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# Cross-Dual on The Golden Optimum Solutions

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**Abstract**

This paper owes its origin to two simple minimization problems. One is a shortest distance problem on the plane. The other is a ratio minimization problem over the unit interval.

We associates each of two minimization problems with a counterpart "maximization problem". Thus we consider two couples of (minimization and maximization) problems. Further we associate the two couples with a third and common *cross-dual* couple. Finally we have *two pairs* between three couples. As a total we have six optimization problems, each of which is to optimize one two-variable quadratic objective function under another quadratic constraint. An optimum solution — optimum point and optimum value — is called Golden if both the slope and the optimum value constitute the Golden ratio. We show two interesting features. One is the Golden optimum solution. All six problems have the Golden optimum slutions. The other is a cross-duality. The first pair has a cross 2-sum property. The second has a cross inverse property. We illustrate a generative one-variable curve. Finally we show that the curve generates a couple of two-variable optimization problems.

## 1 Introduction

It is well known that one of the most beautiful rectangles is the Golden rectangle [9]. This paper considers a class of optimization problems whose optimal solution constitute the Golden rectangle.

We begin with two simple minimization problems. One is a shortest distance problem on the plane $R^2$. It is stated as follows:

$$(\mathrm{O}_1) \qquad \begin{array}{lll} \text{minimize} & \sqrt{x^2+y^2} & \\ \text{subject to} & \text{(i)} & y = x + \sqrt{4-x^2} \\ & \text{(ii)} & -\infty < x, y < \infty. \end{array}$$

The other is a ratio minimization problem over the unit interval $[0, 1)$ :

$$(\mathrm{O}_2) \quad \text{minimize} \quad \frac{u^2 + (1-u)^2}{1-u^2} \quad \text{subject to} \quad \text{(i)} \ \ 0 \le u < 1.$$

We consider two couples of maximization problem and minimization problem. One couple has roots in the shortest distance problem. The other comes from the ratio minimization problem. Further we associate the two couples with a third and common *cross-dual* couple. Finally we have *two pairs* between three couples. As a total we have *six* optimization problems, each of which is to optimize one *two-variable* quadratic objective function under another quadratic constraint [1].

An optimum solution — optimum point $(\tilde{x}, \tilde{y})$ and optimum value $M$ — is called *Golden* if the pair of the slope and the optimum value $\left(\frac{\tilde{y}}{\tilde{x}}, M\right)$ constitutes the Golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$.

We show two interesting features. One is the *Golden optimum solution.* All six problems have the Golden optimum slutions. The other is the *cross-duality.* The first pair has a *cross 2-sum* property. The second has a *cross inverse* property. Further, we illustrate a pivotal one-variable curve which has orthogonal golden optimum points. We show that the curve generates a couple of two-variable minimization problem and maximization problem.

## 2 The Golden Ratio

Throughout this paper we take a basic standard real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

The number $\phi$ is called the *Golden ratio.* $\phi$ is defined as the positive solution of quadratic equation

$$x^2 - x - 1 = 0.$$

A Fibonacci sequence $\{a_n\}$ is defined by second-order linear difference equation

$$a_{n+2} - a_{n+1} - a_n = 0.$$

Then we have a famous relation.

**Lemma 2.1**

$$\begin{aligned} \phi^n &= a_n\phi + a_{n-1} \\ (\phi - 1)^n &= a_{-n}\phi + a_{-n-1} \end{aligned} \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

*where $\{a_n\}$ is the Fibonacci sequence with $a_0 = 0, \ a_1 = 1$.*

The Fibonacci sequence is tabulated in Table 1:

| $n$ | $-10$ | $-9$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | $-55$ | $34$ | $-21$ | $13$ | $-8$ | $5$ | $-3$ | $2$ | $-1$ | $1$ | $0$ | $1$ |

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |

Table 1 Fibonacci sequence $\{a_n\}$

On the other hand, the Fibonacci sequence has the analytic form

**Lemma 2.2**

$$a_n = \frac{1}{2\phi - 1}\{\phi^n - (1-\phi)^n\} \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

We remark that

$$\phi + (1-\phi) = 1, \qquad \phi(1-\phi) = -1. \tag{1}$$

Thus $1 - \phi$ is *conjugate* to $\phi$ and vice verse.

