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Generalized Dual System Structure and Fixed Point Theorems for Multi-valued Mappings

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Abstract

In this paper, the concept of linear (vector space) structure together with the system of duality is generalized so that we may obtain minimal continuity and/or convexity conditions for multi-valued mappings to have fixed points from the viewpoint of the economic equilibrium theory. We use one of the weakest structure for abstract convexity to define an *abstract system of duality,* and we show that the upper semicontinuity as well as the open lower section property may be treated as special features of the *locally fixed direction* or upper demicontinuity defined under the *generalized dual system structure.* We also use the result to show the existence of economic equilibria as one of the most general forms of the Gale-Nikaido-Debreu lemma.

Keywords : Abstract Convexity, Dual Space, Fixed Point Theorem, Fan-Browder Fixed Point Theorem, Gale-Nikaido-Debreu Lemma.

JEL classification: C62; D51

1 Introduction

The purpose of this paper is to show a fixed point theorem which is general as far as possible from the viewpoint of economic equilibrium theory. The special feature of our result is that we have generalized the concept of 'the dual system of linear topological spaces' and used it as an essential tool to describe general conditions on the existence of fixed points for multi-valued mappings of the Fan-Browder type as well as the Kakutani type. To generalize the concept of dual system has an important meaning in the theory of economics since it enables us to reformulate the ordinary fact/value (commodity/price) distinction.

In the social science, it is always important to emphasize the fact that the society cannot be completely described as a rigorous (game theoretic) mechanism constructed by individual agents. The main difficulty consists in the cognitive feature on this problem. The concept, society, is always recognized by its members as the totality including themselves, i.e., their minds, their believes, their expectations for the future, their knowledges about others, and their rationality itself. They are mutually dependent and their thoughts and knowledges are also depending on each other. The economic theory is an attempt to simplify the problem by assuming for each person a rationality based on one belief on the world, i.e., the utility or profit maximizer in the market price mechanism. For this simplification, however, the distinction between the fact (stated on the commodity) and the value (given by the price) plays an essential role. Prices give us a rule for trades and restrict our action in a certain subset of the commodity space. Given a belief on the world and a rationality based on it, the physical situation of demand and supply in the commodity space, gives us the concept of equilibrium through witch we describe what the human society is and/or should be. Mathematically, the equilibrium has been characterized as a fixed point of a mapping defined over the dual system of commodity and price vector spaces, and the abstract treatment of this problem is known as the Gale-Nikaido-Debreu lemma. We think that such a distinction of our believes (value judgements) and the physical situations (factual judgements) is a useful, even necesarry, tool for the economic theory, though the distinction may not be so clear as we have seen in the standard general economic equilibrium theory.[1] To generalize the concept of duality as a mathematical tool for describing such a fact/value distinction is important for the economic theory to have sufficient generosity in describing the world.

Our results in this paper may also have some mathematical significance since some theorems may be considered as a further generalization of the recent most general types of fixed point theorems of the Fan-Browder type as well as the Kakutani type. Theorem 1 and Theorem 2 are extension of Browder's theorem which are essentially the same approaches as treated in Park (2001) (see Corollary 4.5 proved under G-convex space), Ben-El-Mechaiekh et al. (1998) (Proposition 3.8 proved under L-convexity), Ding (2000) (under a special contractible condition), Luo (2001) (see Theorem 3.2 proved under δ-convexity with semi-lattice structure), etc.[2] Theorem 3 may be considered as one of the most general form of Kakutani's fixed point theorem. In this theorem, the necessary condition is given through the distinctive notion of generalized dual system which is defined in this paper without using concepts under the vector space structure. Though Ben-El-Mechaiekh et al. (1998) (Theorem 4.2 and Corollary 4.7) treats the same problem, we can show the result without using the uniform structure. Theorem 3 may also be considered as a generalization of Fan-Browder's fixed point thoerem (see Corollary 1 and 2). Theorem 4 is covered by results in Ben-El-Mechaiekh et al. (1998) though the notion of convexity here is more general. Theorem 5 as a generalization of Kakutani's fixed point theorem has, however, an important meaning different from results in Ben-El-Mechaiekh et al. (1998) since it is based on the concept of generalized dual system structure. Indeed, the result may directly be applied as a general condition on preferences in economic equilibrium arguments as in Urai and Yoshimachi (2004).

2 Abstract Convexity

In the following we use the concept of convexity which may not necessarily depend on the vector space structure. Usually, the property of convexity in a vector space has two different meanings, i.e., (1) it defines the convex hull of a set, the smallest convex set containing it, in the space, and (2) it defines all

[1] We can see many examples in the real society where morals, customs, laws, etc., play significant roles in formulating our value judgements. They may also affect on our factual judgements as the old saying, "habit is second nature."

[2] Our treatments of the abstract convexity is weaker than all the corresponding concepts used in those papers.

convex combinations for each finite subset of the space. For the sake of fixed point arguments, the latter is far more important than the former.[3] To generalize the concept of convexity, we separate the second feature as a structure from the vector space structure.

As we can see in the classical theorem of Eilenberg and Montgomery (1946), and its futher generalization in Begle (1950), the vector structure for the topological space is not a necessary setting for the fixed point argument. Even for the cases with classical Gale-Nikaido-Debreu lemma, for which a certain kind of dual system structure seems to be necessary, it has been known that the necessary setting is "a compact ... set in which the convex linear combination of finitely many points depends continuously on its coefficients," (Nikaido (1959), p.362, Main Theorem). The abstract convexity in this paper is nothing but a generalization for this type of arguments on "linear combination," so that we are not intend to cover all the arguments in recent researches such that Komiya (1981), Horvath (1991), Park and Kim (1996), and Ben-El-Mechaiekh et al. (1998), which are generalizing all the concept including "convex hull." With respect to the concept of "convex combination," however, we shall give the framework which is most general amongst all of these recent arguments.

