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ON THE VERIFICATION THEOREM OF CONTINUOUS-TIME OPTIMAL PORTFOLIO PROBLEMS WITH STOCHASTIC MARKET PRICE OF RISK

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ABSTRACT. In this paper, we study a continuous-time portfolio optimization problem when the market price of risk is driven by linear Gaussian processes. We show sufficient conditions to verify that a solution derived from the Hamilton-Jacobi-Bellman equation is in fact an optimal solution to the portfolio selection problem.

1. INTRODUCTION

In this paper, we study a continuous-time portfolio optimization problem in Kim and Omberg [7]. We show sufficient conditions to verify that a solution derived from the Hamilton-Jacobi-Bellman (HJB) equation is in fact an optimal solution to the portfolio selection problem.

Since Merton's seminal work (Merton [10], [11], [12]), many studies have been done on continuous-time portfolio optimization problems. In particular, there has been increasing interest in finding an optimal portfolio strategy when investment opportunities are stochastic, because many empirical works conclude that investment opportunities are time-varying. In this paper, we study a continuous-time power-utility maximization problem when the market price of risk is driven by linear Gaussian processes. Such a problem has been studied by many authors. See, for example, Kim and Omberg [7], Liu [9], Wachter [14], Bielecki and Pliska [2], Bielecki et al. [3], and Nagai [13]. In this paper, we concentrate on the Kim-Omberg model [7], where the market price of risk is driven by an Ornstein-Uhlenbeck process.

There are two main approaches to solving the continuous-time portfolio optimization problem. One is the stochastic control approach and the other is the martingale approach. Since the market is incomplete in our model, the martingale approach is not applied directly. In this paper, we thus employ the former approach. For an example of the latter approach, see Karatzas and Schreve [6]. In the stochastic control approach, an optimal solution is conjectured by guessing a solution to the HJB equation. It is necessary to verify that the conjectured solution is in fact a solution to the original problem. The solution conjectured from the HJB equation could be an incorrect solution to the original problem. However, as Korn and Kraft [8] pointed out, the verification is often skipped since it is mathematically demanding. For example, Kim and Omberg [7] examined the finiteness of the conjectured value function very carefully, but they did not provide verification

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conditions. Therefore, in this paper, we will give sufficient conditions to verify that the conjectured solution is in fact the solution to the original problem.

2. Formulation of the problem

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ on which a two-dimensional standard Brownian motion $B = (B^1, B^2)^\top$ is defined, and we also fix a time interval $[0, T]$. Let $\mathcal{F}(t)$ be the augmentation of the filtration $\mathcal{F}^B(t) := \sigma(B(s)\,;\, 0 < s < t)$, $0 < t < T$.

Let X be an Ornstein-Uhlenbeck process:

$$
\begin{aligned}
dX(t) &= \lambda(\bar{X} - X(t))dt + \sigma_X \left(\rho dB^1(t) + \sqrt{1-\rho^2}dB^2(t)\right) \\
X(0) &= x_0 \in \mathbb{R}.
\end{aligned} \tag{1}
$$

$\rho \in [-1, 1]$, $\lambda > 0$, $\sigma_X > 0$, and $\bar{X} \in \mathbb{R}$. We call X a state process, because it determines an investment opportunity set in our portfolio problem.

There is one riskless asset and one risky asset. Suppose the price S_0 of the riskless asset satisfies