**Lemma 2.3** *It holds that for any real values $a$, $b$*

$$(a + b\phi)(a + b - b\phi) = a^2 + ab - b^2.$$

Thus the pair of two numbers $a + b\phi$ and $a + b(1-\phi)$ is called each other *conjugate.* We have for any nontrivial pair of real values $(a, b)$

$$\begin{aligned}\frac{1}{a+b\phi} &= \frac{a + b(1-\phi)}{(a+b\phi)(a+b(1-\phi))} \\ &= \frac{a+b}{a^2+ab-b^2} - \frac{b}{a^2+ab-b^2}\phi.\end{aligned}$$

For instance we have a list of linear expressions for fractional forms of two linear forms in $\phi$ as follows :

$$\begin{aligned}\frac{1}{\phi} &= -1 + \phi = \frac{-1+\sqrt{5}}{2} \approx 0.61803 \\ \frac{1}{1+\phi} &= 2 - \phi = \frac{3-\sqrt{5}}{2} \approx 0.38197 \\ \frac{1}{2+\phi} &= \frac{1}{5}(3-\phi) = \frac{5-\sqrt{5}}{10} \approx 0.27639\end{aligned}$$

and

$$
\begin{aligned}
\frac{-7+4\phi}{4-3\phi} &= \frac{-4+3\phi}{3-\phi} = \frac{-1+2\phi}{2+\phi} = -1+\phi \approx 0.61803 \\
\frac{3-\phi}{2+\phi} &= \frac{2+\phi}{3+4\phi} = 2-\phi \approx 0.38197 \\
\frac{1+3\phi}{2+\phi} &= \frac{3+4\phi}{1+3\phi} = \phi \approx 1.61803 \\
\frac{2+\phi}{3-\phi} &= \frac{3+4\phi}{2+\phi} = 1+\phi = \frac{3+\sqrt{5}}{2} \approx 2.61803.
\end{aligned}
$$

Further the Golden ratio is expressed in terms of the Fibonacci sequence as follows.

$$
\cdots = \frac{\phi-1}{2-\phi} = \frac{1}{\phi-1} = \frac{\phi}{1} = \frac{1+\phi}{\phi} = \frac{1+2\phi}{1+\phi} = \cdots = \frac{a_n + a_{n+1}\phi}{a_{n-1} + a_n\phi}
$$
$$
n = \cdots, -2, -1, 0, 1, 2, \cdots
$$

where $\{a_n\}$ is the Fibonacci sequence with $a_0 = 0,\ a_1 = 1$ (Table 1).

**Lemma 2.4** *The following three equations have a unique common solution $x = \phi$ :*

$$
\begin{aligned}
\sqrt{x} + \sqrt{x-1} &= \sqrt{2x+1} \\
\sqrt{x} - \sqrt{x-1} &= \sqrt{2x-3} \qquad (2) \\
x\sqrt{x-1} &= \sqrt{x}.
\end{aligned}
$$

## 3 A Minimum Distance Problem

Let us consider the problem of finding minimum distance from the origin to the graph of

$$
y = x + \sqrt{4-x^2}. \qquad (3)
$$

This is stated as follows:

$$
(\mathrm{O}_1) \qquad \begin{array}{lll} \text{minimize} & \sqrt{x^2+y^2} & \\ \text{subject to} & \text{(i)} & y = x + \sqrt{4-x^2} \\ & \text{(ii)} & -\infty < x, y < \infty. \end{array}
$$

An equivalent criterion is the square of distance $x^2 + y^2$. The graph (3) is the upper part $(y \geq x)$ of the quadraric curve (ellipse)

$$
2x^2 - 2xy + y^2 = 2^2
$$

which is equivalent, from the viewpoint of optimization, to the ellipse with unit radius

$$x^2 + (x - y)^2 = 1^2.$$

Thus we have the following quadratic minimization problem

$$(\mathrm{m}_1) \qquad \begin{array}{ll} \text{minimize} & x^2 + y^2 \\ \text{subject to} & \text{(i)} \ \ x^2 + (x-y)^2 = 1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty. \end{array}$$

We also consider the corresponding maximum distance problem, which is equivalently stated by the quadratic maximization problem

$$(\mathrm{m}_2) \qquad \begin{array}{ll} \text{Maximize} & x^2 + y^2 \\ \text{subject to} & \text{(i)} \ \ x^2 + (x-y)^2 = 1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty. \end{array}$$

Thus we have a couple $(\mathrm{M}_1)$ of maximum problem $(\mathrm{m}_2)$ and minimum problem $(\mathrm{m}_1)$ .