For a finite set A, denote by $\sharp A$ the number of elements of A. By Δ^A, we denote the set of all function $e : A \to R_+$ such that $\sum_{a \in A} e(a) = 1$ for each non-empty finite set A. Denote by e^a the element of Δ^A such that $e^a(a) = 1$ and $e^a(a') = 0$ for each $a' \in A \setminus \{a\}$. We identify Δ^A with the $(\sharp A - 1)$-dimensional standard simplex in $R^{\sharp A}$ by identifying e^A with an appropriately chosen element of the standard basis of $R^{\sharp A}$. Moreover, for each non-empty finite set A and a finite set $A' \supset A$, we identify Δ^A as a subset of $\Delta^{A'}$ by identifying $e \in \Delta^A$ with the element, $e' \in \Delta^{A'}$ such that $e'(a) = e(a)$ for each $a \in A$ and $e'(a') = 0$ for each $a' \in A' \setminus A$.

Let X be a topological space. Denote by $\mathscr{F}(X)$ the set of all non-empty finite subset of X. We say that on the space X an *(abstract) convex structure* is defined if for each non-empty finite set $A \in \mathscr{F}(X)$, there is a finite set $\hat{A} \in \mathscr{F}(X)$, $A \subset \hat{A}$, together with a continuous function $f_A : \Delta^A \to X$. A subset Z of X is said to be *(abstract) convex* if for each non-empty finite set $B \subset Z$ and for each $B' \supset \hat{B}$ such that $B' \in \mathscr{F}(X)$, we have $f_{B'}(\Delta^B) \subset Z$.[4]

It is easily seen that an arbitrary intersection of abstract convex sets are abstract convex sets. Let X be a set on which an abstract convex structure is defined. For each non-empty finite subset $A \subset X$, define $C(A)$, a generalized concept of the set of *convex combinations* among points in A, as

$$C(A) = \bigcup_{A' \supset \hat{A}, A' \in \mathscr{F}(X)} f_{A'}(\Delta^A).$$

Then, for each convex subset Z of X, it is clear that $C(A) \subset Z$ for each non-empty finite subset $A \subset Z$, though it is not always the case that the union of all $C(A)$'s, $A \subset Z$, $0 < \sharp A < \infty$, forms a convex set.[5] Therefore, since an arbitrary intersection of abstract convex sets is abstract convex, the smallest abstract convex set containing Z, the *convex hull*, $\mathrm{co}\, Z$, of Z, exists. Note that the convex hull, $\mathrm{co}\, Z$, may not be identified with the set of all finite convex combinations of points in Z, $C(Z)$. When Z is an abstract convex set, $\mathrm{co}\, Z$ coincides with $C(Z)$.

[3] Indeed, to define the convex hull is nothing but to define the family of all convex sets in the space, so that the condition is always a restriction for a convex structure of the space to be a coarse one. For fixed point arguments, the former is usually unnecessary since the latter automatically defines one of the finest convex structure, and the necessary condition of the convexity for fixed point theorems becomes weaker if the convex structure becomes finer.

[4] In the above condition, if we replace "for each $B' \supset \hat{B}$" with "for each $B' \supset B$," we obtain the concept of *L-convexity* defined in Ben-El-Mechaiekh et al. (1998). See also Komiya (1999) for an excellent brief introduction to the theory of abstract convexity.

[5] In this sense, a single point may not necessarily be an abstract convex set.

3 Fixed Point Theorems and Dual System Structure

At first, we show the following extension of the fixed point theorem of Browder (1968) in abstract convex spaces.

THEOREM 1 : (Browder's Theorem under Abstract Convexity) Let X be a compact Hausdorff space on which an abstract convex structure is defined. Then, every non-empty convex valued correspondence, $\varphi : X \to 2^X$, having open lower section at each point has a fixed point.

PROOF : Since φ is non-empty valued, $\{\varphi^{-1}(y)|y \in X\}$ is an open covering of X. Since X is compact, there is a finite subcovering $\{\varphi^{-1}(y^i)|i = 0, \ldots, n\}$ of $\{\varphi^{-1}(y)|y \in X\}$. Let A be the finite set $\{y^0, \ldots, y^n\}$ and let $\bar{A}$ be a set containing $\hat{B}$ for all $B \subset A$. Then, by the definition of abstract convexity, if Z is convex and if Z contains a set $B \subset A$, we have $Z \supset f_{\bar{A}}(\Delta^B)$. Denote by $\beta^0, \ldots, \beta^n$ the partition of unity subordinated to $\varphi^{-1}(y^0), \ldots, \varphi^{-1}(y^n)$ and define a mapping, $F : \Delta^A \to \Delta^A$, as

$$F(e) = \sum_{i=0}^{n} \beta^i(f_{\bar{A}}(e))e^{y^i}.$$

F is a continuous function on Δ^A to itself so that F has a fixed point, $e^* = F(e^*)$, by the fixed point theorem of Brouwer. Define x^* as $x^* = f_{\bar{A}}(e^*)$. Then, by the property of partition of unity, we have

$$\beta^i(x^*) > 0 \Longrightarrow x^* \in \varphi^{-1}(y^i) \Longrightarrow y^i \in \varphi(x^*).$$