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$

where $r \geq 0$ is constant. The risky asset price S satisfies the stochastic differential equation

$$dS(t) = S(t)\mu(X(t))dt + S(t)\sigma dB^1(t), \quad S(0) = s > 0, \tag{2}$$

where $\mu\colon \mathbb{R} \to \mathbb{R}$ satisfies $(\mu(x) - r)/\sigma = x$ for $x \in \mathbb{R}$. Then (2) can be written by

$$dS(t) = S(t)(r + \sigma X(t))dt + S(t)\sigma dB^1(t).$$

We consider an investor who can divide his wealth between the riskless asset and the risky assets. Let $\mathcal{L}^2(t_0, t_1)$ be a set of $\mathcal{F}(t)$-progressively measurable processes $\phi\colon \Omega \times [t_0, t_1] \to \mathbb{R}$ such that

$$P\left(\int_{t_0}^{t_1} \phi(t)^2 dt < \infty\right) = 1. \tag{3}$$

We call an element of $\mathcal{L}^2(t_0, t_1)$ a portfolio strategy. We regard $\phi_i(t)$ as a fraction of the wealth invested in the risky asset at time t. The investor's wealth process W^ϕ corresponding to $\phi \in \mathcal{L}^2(0, T)$ is given by $W^\phi(0) = w_0 > 0$ and

$$
\begin{aligned}
dW(t) &= W(t)[\phi(t)(\mu(X(t)) - r) + r]dt + W(t)\phi(t)\sigma dB^1(t) \\
&= W(t)[\phi(t)\sigma X(t) + r]dt + W(t)\phi(t)\sigma dB^1(t).
\end{aligned} \tag{4}
$$

The market is incomplete in the sense that there are some random processes that are not replicated by the self-financing portfolio strategy ϕ.

The investor maximizes the expected utility of his wealth at terminal date T. We assume that the investor has a power utility function with a relative risk aversion coefficient γ:

$$\max_{\phi \in \mathcal{A}_\gamma(0,T)} \quad E\left[\frac{W^\phi(T)^{1-\gamma}}{1-\gamma}\right]. \tag{5}$$

Here $\mathcal{A}_\gamma$ denotes the set of admissible portfolio strategies defined as follows. A stochastic process ϕ is said to be an admissible portfolio strategy on $[t_0, t_1]$ if

(i) $\phi \in \mathcal{L}^2(t_0, t_1)$, when $0 < \gamma < 1$

(ii) for some function $\tilde{\phi}: [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfying the linear growth condition[1], $\phi(t)=\tilde{\phi}(t,X(t))$ on $[t_0,t_1]$, when $\gamma>1$.

The set of all admissible strategies on $[t_0,t_1]$ is denoted by $\mathcal{A}_\gamma(t_0,t_1)$. The choice of our set of portfolio strategies seems to be restrictive. The reason why such a restrictive definition is needed will be explained in the end of Section 4.

Since the market is incomplete, there is no unique equivalent martingale measure, and we cannot apply the so-called martingale approach directly. It is thus common to apply the dynamic programming approach using the HJB equation. Let

$$J(t,w,x;\phi)=E^{t,w,x}\left[\frac{W^\phi(T)^{1-\gamma}}{1-\gamma}\right],$$

Here and in the sequel, we use the notation $E^{t,w,x}[\,\cdot\,]=E[\,\cdot\mid W(t)=w,X(t)=x]$. Let

$$Q=[0,T)\times(0,\infty)\times\mathbb{R}.$$

We then define $V\colon Q\to\mathbb{R}$ by

$$V(t,w,x)=\sup_{\phi\in\mathcal{A}_\gamma(t,T)}J(t,w,x;\phi).$$

The function V is called a value function. The HJB equation related to the problem (5) is

$$\sup_{\phi\in\mathbb{R}}\mathcal{D}^\phi G(t,w,x)=0 \tag{6}$$

with the boundary condition

$$G(T,w,x)=\frac{w^{1-\gamma}}{1-\gamma}, \tag{7}$$

where

$$\begin{aligned}\mathcal{D}^\phi G(t,w,x)&=G_t+w(\phi\sigma x+r)G_w+\lambda(\bar{X}-x)G_x\\&\quad+\frac{1}{2}w^2\phi^2\sigma^2G_{ww}+\frac{1}{2}\sigma_X^2G_{xx}+\sigma_X w\phi\sigma\rho G_{wx}.\end{aligned}$$

It is well-known from Kim and Omberg [7], Liu [9], and others that the function G is separable and has the following form:

$$G(t,w,x)=\frac{w^{1-\gamma}}{1-\gamma}f(t,x), \tag{8}$$

where

$$f(t,x)=\exp\left\{a(t)+b(t)x+\frac{1}{2}c(t)x^2\right\}$$

with the boundary conditions $a(T)=b(T)=c(T)=0$.