$$(\mathrm{M}_1) = (\mathrm{m}_1, \mathrm{m}_2).$$

We write this couple as follows.

$$(\mathrm{M}_1) \qquad \begin{array}{ll} \text{Maximize and minimize} & x^2 + y^2 \\ \text{subject to} & \text{(i)} \ \ x^2 + (x-y)^2 = 1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty. \end{array}$$

Through the rotation $\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix}$ ; $T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\tan\alpha = \phi$, $\alpha \approx 58.3°$, we have

$$\begin{aligned} x^2 + (x-y)^2 &= 2x^2 - 2xy + y^2 \\ &= (2-\phi)X^2 + (1+\phi)Y^2 \\ &= (-1+\phi)^2 X^2 + \phi^2 Y^2 \\ &= \frac{X^2}{\phi^2} + \frac{Y^2}{(-1+\phi)^2}. \end{aligned}$$

It is easily shown that Couple $(\mathrm{M}_1)$ has the maximum value $M = 1 + \phi$ at the point

$$(x^*, y^*) = \pm \frac{1}{\sqrt{3-\phi}}(1, \phi)$$

and the minimum value $m = 2 - \phi$ at the point

$$(\hat{x}, \hat{y}) = \pm \frac{1}{\sqrt{2+\phi}}(1, 1-\phi).$$

## 3.1 Circle inscribes and circumscribes Ellipse

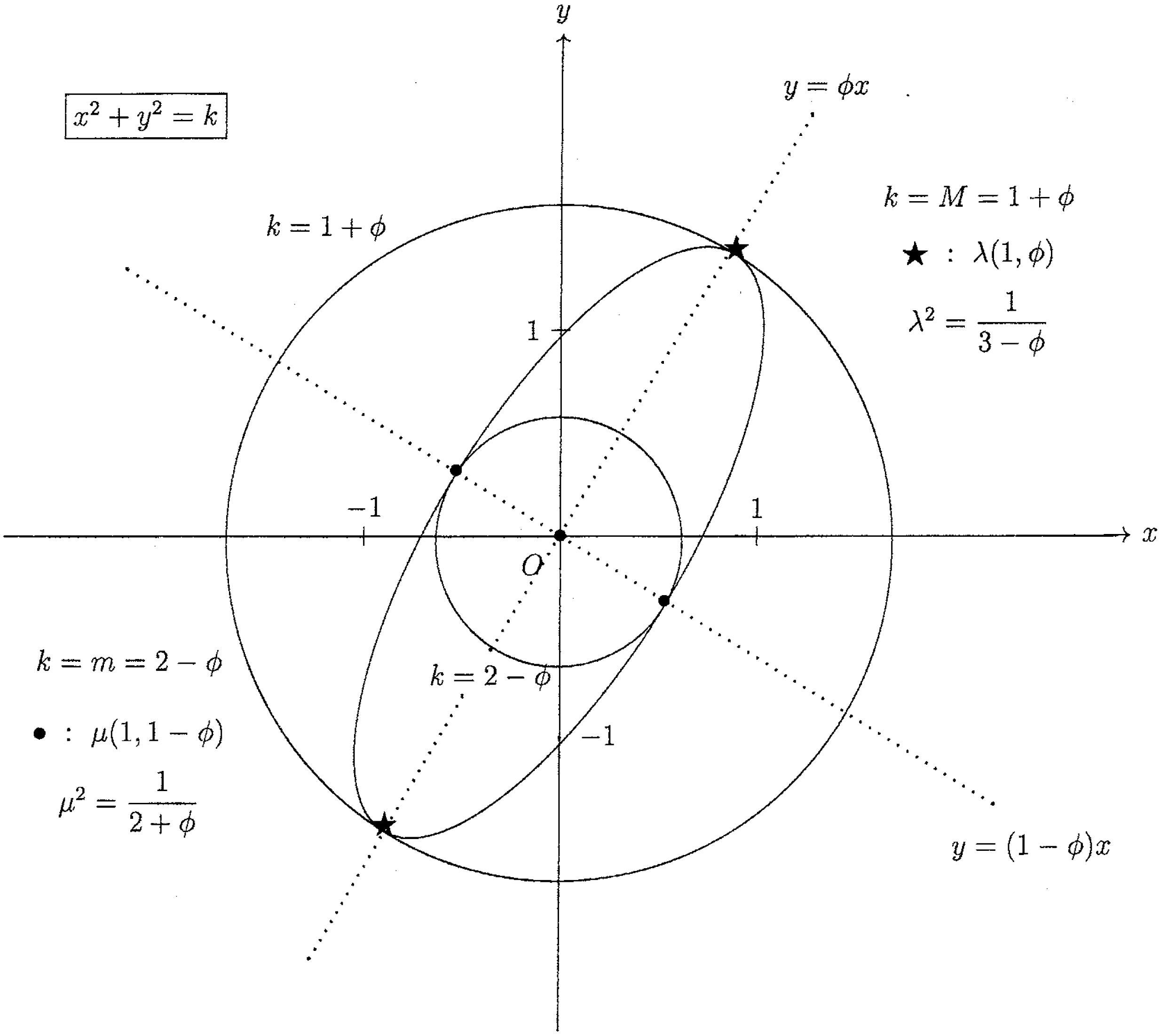


Fig.1 Ellipse $x^2 + (y - x)^2 = 1$ has golden optimum points
— the longest points ★ and the shortest points • —