Denote by B the set $\{y^i|\beta^i(x^*) > 0\}$. Apply $f_{\bar{A}}$ on each side of the equation, $e^* = \sum_{i=0}^{n} \beta^i(f_{\bar{A}}(e^*))e^{y^i}$. By considering the fact that $x^* = f_{\bar{A}}(e^*)$ and $x^* = f_{\bar{A}}(\sum_{i=0}^{n} \beta^i(e^*)e^{y^i})$, we have $x^* \in f_{\bar{A}}(\Delta^B)$. Since $\varphi(x^*)$ is convex, $f_{\bar{A}}(\Delta^B) \subset \varphi(x^*)$, so that we have

$$x^* \in f_{\bar{A}}(\Delta^B) \subset \varphi(x^*),$$

i.e., x^* is a fixed point of φ. ■

The next theorem may also be considered as an extension of Browder's fixed point theorem.[6] This type of theorems including cases with mappings of the Kakutani type has been treated in Urai and Hayashi (2000), Urai (2000), Urai and Yoshimachi (2004), under the settings of linear topological spaces. We shall develop the results into the topological spaces with or without linear structures. In the following, we denote the structure of (abstract) convexity on a set X by the family of mappings, $\{f_A|A \in \mathscr{F}(X)\}$, i.e., the operator $A \mapsto \hat{A}$ on $\mathscr{F}(X)$ to itself will not be referred to as long as there is no fear of confusion. We also denote by $\mathcal{F}ix(\varphi)$ the set of all fixed point of φ, $\{x \in X|x \in \varphi(x)\}$, for a mapping, $\varphi : X \to 2^X$.

THEOREM 2 : (Extension of Browder's Theorem) Let X be a non-empty compact Hausdorff space having a convex structure $\{f_A|A \in \mathscr{F}(X)\}$. If a convex valued correspondence, $\varphi : X \to 2^X \setminus \{\emptyset\}$, satisfies that for all $x \notin \mathcal{F}ix(\varphi)$, there are a point $y^x \in \varphi(x)$ and an open neighbourhood U^x of x such that for all $z \in U^x \setminus \mathcal{F}ix(\varphi)$, we have $y^x \in \varphi(z)$. Then, $\mathcal{F}ix(\varphi) \neq \emptyset$.

[6]Indeed, the open lower section property is imposed merely on a certain element y^x of $\varphi(x)$.

PROOF : Assume that $\mathcal{Fix}(\varphi) = \emptyset$. Then, $\{U^x | x \in X\}$ covers X, so that there is a finite subcovering $\{U^{x^1}, \dots, U^{x^n}\}$ together with points $y^1 = y^{x^1}, \dots, y^n = y^{x^n}$ satisfying conditions in the theorem. Let A be the finite set $\{y^1, \dots, y^n\}$ and let $\beta^0, \dots, \beta^n$ be the partition of unity subordinated to $U^{x^1}, \dots, U^{x^n}$. Take $\bar{A} \in \mathscr{F}(X)$ as $\bar{A} = \bigcup\{\hat{B} | B \subset A\}$, and define the continuous mapping F on Δ^A to itself as

$$F(e) = \sum_{t=1}^{n} \beta^t(f_{\bar{A}}(e)) e^{y^t}.$$

Then, F has a fixed point $e^* = F(e^*)$ by Brouwer's fixed point theorem. Let $x^* = f_{\bar{A}}(e^*)$ and $B = \{y^t | \beta^t(e^*) > 0\}$. Since $\beta^t(x^*) > 0$ implies that $y^t \in \varphi(x^*)$, we have $f_{\bar{A}}(\Delta^B) \subset \varphi(x^*)$. It follows that $x^* = f_{\bar{A}}(e^*) = f_{\bar{A}}(\sum_{t=1}^{n} \beta^t(f_{\bar{A}}(e^*)) e^{y^t}) \in f_{\bar{A}}(\Delta^B) \subset \varphi(x^*)$. Hence, φ has a fixed point contrary to the assumption. ∎

We say that a correspondence $\varphi : X \to 2^X$ has a *locally common element* on $X \setminus \mathcal{Fix}(\varphi)$ if the condition for φ in Theorem 2, "for all $x \notin \mathcal{Fix}(\varphi)$, there are an open neighbourhood U^x of x and a point $y^x \in \varphi(x)$ such that for all $z \in U^x \setminus \mathcal{Fix}(\varphi)$, we have $y^x \in \varphi(z)$," is satisfied.

A further generalization of Theorem 2 will be given through the following concept on the *direction* of $\varphi(x)$ from x of mapping $\varphi : X \to 2^X$. Assume that X and W are sets having (abstract) convex structures, $\{f_A | A \in \mathscr{F}(X)\}$ and $\{g_A | A \in \mathscr{F}(W)\}$, respectively. The set, W, together with a mapping, $V : X \times W \to 2^X$, is called a *generalized dual system structure* on X, (or a *directional structure* on X), if V satisfies the following three conditions.[7]

(V1) $V(x, w)$ is a convex subset of X.

(V2) $x \notin V(x, w)$.

(V3) $y \in V(x, w^0) \wedge y \in V(x, w^1) \wedge \dots \wedge y \in V(x, w^n) \Longrightarrow y \in V(x, w)$ for all $w \in C(\{w^0, w^1, \dots, w^n\})$.

