** (2003), 439-445.

[32] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[33] S. Reich, *Constructive techniques for accretive and monotone operators*, in Applied Nonlinear Analysis (Proc. Third Internat. Conf. Univ. Texas, Arlington, Tex., 1978), Academic Press, New York, 1979, pp. 335-345.

[34] ———, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[35] ———, *A weak convergence theorem for the alternative method with Bregman distance*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 313-318.

[36] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[37] ———, *Monotone operators and the proximal point algorithm*, SIAM J. Control and Optim. **14** (1976), 877–898.

[38] M. V. Solodov and B. F. Svaiter, *A hybrid projection – proximal point algorithm*, J. Convex Anal. **6** (1999), 59–70.

[39] ———, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), 189–202.

[40] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris **258** (1964), 4413-4416.

[41] W. Takahashi, *Nonlinear variational inequalities and fixed point theorems*, J. Math. Soc. Japan **28** (1976), 168-181.

[42] ———, *Nonlinear complementarity problem and systems of convex inequalities*, J. Optim. Theory Appl. **24** (1978), 493-508.

[43] ———, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[44] ———, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[45] ———, *Iterative methods for approximation of fixed points and their applications*, J. Oper. Res. Soc. Japan **43** (2000), 87-108.

[46] ———, *Approximation of fixed points and proximal point algorithms*, in Mathematical Economics (T. Maruyama Ed.), RIMS Kokyuroku 1337, 2003, pp. 19-31.

[47] W. Takahashi and T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal. **5** (1998), 45-56.

[48] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory App. **118** (2003), 417-428.

[49] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.

[50] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in Inherently Parallel Algorithms

for Feasibility and Optimization and Their Applications (D. Butnariu, Y. Censor and S. Reich Eds.), Elsevier, 2001, pp. 473-504.