## 3.2 A Cross-Dual Couple

As a kind of dual, we associate the couple $(M_1)$ with a cross-dual couple as follows.

$$
(\mathrm{R}) \qquad \begin{array}{ll} \text{Maximize and minimize} & -x^2+y^2 \\ \text{subject to} & \text{(i)}\ \ x^2+(x-y)^2=1 \\ & \text{(ii)}\ \ -\infty < x, y < \infty. \end{array}
$$

Thus we have a pair of couple $(M_1)$ and couple (R).

Then we see that Couple (R) has the maximum value $M = \phi$ at the point

$$
(x^*, y^*) = \pm\frac{1}{\sqrt{2+\phi}}(1, 1+\phi)
$$

and the minimum value $m = 1-\phi$ at the point

$$
(\hat{x}, \hat{y}) = \pm\frac{1}{\sqrt{3-\phi}}(1, 2-\phi)
$$

(Table 2).

| Solution/Couple | Main $(M_1)$ | Cross-Dual (R) |
|---|---|---|
| Maximization | Max $x^2+y^2$<br>s.t. $x^2+(x-y)^2=1$<br>$-\infty < x, y < \infty$ | Max $-x^2+y^2$<br>s.t. $x^2+(x-y)^2=1$<br>$-\infty < x, y < \infty$ |
| maximum value $M$<br>*slope* $a^*$ | $1+\phi$<br>$\phi$ | $\phi$<br>$1+\phi$ |
| minimization | min $x^2+y^2$<br>s.t. $x^2+(x-y)^2=1$<br>$-\infty < x, y < \infty$ | min $-x^2+y^2$<br>s.t. $x^2+(x-y)^2=1$<br>$-\infty < x, y < \infty$ |
| minimum value $m$<br>*slope* $\hat{a}$ | $2-\phi$<br>$1-\phi$ | $1-\phi$<br>$2-\phi$ |