In condition (V3), $C(\{w^0, w^1, \dots, w^n\})$ denotes the set

$$\bigcup_{A' \in \mathscr{F}(X), A' \supset \hat{A}} g_{A'}(\Delta^A),$$

where $A = \{w^0, w^1, \dots, w^n\}$.[8] We say that the dual system structure is *topological* if, adding to (V1), (V2), and (V3), the following condition is satisfied.

(V4) $V(x, w) \neq \emptyset \Longrightarrow \exists y^x \in V(x, w), \exists U^x \subset X, x \in U^x, U^x$ is open, and $\forall z \in U^x, y^x \in V(z, w)$.[9]

Condition (V4) is not necessary for fixed point theorems of the Browder type. For the Kakutani type, however, (V4) is essential. (When the space has a uniform structure, however, (V4) is not necessary even for cases with the Kakutani type. See Theorem 4 and Theorem 5 below.)

A correspondence $\Phi : X \to 2^W$ is said to be the *dual space (directional) representation of* φ *under* V *and* W if for all $x \notin \mathcal{Fix}(\varphi)$ and for all $w \in \Phi(x)$, $\varphi(x) \subset V(x, w)$. Of course, we may not expect that for

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[7] Strictly speaking, the usage of the word "structure" in the above sense is an abuse of language. Indeed, it is V that should be called as a structure on two base sets X and W.

[8] If X and W are linear spaces, and if there is a canonical bilinear form, $f : X \times W \to R$, such that (X, W, f) forms a dual system, we may define V as $V(x, w) = \{y \in X | f(y - x, w) > 0\}$ so that the set W together with the mapping V on $X \times W$ is recognized as a generalized dual system structure on X. Note also that condition (V3) holds even when $w^0 = w^1 = \dots = w^n$.

[9] Of course, the condition generalizes the concept of partial continuity of the canonical bilinear form and the topological dual space. Condition (V4) is weaker than the *lower topological condition*, (A3), in Urai and Yoshimachi (2004).

each $\varphi : X \to 2^X$ there is a generalized dual system structure, W and $V : X \times W \to 2^X$, under which φ has a dual space representation. If such exists, however, we may obtain various ways to characterize the mapping, φ, and the set of its fixed points, $\mathcal{F}ix(\varphi)$. A mapping $\varphi : X \to 2^X$ is said to have a *locally fixed direction* if it has a dual space representation, Φ, satisfying that for each $x \in X \setminus \mathcal{F}ix(\varphi)$, there are a point (direction), $w^x \in W$, and an open set, U^x, of x such that $\forall z \in U^x$, $w^x \in \Phi(z)$.[10]

The special case of generalized dual system structure such that $X = W$ was originally given by one of the authors in Urai (2000; Section 6, Theorem 21), and was further developed in Urai and Yoshimachi (2004) as a structure of "directions" in topological vector spaces.[11] Here, we have generalized the concept in two respects:

(i) The concept of "convexity" may not necessarily depend on the linear structure on the base set.

(ii) The set, W, is not restricted to the subset of X.

The second point is particularly important since for cases with vector spaces, by considering W as a subset of the dual vector space of the vector space including X, we may recognize the directional structure as a generalized concept of the vector space duality. Then, since the condition, closedness and convexity, on the values of mappings of the Kakutani type has a special relation to the set of continuous real linear forms (the topological dual space), we may obtain a unified viewpoint for fixed point theorems of the Kakutani (1941) type and the Browder (1968) type.[12] See the next theorem.

THEOREM 3 : (Fixed Point Theorem under Dual System Structure) Let X be a non-empty compact Hausdorff space having a convex structure, $\{f_A | A \in \mathscr{F}(X)\}$, and let W be a set having a convex structure, $\{g_A | A \in \mathscr{F}(W)\}$. Assume that a non-empty valued correspondence, $\varphi : X \to 2^X$, has a locally fixed direction under a topological dual system structure, $(W, V : X \times W \to 2^X)$, on X. Then, φ has a fixed point.

Before proving the theorem, it will be convenient to define the following notion. We say that $\hat{\varphi} : X \to 2^X$ is a *directional extension* of φ if there is at least one dual space representation, Φ, of φ, under the dual system structure, (W, V), such that $\hat{\varphi}(x) = \bigcap_{w \in \Phi(x)} V(x, w)$ for all $x \in X \setminus \mathcal{F}ix(\varphi)$. Condition (V4) assures, then, that if φ has a locally fixed direction, the directional extension $\hat{\varphi}$ has a *locally common element* at each $x \in X \setminus \mathcal{F}ix(\varphi)$, i.e., there are an element $y^x \in X$ and an open set U^x of x such that $\forall z \in U^x, y^x \in \hat{\varphi}(z)$.

PROOF : As stated above, since φ has a locally fixed direction under the topological dual system structure, it also has the directional extension, $\hat{\varphi}$, having a locally common element at each $x \in X \setminus \mathcal{F}ix(\varphi)$. Assume

[10] We also say that φ has a *locally continuous* (or *locally upper semicontinuous*) *direction*, if there is a dual space representation, Φ, of φ such that for each $x \in X \setminus \mathcal{F}ix(\varphi)$, there is a neighbourhood U^x of x and a continuous mapping (upper semicontinuous correspondence, resp.,) $p^x : U^x \setminus \mathcal{F}ix(\varphi) \to W$ that is a selection of Φ on $U^x \setminus \mathcal{F}ix(\varphi)$. Fixed point theorem for these mappings are treated in Urai and Yoshimachi (2002) and Urai and Yoshimachi (2004) under the linear structure on $X = W$.

[11] In those papers, condition (V3) was written in a stronger form under the special setting of $X = W$. (See footnote 2 of Urai and Yoshimachi (2004).)