It follows from the first order condition for (6) that the candidate optimal portfolio strategy is given by

$$\phi^*(t)=\frac{1}{\gamma}\frac{X(t)}{\sigma}+\frac{1}{\gamma}\frac{\rho\sigma_X}{\sigma}(b(t)+c(t)X(t)). \tag{9}$$

[1] A function $h\colon[0,T]\times\mathbb{R}^K\to\mathbb{R}^L$ is said to satisfy the linear growth condition if $|h(t,x)|\le k(1+|x|)$ for some $k>0$, where $|\cdot|$ is the Euclidean norm

Substituting this conjectured solution into the HJB equation, we obtain the differential equation for $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ as follows:

$$\dot{c}(t) = -\sigma_X^2\left(\frac{1-\gamma}{\gamma}\rho^2+1\right)c(t)^2 - 2\left(\frac{1-\gamma}{\gamma}\sigma_X\rho-\lambda\right)c(t) - \frac{1-\gamma}{\gamma} \tag{10}$$

$$\dot{b}(t) = -\sigma_X^2\left(\frac{1-\gamma}{\gamma}\rho^2+1\right)b(t)c(t) - \left(\frac{1-\gamma}{\gamma}\sigma_X\rho-\lambda\right)b(t) - \lambda\bar{X}c(t) \tag{11}$$

$$\dot{a}(t) = -\frac{1}{2}\sigma_X^2\left(\frac{1-\gamma}{\gamma}\rho^2+1\right)b(t)^2 - \frac{1}{2}\sigma_X^2 c(t) - \lambda\bar{X}b(t) - (1-\gamma)r. \tag{12}$$

The first term of (9) is the usual mean-variance portfolio in a continuous-time model. The second term is a so-called hedging portfolio, which is held by investors in order to hedge against an unfavorable shift in the state variables. Both terms turned out to be linear with respect to state variable X. In general, it is difficult to solve optimal portfolio problems when the market is incomplete. Portfolio (9) is an interesting exception that solves the HJB equation when the market is incomplete.

In order to complete the whole story, we need to verify that $G = V$ and that the candidate optimal portfolio strategy ϕ^* is indeed a solution to (5). In the next section, we will prove a verification theorem.

3. Verification theorem

Verification theorems, such as those in Fleming and Rishel [4] and Fleming and Soner [5], ensure that the solution to the HJB equation coincides with the value function and the candidate portfolio is indeed the optimal portfolio strategy. However, since the wealth process (4) and the conjectured value function (8) do not satisfy the usual assumptions, such as the Lipschitz condition and the polynomial growth condition, we cannot apply standard verification theorems directly to our model. We therefore use the method employed in Nagai [13] and the references therein.

Theorem 1 (Verification Theorem). *Assume that the solution to* (10) *exists on* $[0,T]$. *Then, the function G defined by* (8) *satisfies* $G = V$. *Further,* $\phi^*(t, X(t))$, *defined by* (9), *is an optimal portfolio strategy.*

The following lemma is crucial to the proof of the verification theorem. This result is proven essentially in Bensoussan [1, Lemma 4.1.1]. For a stochastic process g, define

$$\xi(t,g) := \exp\left\{\int_0^t g(u)^\top dB(u) - \frac{1}{2}\int_0^t |g(u)|^2 du\right\} \quad \text{and} \quad \xi_s(t,g) := \frac{\xi(t,g)}{\xi(s,g)}.$$

Lemma 2. *Let* $g(t) := \tilde{g}(t, X(t))$, *where* $\tilde{g}\colon [0,T]\times\mathbb{R}\to\mathbb{R}^2$ *satisfies the linear growth condition. Then*

$$E\left[\xi(T,g)\right] = 1.$$

Using this lemma, we can show the theorem.