Table 2 Main $(M_1)$ and Cross-Dual (R) have the Golden optimum solutions

### 3.3 Hyperbola tangent to Ellipse

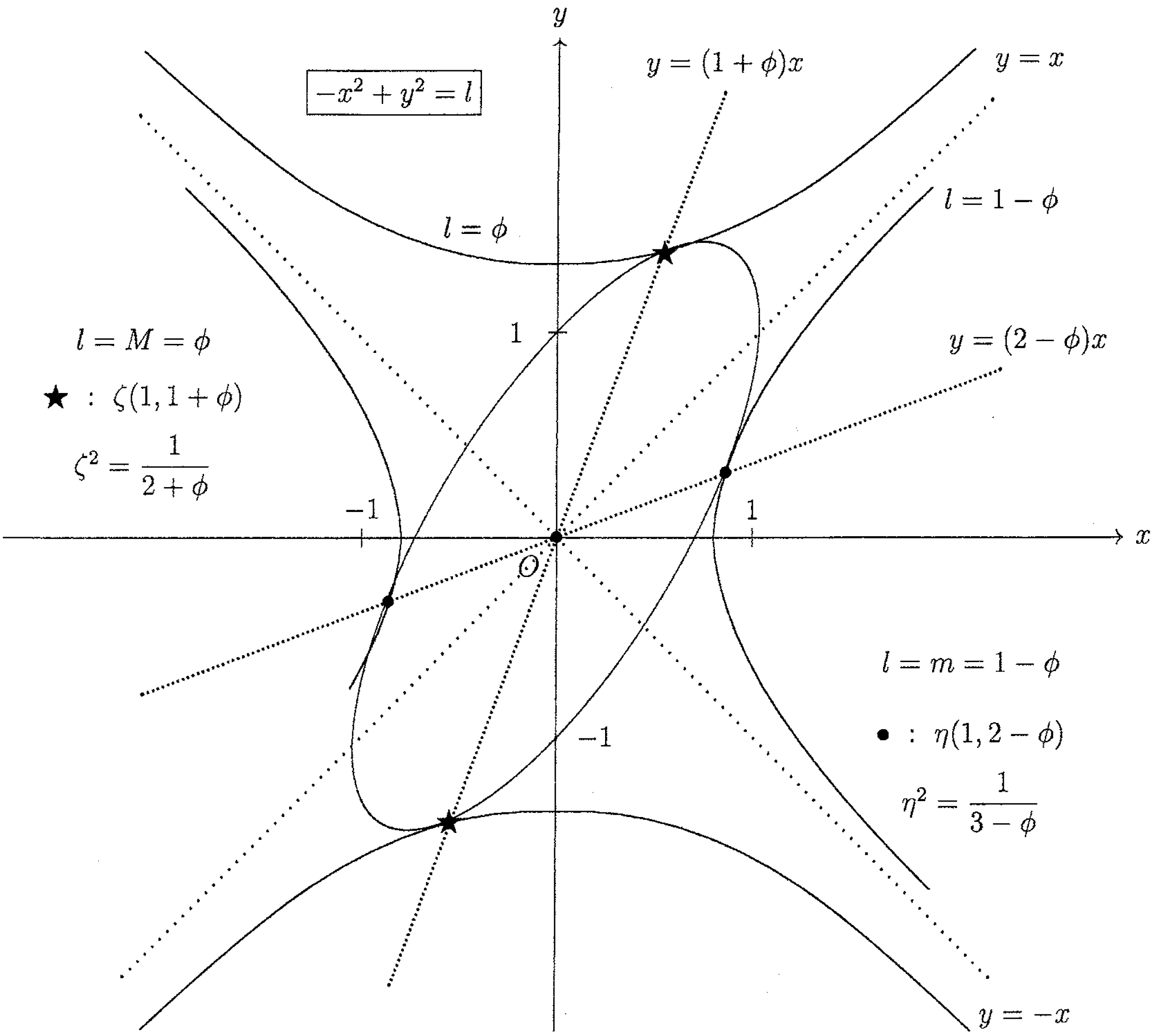


Fig.2 Ellipse $x^2 + (y-x)^2 = 1$ has golden optimum points

## 3.4 The Golden Optimum and Cross Two-Sum

Our problem is to derive what happens between the paired couples.

| Couple / Objective / Optimization | | Main $(\mathrm{M}_1)$ $x^2+y^2$ | | Cross-Dual $(\mathrm{R})$ $-x^2+y^2$ |
|---|---|---|---|---|
| Max | Value | $1+\phi$ | Cross | $\phi$ |
| | Slope | $\phi$ | | $1+\phi$ |
| | | | Two Sum | |
| min | value | $2-\phi$ | Cross | $1-\phi$ |
| | slope | $1-\phi$ | | $2-\phi$ |
| Constraint | | $x^2+(y-x)^2=1$ | | |

Table 3 The Golden optimum solutions have cross two-sum property

**Theorem 3.1** (Cross Two-Sum Theorem)
*Main Couple* $(\mathrm{M}_1)$ *has the maximum value* $M = 1+\phi$ *at the point* $(x^*, y^*) = \lambda(1, \phi)$ *and the minimum value* $m = 2-\phi$ *at the point* $(\hat{x}, \hat{y}) = \mu(1, 1-\phi)$ *if and only if Cross-Dual Couple* $(\mathrm{R})$ *has the maximum value* $M = \phi$ *at the point* $(x^*, y^*) = \zeta(1, 1+\phi)$ *and the minimum value* $m = 1-\phi$ *at the point* $(\hat{x}, \hat{y}) = \eta(1, 2-\phi)$.