[12] If X is a compact convex subset of a locally convex topological vector space, a non-empty closed valued upper semicontinuous correspondence, $\varphi : X \to X$, is easily seen to have a locally fixed direction under the structure of standard topological dual system. Indeed, for all $x \in X \setminus \mathcal{F}ix(\varphi)$, the second separation theorem (c.f. Schaefer (1971)) assures the existence of a closed hyper plane (i.e., a continuous linear form on the locally convex space) which strictly separates x and $\varphi(x)$. Define $\Phi(x)$ as the set of such linear forms. Then, the upper semicontinuity of φ means that there is a neighbourhood, U^x, of x on which the hyper plane, (an element of $\Phi(x)$), gives the fixed local direction of $\varphi(z)$ for each $z \in U^x$.

that φ does not have a fixed point. Then, it is also clear that $\hat{\varphi}$ has a non-empty convex valued on X, so that by Theorem 2, has a fixed point $x^* \in \hat{\varphi}(x^*)$. By the definition of directional extension, however, we have $x^* \in \hat{\varphi}(x^*) = V(x^*, y)$ for some $y \in \Phi(x^*)$, which contradicts the condition, (V2). ■

Given a convex valued correspondence, $\varphi : X \to 2^X$, we may define a dual system structure, $(W = X, V_\varphi : X \times X \to X)$, as $V_\varphi(x,y) = \varphi(x)$ for each (x,y) such that $x \neq y$ and $y \in \varphi(x)$, and $V_\varphi(x,y) = \emptyset$, otherwise. The structure, $(W = X, V_\varphi)$, is called the ***dual system structure induced by*** φ. Then, if φ has a locally common element, as in Theorem 1 and Theorem 2, φ has a locally fixed direction under $(W = X, V_\varphi)$, and if φ has a locally fixed direction under $(W = X, V_\varphi)$, say y near x, by taking $z^y \in \varphi(x) = V_\varphi(x,y)$ arbitrarily, we may also check that $(W = X, V_\varphi)$ satisfies (V4). Hence, Theorem 3 is indeed an extension of Theorem 1 and Theorem 2. More generally, we have the following corollary to Theorem 3 as an extension of the fixed point theorem of the Browder type.

COROLLARY 1 : (Most General Type of Browder's Theorem) Let X be a compact Hausdorff space having the convex structure, $\{f_A | A \in \mathscr{F}(X)\}$, and let $\varphi : X \to 2^X$ be a non-empty valued map. Suppose that φ has a locally common element on $X \setminus \mathcal{F}ix(\varphi)$. Moreover, assume that for each $x \in X \setminus \mathcal{F}ix(\varphi)$, there is a convex set $\Psi(x)$ such that $x \notin \Psi(x)$ and $\varphi(x) \subset \Psi(x)$. Then, φ has a fixed point.

PROOF : If φ does not have a fixed point, then the non-empty convex valued correspondence Ψ on X to itself satisfies (V4) under the dual system structure, $(W = X, V_\Psi)$, induced by Ψ. Hence, Ψ has a fixed point by Theorem 3, though it is impossible since $x \notin \Psi(x)$ for all $x \in X$. ■

The next result is also an immediate consequence of Theorem 3. This corollary may also be considered as an extension of main theorems in Urai (2000; Theorem 1, (K*)) and Urai and Yoshimachi (2004; Theorem 1).

COROLLARY 2 : (Generalization of (K^*) in Urai (2000)) Let X be a compact Hausdorff space having the convex structure, $\{f_A | A \in \mathscr{F}(X)\}$, and let $\varphi : X \to 2^X$ be a non-empty valued map. Assume that there is a convex valued map $\Phi : X \to 2^X$ such that for all $x \in X \setminus \mathcal{F}ix(\varphi)$, $x \notin \Phi(x)$ and $\varphi(x) \subset \Phi(x)$. If, for all $x \in X \setminus \mathcal{F}ix(\varphi)$, Φ has a locally common element, φ has a fixed point.

PROOF : By considering the dual system structure, $(W = X, V_\Phi)$, induced by Φ, the dual space representation of Φ, the directional extension of Φ, and Φ are identical. Hence, $(W = X, V_\Phi)$ is topological and Φ has a fixed point x^* by Theorem 3. Since $x \notin \mathcal{F}ix(\varphi)$ implies $x \notin \Phi(x)$, we have $x^* \in \mathcal{F}ix(\varphi)$. ■

As we have seen, if X is a compact convex subset of a locally convex topological vector space, a non-empty closed valued upper semicontinuous correspondence, $\varphi : X \to X$, has a locally fixed direction under the standard topological dual system structure (see footnote 12). Therefore, it is immediate that Theorem 3 is an extension of Kakutani-Fan-Glicksberg's fixed point theorem (Kakutani (1941), Fan (1952), Glicksberg (1952)). It should also be noted, however, that this type of theorems is important since it includes the cases for all (single valued) continuous functions in locally convex topological vector spaces. The next corollary, though it is merely a special (single valued mapping) case of Theorem 3, may be considered as a generalization of Brouwer's fixed point theorem to the space without locally convex vector space structures and/or metrizability.[13]

COROLLARY 3 : (Fixed Point Theorem for Single Valued Mapping) Let X be a compact Hausdorff space having the convex structure, $\{f_A | A \in \mathscr{F}(X)\}$, and let $f : X \to X$ be a single valued function. Suppose that there is a topological dual system structure, (W, V), on X such that for each $x \in X \setminus \mathcal{F}ix(f)$, there are a $w^x \in W$ and an open neighbourhood U^x of x satisfying that for all $z \in U^x$, $f(z) \in V(z, w^x)$. Then, f has a fixed point.