Proof of Theorem 1. Using Itô's formula, we obtain

$$\begin{aligned} &dG\left(t, W^\phi(t), X(t)\right) \\ &\quad = \mathcal{D}^\phi G\left(t, W^\phi(t), X(t)\right)dt + G\left(t, W^\phi(t), X(t)\right)g^\phi(t)^\top dB(t) \end{aligned} \tag{13}$$

where

$$g^{\phi}(t) := (1-\gamma)\phi(t)(\sigma, 0)^{\top} + \sigma_X(b(t) + c(t)X(t))\left(\rho, \sqrt{1-\rho^2}\right)^{\top}$$

for all $t \in [0, T]$ and $\phi \in \mathcal{A}_{\gamma}(t, T)$. Let $(t, w, x) \in [0, T) \times [0, \infty) \times \mathbb{R}$ be fixed. Since G is the solution to the HJB equation (6) and ϕ^* is the maximizer in (6), it follows that

$$G(T, W^{\phi^*}(T), X(T)) = G(t, w, x)\xi_t(T, g^{\phi^*}) \tag{14}$$

Using (9), we have

$$\begin{aligned} g^{\phi^*}(t) =& b(t)\left(\left(\frac{1-\gamma}{\gamma}+1\right)\rho\sigma_X, \sqrt{1-\rho^2}\sigma_X\right)^{\top} \\ &+ c(t)X(t)\left(\frac{1-\gamma}{\gamma}+\left(\frac{1-\gamma}{\gamma}+1\right)\rho\sigma_X, \sqrt{1-\rho^2}\sigma_X\right)^{\top} \end{aligned}$$

Then, it follows from Lemma 2 that the process $\xi(t, g^{\phi^*})$ is a martingale. Hence, from (14), we have

$$E^{t,w,x}\left[\frac{W^{\phi^*}(T)^{1-\gamma}}{1-\gamma}\right] = E^{t,w,x}\left[G(T, W^{\phi^*}(T), X(T))\right] = G(t, w, x). \tag{15}$$

On the other hand, it follows from Lemma 2 and the definition of admissible portfolio strategies that the process

$$H_t(u) := G(t, w, x)\xi_t(u, g^{\phi}), \quad t \le u \le T$$

is a supermartingale for all $\phi \in \mathcal{A}_{\gamma}(t, T)$. Then, using (6) and (13), we obtain

$$\begin{aligned} E^{t,w,x}\left[\frac{W^{\phi}(T)^{1-\gamma}}{1-\gamma}\right] = E^{t,w,x}\left[G\left(T, W^{\phi}(T), X(T)\right)\right] &\le E^{t,w,x}[H_t(T)] \\ &\le G(t, w, x) \end{aligned} \tag{16}$$

for all $\phi \in \mathcal{A}_{\gamma}(t, T)$.

Combining (15) and (16), we see that $G = V$ and $\phi^*(t, X(t))$ is an optimal portfolio strategy. □

From Theorem 1, we see that if a solution to the Riccati equation (10) exist, then the conjectured function (8) is in fact the value function. In the following, we can concentrate on if a solutions to the Riccati equation (10) exists. We however emphasize that the choice of portfolio trading strategies set plays an important role here. A key property is if $\{H_t(u)\}$ is a martingale for ϕ^* and a supermartingale for all $\phi \in \mathcal{A}_{\gamma}(t, T)$. When $\gamma > 1$, $\{\xi_t(u, g^{\phi})\}$ is a martingale for all $\phi \in \mathcal{A}_{\gamma}(t, T)$ because of Lemma 2. Thus $\{H_t(u)\}$ is a (super)martingale for all $\phi \in \mathcal{A}_{\gamma}(t, T)$. However, for a broader set of portfolio strategies, say $\phi \in \mathcal{L}^2$, $\{H_t(u)\}$ may not be a supermartingale even if $\{\xi_t(u, g^{\phi})\}$ is supermartingale, since $G(t, w, x)$ is negative when $\gamma > 1$. This is why we restrict the set of admissible portfolio strategies when $\gamma > 1$. Further, it is easy to see that another possible definition of admissible portfolio strategies for $\gamma > 1$ is that $\phi \in \mathcal{L}^2$ and $\{\xi(t, g^{\phi})\}$ is a martingale.