Here the cross two-sum means that

$$(1+\phi)+(1-\phi) = 2, \qquad (2-\phi)+\phi = 2$$
$$\phi+(2-\phi) = 2, \qquad (1-\phi)+(1+\phi) = 2.$$

## 3.5 The Golden Triplet

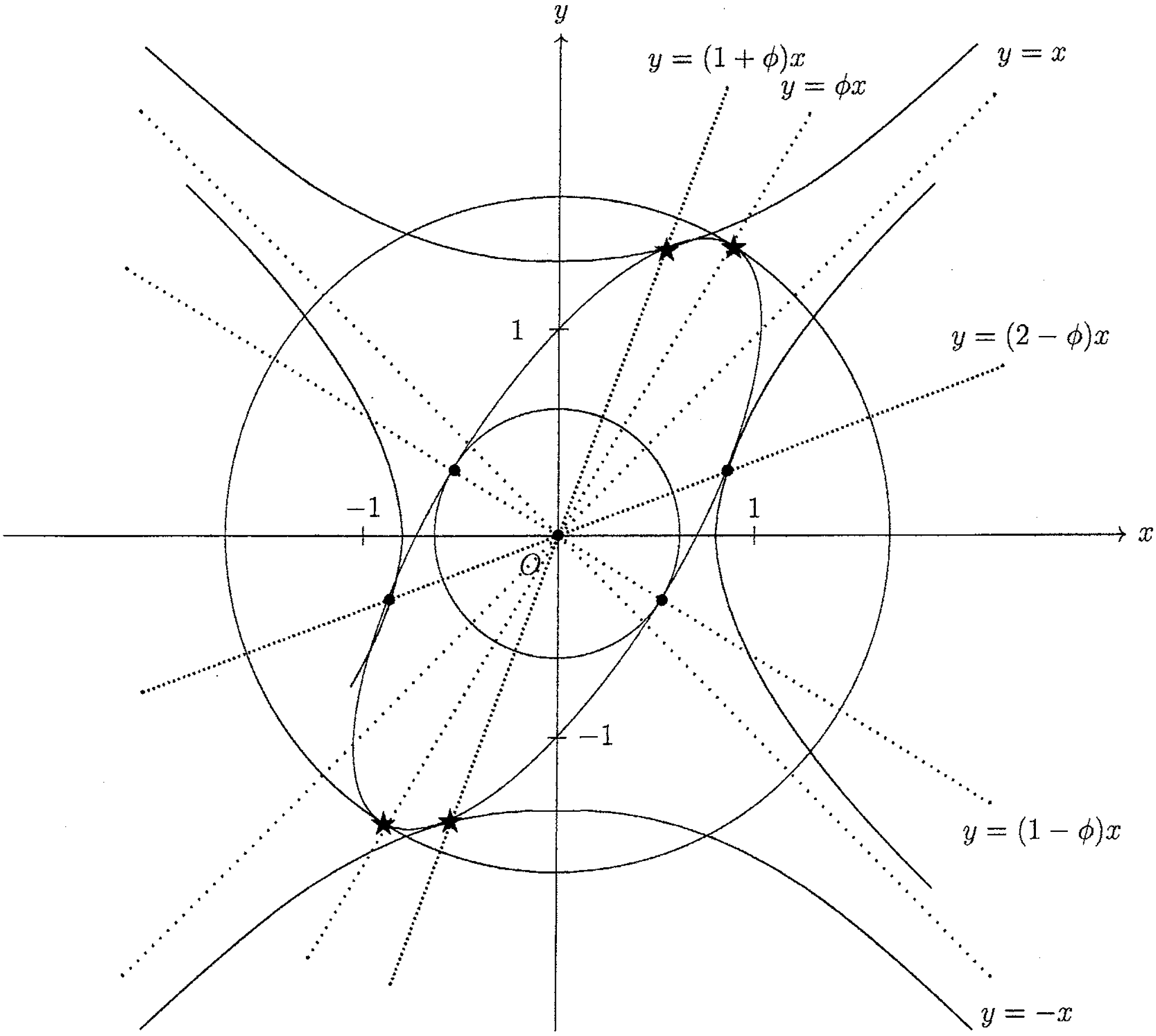


Fig.3 Triplet $x^2 + y^2$, $x^2 + (y-x)^2$, $x^2 - y^2$ yields the Golden optimum points

# 4 A Minimum Ratio Problem

Let us consider the following minimization problem over the unit interval:

$$(\mathrm{O}_2) \quad \text{minimize} \quad \frac{u^2+(1-u)^2}{1-u^2} \quad \text{subject to} \quad \text{(i)} \ \ 0 \le u < 1.$$