We also note that not only the case with the vector space structure but also the theorem of the Kakutani-Fan-Glicksberg type under the uniform space structure may be treated in the same framework of the system of duality of our results. Let X be a (topological) uniform space whose topology is given by a symmetric open base, $\{U_\mu \subset X \times X : \mu \in \mathscr{M}\}$, for the uniformity for X.[14] Suppose that there is on X a convex structure, $\{f_A : A \in \mathscr{F}(X)\}$. Moreover, assume that for each member (vicinity), U_μ, of the base for the uniformity, (1) $U_\mu(x) = \{y | (x, y) \in U_\mu\}$ is convex, and (2) for each convex subset Z of X, $U_\mu(Z) = \{y | (z, y) \in U_\mu$ for some $z \in Z\}$ is convex. We call such a space a *locally convex uniform space.*[15]

THEOREM 4 : (Kakutani's Theorem in Locally Convex Uniform Space) Let X be a non-empty compact Hausdorff locally convex uniform space, and let $\{f_A : A \in \mathscr{F}(X)\}$ be the convex structure on X. If $\varphi : X \to 2^X$ is a non-empty convex valued mapping having closed graph, φ has a fixed point.

PROOF : Assume the contrary. Then, for each $x \in X$, we have $x \notin \varphi(x)$. For each x and $y \in \varphi(x)$, there is a vicinity U such that $(x, y) \notin U$. Take a symmetric vicinity $V_{(x,y)}$ such that $(V_{(x,y)} \circ V_{(x,y)}) \circ (V_{(x,y)} \circ V_{(x,y)}) \subset U$. Then, $V_{(x,y)}(x) \times V_{(x,y)}(y) \cap V_{(x,y)} = \emptyset$ since $((z_1, z_2) \in V_{(x,y)}(x) \times V_{(x,y)}(y) \cap V_{(x,y)}) \Longrightarrow ((x, z_1) \in V_{(x,y)} \wedge (z_1, z_2) \in V_{(x,y)} \wedge (z_2, y) \in V_{(x,y)}) \Longrightarrow ((x, y) \in V_{(x,y)} \circ V_{(x,y)} \circ V_{(x,y)} \subset U)$. Since the graph of φ is compact, there are finite points $(x_1, y_1), \ldots, (x_n, y_n)$ in the graph of φ such that $\bigcup_{i=1}^n V_{(x_i,y_i)}(x) \times V_{(x_i,y_i)}(y)$ covers the graph of φ. Let U^* be a vicinity such that $U^* \subset \bigcap_{i=1}^n V_{(x_i,y_i)}$ and V^* be a symmetric vicinity such that $(V^* \circ V^*) \circ (V^* \circ V^*)$. Then, for all (x, y) in the graph of φ, we have $V^*(x) \times V^*(y) \cap V^* = \emptyset$.

Let $U \subset V^*$ be an arbitrary open vicinity. Since X is compact, the covering $\{U(x) | x \in X\}$ has a finite subcovering, $\{U(a_1^U), \ldots, U(a_n^U)\}$. Denote by $\beta_1^U : X \to [0,1], \ldots, \beta_n^U : X \to [0,1]$ the partition of unity subordinated to $\{U(a_1^U), \ldots, U(a_n^U)\}$. Take $b_1^U, \ldots, b_n^U$ as arbitrary points of $\varphi(a_1^U), \ldots, \varphi(a_n^U)$, respectively, and define a mapping g^U on Δ^{B^U}, $B^U = \{b_1^U, \ldots, b_n^U\}$, to itself as

$$g^U(e) = \sum_{i=1}^n \beta_i^U(f_{B^U}(e))e^i,$$

where f_{B^U} denote the function given under the convex structure on X, and e^i denotes the member of Δ^{B^U} such that the value of b_i^U is 1. It is clear that g^U is a continuous function on Δ^{B^U} to itself so that has a

[13] Under the vector space structure, we have a further development of the theorem in Urai and Yoshimachi (2004).

[14] See Kelley (1955; Chapter 6) for details on these notions with respect to the uniform space.

[15] Note that there may not exist on X a vector space structure. Of course, every locally convex topological vector space is a locally convex uniform space under the standard vector space convex structure.

fixed point e^U by Brouwer's fixed point theorem. Let $x^U = f_{\dot{B}^U}(e^U)$. Since X is compact, by considering the uniformity as a directed set, we may consider that there is a converging subnet, $x^U \to x^*$.

Take an open vicinity $U \subset V^*$ satisfying conditions (1) and (2). Since $U \subset V^*$, we have for each $x \in X$ and $\varphi(x)$,

$$U(\varphi(x)) \cap U(x) = \emptyset.$$

By the upper semicontinuity of φ, we may take a symmetric open vicinity $\bar{U} \subset U$ such that for all $z \in \bar{U}(x^*)$, $\varphi(z) \subset U(\varphi(x^*))$. Moreover, take a symmetric open vicinity U_0 such that $U_0 \circ U_0 \subset \bar{U}$ and a symmetric open vicinity $\bar{V} \subset U_0$ such that $x^{\bar{V}} \in U_0(x^*) \subset \bar{U}(x^*)$, where, the point $x^{\bar{V}}$, (implicitly, together with points, $a_1, \ldots, a_k$, and $b_1, \ldots, b_k$, depending on $\bar{V}$), is taken as in the argument of the previous paragraph. (That is, $\{\bar{V}(a_1), \ldots, \bar{V}(a_k)\}$ covers X, and $b_1, \ldots, b_k$ are points of $\varphi(a_1), \ldots, \varphi(a_n)$, respectively.) Denote by $\beta^1 : X \to [0,1], \ldots, \beta^n : X \to [0,1]$ the partition of unity subordinated to $\{\bar{V}(a_1), \ldots, \bar{V}(a_k)\}$, and denote by B the finite set $\{b_1, \ldots, b_k\}$. Now, the point $x^{\bar{V}} = f_{\dot{B}}(e^{\bar{V}})$ satisfies