4. The Riccati equation

It follows from Theorem 1 that the solution to the Riccati equation (10), if it exists, give us the solution to the original problem. In this section, we discuss a sufficient condition for the existence of the solution to the Riccati equation (10). The method for solving (10) is standard. See Kim and Omberg [7] for details.

Let

$$C_0 = \frac{1-\gamma}{\gamma}, \quad C_1 = 2\left(\frac{1-\gamma}{\gamma}\sigma_X\rho - \lambda\right), \quad C_2 = \sigma_X^2\left(\frac{1-\gamma}{\gamma}\rho^2 + 1\right),$$

$$q = C_1^2 - 4C_0C_2 = 4\lambda^2\left\{1 - \frac{1-\gamma}{\gamma}\left(\frac{\sigma_X^2}{\lambda^2} + \frac{2\rho\sigma_X}{\lambda}\right)\right\}, \tag{17}$$

and

$$\eta = \begin{cases} \sqrt{q}, & q \geq 0 \\ \sqrt{-q}, & q < 0. \end{cases}$$

Then, the solution to (10) is given by

$$c(t) = \begin{cases} \dfrac{2C_0(1-e^{-\eta(T-t)})}{2\eta - (C_1+\eta)(1-e^{-\eta(T-t)})} & (q > 0) \\ -\dfrac{1}{C_2\left(T-t-\frac{2}{C_1}\right)} - \dfrac{C_1}{2C_2} & (q = 0, C_1 \neq 0) \\ \dfrac{1-\gamma}{\gamma}(T-t) & (q = 0, C_1 = 0) \\ \dfrac{\eta}{2C_2}\tan\left(\dfrac{\eta}{2}(T-t) + \tan^{-1}\left(\dfrac{C_1}{\eta}\right)\right) - \dfrac{C_1}{2C_2} & (q < 0). \end{cases} \tag{18}$$

Note that, by (11) and (12), $\sup_{t\in[0,T]}|c(t)| < \infty$ implies that $\sup_{t\in[0,T]}|b(t)| < \infty$ and $\sup_{t\in[0,T]}|a(t)| < \infty$.

We can easily show that if $\gamma > 1$, then the solution to (10) always exists on $[0, T]$ since $\gamma > 1$ implies $q > 0$. Then we can obtain the following result.

Proposition 3. *If $\gamma > 1$, then the solution to* (5) *exists.*

If $0 < \gamma < 1$, the solution to (10) may not exist. If $q < 0$ and

$$0 < \frac{2}{\eta}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{C_1}{\eta}\right)\right) < T, \tag{19}$$

then (18) takes infinite value at some point on $[0, T)$. Therefore, there is no solution to (10) on $[0, T]$ in this case. However, we can obtain the following result.

Proposition 4. *If $0 < \gamma < 1$ and $q > 0$, then the solution to* (5) *exists.*

5. Conclusion

In this paper, we have derived sufficient conditions that confirm the conjectured solution in Kim and Omberg [7] to be in fact a solution to the original problem. We have shown that if the Riccati equation related to the HJB equation has a solution, then the conjectured solution is in fact a solution to the original problem. If $0 < \gamma < 1$, then the related Riccati equation does not always have a solution. On the other hand, if $\gamma > 1$, then the related Riccati equation always have a solution. However, portfolio strategies are chosen from a rather restrictive set of stochastic processes.

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