This comes from an abstract *optimality equation* in *non-deterministic dynamic programming* [5,7]:

$$v(x) \;=\; \min_{u \in U(x)} \left[ r(x,u) + \lambda \int_{T(x,u)} \beta(x,u,y) v(y) dy \right] \qquad x \in X.$$

The case

$$X = (0,\infty), \quad T(x,u) = [0,u], \quad U(x) = (0,x)$$
$$\lambda = \alpha, \quad r(x,u) = cu^2 + d(x-u)^2, \quad \beta(x,u,y) = \frac{2}{y}$$

yields the controlled integral equation

$$v(x) \;=\; \min_{0<u<x} \left[ cu^2 + d(x-u)^2 + \int_0^u \frac{2\alpha v(y)}{y} dy \right] \qquad x > 0$$

where

$$\alpha,\ c,\ d > 0.$$

In particular, we take

$$\alpha = c = d = 1.$$

Then we have

$$v(x) \;=\; \min_{0<u<x} \left[ u^2 + (x-u)^2 + 2\int_0^u \frac{v(y)}{y} dy \right] \qquad x > 0.$$

Let us now consider a *proportional* policy $\pi = \{u_0, u_1, \ldots, u_n, \ldots\}$ with $u_n(x) \equiv u(x) = ux$, where $0 \le u < 1$ is called a *proportinal rate*. The proportional policy $\pi$ is identified with a real constant $u$ in interval $[0,1)$. Let the proportional policy $\pi$ with rate $u$ yield the corresponding quadratic *minimum value function* $v(x) = vx^2$. Then we have

$$vx^2 \;=\; \min_{0 \le u < 1} \left[ \{u^2 + (1-u)^2\}x^2 + 2\int_0^{ux} \frac{vy^2}{y} dy \right].$$

Calculating the integral part and dividing both sides by $x^2$, we get

$$v \;=\; \min_{0 \le u < 1} \left[ u^2 + (1-u)^2 + u^2 v \right].$$

Thus we have the equality

$$v \;=\; \min_{0 \le u < 1} \frac{u^2 + (1-u)^2}{1-u^2}.$$

Therefore we have the original problem ($O_2$) at the right hand.

Now we minimize this ratio over the open inteval $(-1, 1)$. This is stated as follows.

$$(\mathrm{O}_3) \quad \text{minimize} \quad \frac{u^2 + (1-u)^2}{1-u^2} \quad \text{subject to} \quad \text{(i)} \quad -1 < u < 1.$$

Letting $u = \dfrac{y}{x}$ we have the following equivalnt two-variable minimization problem (see Section 6).

$$(\mathrm{m}_3) \qquad \begin{array}{ll} \text{minimize} & y^2 + (x-y)^2 \\ \text{subject to} & \text{(i)} \ \ x^2 - y^2 = 1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty. \end{array}$$

We associates this minimum problem a maximum problem as follows (see Section 6).

$$(\mathrm{m}_4) \qquad \begin{array}{ll} \text{Maximize} & -y^2 - (x-y)^2 \\ \text{subject to} & \text{(i)} \ \ x^2 - y^2 = -1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty. \end{array}$$

Thus we have a couple ($M_2$) of maximum problem ($m_4$) and minimum problem ($m_3$):

$$(\mathrm{M}_2) = (\mathrm{m}_3, \mathrm{m}_4)$$

## 4.1 A Cross-Dual Couple

We also associate the couple $(\mathrm{M}_2) = (\mathrm{m}_3, \mathrm{m}_4)$ with the cross-dual couple (R):

$$(\mathrm{R}) \qquad \begin{array}{ll} \text{Maximize and minimize} & -x^2 + y^2 \\ \text{subject to} & \text{(i)} \ \ x^2 + (x-y)^2 = 1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty. \end{array}$$

Thus we have in turn a pair of couple ($M_2$) and couple (R).