$$x^{\bar{V}} = f_{\dot{B}}(g^{\bar{V}}(e^{\bar{V}})) = f_{\dot{B}}(\sum_{i=1}^{k} \beta^i(x^{\bar{V}})e^i),$$

where e^i denotes the member of Δ^B such that the value of b_i is 1. It should be noted that $\beta^i(x^{\bar{V}}) > 0$ means that $x^{\bar{V}} \in \bar{V}(a_i)$, i.e., $a_i \in \bar{V}(x^{\bar{V}}) \subset U_0(x^{\bar{V}})$. Since $x^{\bar{V}} \in U_0(x^*)$, $\beta^i(x^{\bar{V}}) > 0$ means that $(x^*, a_i) = (x^*, x^{\bar{V}}) \circ (x^{\bar{V}}, a_i) \in U_0 \circ U_0 \subset \bar{U}$. Therefore, we have $b_i \in \varphi(a_i) \subset U(\varphi(x^*))$ as long as $\beta^i(x^{\bar{V}}) > 0$. This means, however, that $x^{\bar{V}} = f_{\dot{B}}(\sum_{i=1}^{k} \beta^i(x^{\bar{V}})e^i)$ is an element of $U(\varphi(x^*))$ since $U(\varphi(x^*))$ is convex by condition (2). Since $x^{\bar{V}} \in U_0(x^*) \subset U(x^*)$, we also have $x^{\bar{V}} \in U(x^*) \cap U(\varphi(x^*))$, a contradiction. ■

In the above proof, we may define a directional structure on X as follows. For each $x \in X$ and $e \in \Delta^B$, define $V(x,e)$ as

$$V(x,e) = \bigcap_{i \in \{j : e(b^j) > 0\}} U(\varphi(a_i)).$$

Then, it is easy to check that $V : X \times \Delta^B$ satisfies (V1), (V2), and (V3) of the axiom for the directional structure, and φ has a locally fixed direction under (Δ^B, V). More strongly, the closed valued upper semicontinuity of φ means that for each $x \notin \varphi(x)$, there is an open neighbourhood $U(x)$ of x such that for all $z \in U(x)$, $\varphi(z)$ is a subset of $V(x,e)$ which is disjointed from $U(x)$. We call this situation as *closed valued upper demicontinuity of φ at x* in the generalized sense.[16] Though condition (V4) may not necessarily be satisfied, in this case, the fixed point argument on g^U which is based on the uniformity U and the mapping φ together with the limit argument for point x^* under the uniformity $\bar{V}$, show (V1)–(V3) to be sufficient for the existence of fixed points.

THEOREM 5 : (Upper Demicontinuous Extension of Kakutani's Theorem in Locally Convex Uniform Space) Let X be a non-empty compact Hausdorff locally convex uniform space with convex structure $\{f_A : A \in \mathscr{F}(X)\}$, and let W be a set having convex structure $\{g_A | A \in \mathscr{F}(W)\}$. If a non-empty valued correspondence, $\varphi : X \to 2^X$, is closed valued upper demicontinuous under the dual system structure, $(W, V : X \times W \to 2^X)$, then φ has a fixed point.

[16] The upper demicontinuity is a requirement for φ in a Hausdorff topological vector spaces such that if $\varphi(x)$ is contained in a open half space defined by a closed hyperplane H, then $\varphi(z)$ is also contained in the open half space for all z near x. (See Fan (1969).) If X is compact Hausdorff locally convex uniform space and if $\varphi : X \to X$ is closed valued, $x \notin \varphi(x)$ means that x and $\varphi(x)$ are separated by two open sets since X is normal. If $\varphi(x)$ is convex, such two open sets may also be taken as convex. Therefore, by considering the convex open set containing $\varphi(x)$ as the direction, $V(x,y)$, for each $y \in \varphi(y)$, the concept of upper demicontinuity for a closed valued mapping may be generalized as above.

PROOF : Define g^U, x^U for each vicinity U, and a limit point x^*, in, exactly, the same way as in the previous proof. (Note that the supposition $U \subset V^*$ and the definition of V^* are not essential for these definitions.) Assume that there is no fixed point of φ. Then, by considering the upper demicontinuity of φ at x^*, there is an open vicinity $\bar{U}$ satisfying condition (1), condition (2),

$$\begin{aligned} \forall z \in \bar{U}(x^*), \quad \varphi(z) &\subset V(x^*, w^*), \text{ and} \\ V(x^*, w^*) \cap \bar{U}(x^*) &= \emptyset, \end{aligned}$$

where w^* is an element of W satisfying the condition of upper demicontinuity. Take a symmetric open vicinity U_0 such that $U_0 \circ U_0 \subset \bar{U}$ and a symmetric open vicinity $\bar{V} \subset U_0$ such that $x^{\bar{V}} \in U_0(x^*) \subset \bar{U}(x^*)$, where, the point $x^{\bar{V}}$, (implicitly, together with points, $a_1, \ldots, a_k$, and $b_1, \ldots, b_k$, depending on $\bar{V}$), is taken as in the argument of the previous proof. (That is, $\{\bar{V}(a_1), \ldots, \bar{V}(a_k)\}$ covers X, and $b_1, \ldots, b_k$ are points of $\varphi(a_1), \ldots, \varphi(a_n)$, respectively.) Denote by $\beta^1 : X \to [0,1], \ldots, \beta^n : X \to [0,1]$ the partition of unity subordinated to $\{\bar{V}(a_1), \ldots, \bar{V}(a_k)\}$, and denote by B the finite set $\{b_1, \ldots, b_k\}$. Now, the point $x^{\bar{V}} = f_{\hat{B}}(e^{\bar{V}})$ satisfies

$$x^{\bar{V}} = f_{\hat{B}}(g^{\bar{V}}(e^{\bar{V}})) = f_{\hat{B}}(\sum_{i=1}^{k} \beta^i(x^{\bar{V}})e^i),$$

where e^i denotes the member of Δ^B such that the value of b_i is 1. It should be noted that $\beta^i(x^{\bar{V}}) > 0$ means that $x^{\bar{V}} \in \bar{V}(a_i)$, i.e., $a_i \in \bar{V}(x^{\bar{V}}) \subset U_0(x^{\bar{V}})$. Since $x^{\bar{V}} \in U_0(x^*)$, $\beta^i(x^{\bar{V}}) > 0$ means that $(x^*, a_i) = (x^*, x^{\bar{V}}) \circ (x^{\bar{V}}, a_i) \in U_0 \circ U_0 \subset \bar{U}$. Therefore, we have $a_i \in \bar{U}(x^*)$ and $b_i \in \varphi(a_i) \subset V(x^*, w^*)$ as long as $\beta^i(x^{\bar{V}}) > 0$. This means, however, that $x^{\bar{V}} = f_{\hat{B}}(\sum_{i=1}^{k} \beta^i(x^{\bar{V}})e^i)$ is an element of $\hat{V}(x^*, w^*)$ under (V1). Since $x^{\bar{V}} \in U_0(x^*) \subset \bar{U}(x^*)$, we also have $x^{\bar{V}} \in \bar{U}(x^*) \cap V(x^*, w^*)$, a contradiction. ■

The same argument may also be possible as long as the topological space, X, is approximated by a limit of open coverings and the mapping, φ, has a locally fixed direction for each member of sufficiently small coverings.[17]

We will end up this paper with a corollary to the theorem on the coincidence of two mappings. The result may be interpreted as the coincidence of *demand* and *supply* correspondences in the economic equilibrium theory, i.e., a sort of Gale-Nikaido-Debreu's lemma. Mathematically, the result may also be classified in a generalized form of the variational inequality problem under the generalized dual system structure.

COROLLARY 4 : (Gale-Nikaido-Debreu's Lemma under Generalized Dual System Structure) Let X be a compact Hausdorff space having convex structure $\{f_A | A \in \mathscr{F}(X)\}$ and topological dual system structure (W, V) on X. Assume that W is also a compact Hausdorff space having convex structure $\{g_A | A \in \mathscr{F}(W)\}$. Let $D : W \to 2^X$ and $S : W \to 2^X$ be two non-empty multi-valued mappings such that if $D(w) \cap S(w) = \emptyset$, then there are an open neighbourhood U^w of w and a point $\theta(w) \in W$ satisfying that for all $w' \in U^w$ and $s \in S(w')$,

$$D(w') \subset V(s, \theta(w)). \quad \text{(Generalized Continuity)}$$

Moreover, suppose that for all $w \in W$, $\exists s \in S(w)$,

$$D(w) \subset X \setminus V(s, w). \quad \text{(Weak Walras' Law)}$$

Then, there is at least one $w^* \in W$ such that $D(w^*) \cap S(w^*) \neq \emptyset$.

[17] Hence, we may relate our results to Čech and Vietoris Homology Theory. We will treat the problem in Urai et al. (2005).

PROOF : Assume the contrary, i.e., for all $w \in W$, $D(w) \cap S(w) = \emptyset$. Then, since W is compact, there are finite $w^1, \dots, w^n$ and $U^1 = U^{w^1}, \dots, U^n = U^{w^n}$ covering W satisfying the condition stated in the theorem. Let us consider the partition of unity subordinated to $U^1, \dots, U^n$, $\beta^1 : U^1 \to [0,1], \dots, \beta^n : U^n \to [0,1]$. Define a multi-valued mapping, φ, on W to itself as

$$\varphi : W \ni w \mapsto \{w' \in W | \forall s \in S(w), D(w) \subset V(s, w')\} \in 2^W.$$

Since $w \in U^t$ means that $\theta(w^t) \in \varphi(w)$, φ is a non-empty valued correspondence. It is convex valued by condition (V3) for V. It is also clear that for all $w \in W \setminus \mathcal{F}ix(\varphi)$, there are a point $y^w \in \varphi(w)$ and an open neighbourhood U^w of w such that for all $z \in U^w \setminus \mathcal{F}ix(\varphi)$, we have $y^w \in \varphi(z)$. (Indeed, if $w \in U^t$, let y^w be the element $\theta(w^t)$ and U^w be U^t.) Therefore, φ is a mapping satisfying the condition in Theorem 2. Let w^* be a fixed point of φ. Then, we have $\forall s \in S(w^*), D(w^*) \subset V(s, w^*)$, which contradicts to the Walras' Law. ■

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