The Couple $(\mathrm{M}_2) = (\mathrm{m}_3, \mathrm{m}_4)$ has the following maximum solution and minimum solution: The Maximum Problem ($m_4$)

$$(\mathrm{m}_4) \qquad \begin{array}{ll} \text{Maximize} & -y^2 - (x-y)^2 \\ \text{subject to} & \text{(i)} \ \ x^2 - y^2 = -1 \\ & \text{(ii)} \ \ -\infty < x, y < \infty \end{array}$$

has the maximum value $M = -\phi$ at the point

$$(x^*, y^*) = \pm \frac{1}{\sqrt{1+3\phi}}(1, 1+\phi).$$

The Minimum Problem ($m_3$)

$$(m_3) \qquad \begin{array}{ll} \text{minimize} & y^2 + (x-y)^2 \\ \text{subject to} & \text{(i)} \ x^2 - y^2 = 1 \\ & \text{(ii)} \ -\infty < x, y < \infty \end{array}$$

has the the minimum value $m = -1 + \phi$ at the point

$$(\hat{x}, \hat{y}) = \pm \frac{1}{\sqrt{-4+3\phi}}(1, 2-\phi).$$

As we have stated, Couple (R) has the maximum value $M = \phi$ at the point

$$(x^*, y^*) = \pm \frac{1}{\sqrt{2+\phi}}(1, 1+\phi)$$

and the minimum value $m = 1 - \phi$ at the point

$$(\hat{x}, \hat{y}) = \pm \frac{1}{\sqrt{3-\phi}}(1, 2-\phi).$$

We see that Maximum Problem ($m_4$) has the maximum value $M = -\phi$ at the slope $\frac{y^*}{x^*} = 1 + \phi$ and that Minimum Problem ($m_3$) has the minimum value $m = -1+\phi$ at the slope $\frac{\hat{y}}{\hat{x}} = 2 - \phi$. Thus Main Couple ($M_2$) has the dual Golden optimum solutions.

Our next problem is to derive what happens between the pair of couple ($M_2$) and couple (R).

**Theorem 4.1** (Cross Inverse Theorem)
*Main Couple* ($M_2$) *has the maximum value* $M = -\phi$ *at the point* $(x^*, y^*) = \lambda(1, 1+\phi)$ *and the minimum value* $m = -1+\phi$ *at the point* $(\hat{x}, \hat{y}) = \mu(1, 2-\phi)$ *if and only if Cross-Dual Couple* (R) *has the maximum value* $M = \phi$ *at the point* $(x^*, y^*) = \zeta(1, 1+\phi)$ *and the minimum value* $m = 1 - \phi$ *at the point* $(\hat{x}, \hat{y}) = \eta(1, 2-\phi)$.

Here the cross inverse says that

$$\frac{1}{-\phi} = 1 - \phi, \qquad \frac{1}{-1+\phi} = \phi$$
$$\frac{1}{1+\phi} = 2 - \phi, \qquad \frac{1}{2-\phi} = 1 + \phi.$$

Both the Golden optimum solutions and the cross inverse relation are shown in Table 4.

## 4.2 The Golden Optimum and Cross Inverse

| | | Main ($M_2$) | | Cross-Dual (R) |
|---|---|---|---|---|
| Max | Problem | Max $-y^2-(y-x)^2$<br>s.t. $x^2-y^2=-1$<br>$-\infty < x, y < \infty$ | | Max $-x^2+y^2$<br>s.t. $x^2+(x-y)^2=1$<br>$-\infty < x, y < \infty$ |
| | Value | $-\phi$ | minus $\longleftrightarrow$ | $\phi$ |
| | Slope | $1+\phi$ | $\longleftrightarrow$ equal | $1+\phi$ |
| | | | Inverse | |
| min | value | $-1+\phi$ | minus $\longleftrightarrow$ | $1-\phi$ |
| | slope | $2-\phi$ | $\longleftrightarrow$ equal | $2-\phi$ |
| | problem | min $y^2+(y-x)^2$<br>s.t. $x^2-y^2=1$<br>$-\infty < x, y < \infty$ | | min $-x^2+y^2$<br>s.t. $x^2+(x-y)^2=1$<br>$-\infty < x, y < \infty$ |

Table 4 The Golden optimum solutions have cross inverse poroperty

# 5 A generative curve

Let us now describe an illustrative one-variable graph. Then we will appreciate the Golden local-optimum there (Fig.4).

## 5.1 $x = f(u)$

$$f(u) = \frac{u^2 + (1-u)^2}{1-u^2}$$

$$f'(u) = (-2)\frac{u^2 - 3u + 1}{(u^2-1)^2}$$

$$f''(u) = 2\frac{2u^3 - 9u^2 + 6u - 3}{(u^2-1)^3}$$

The golden ratio

$$\frac{1+\phi}{-(-\phi)} = \frac{-1+\phi}{2-\phi} = \frac{\phi}{1}$$

$$\frac{-(-\phi)}{1} = \frac{1}{-1+\phi} = \frac{\phi}{1}$$

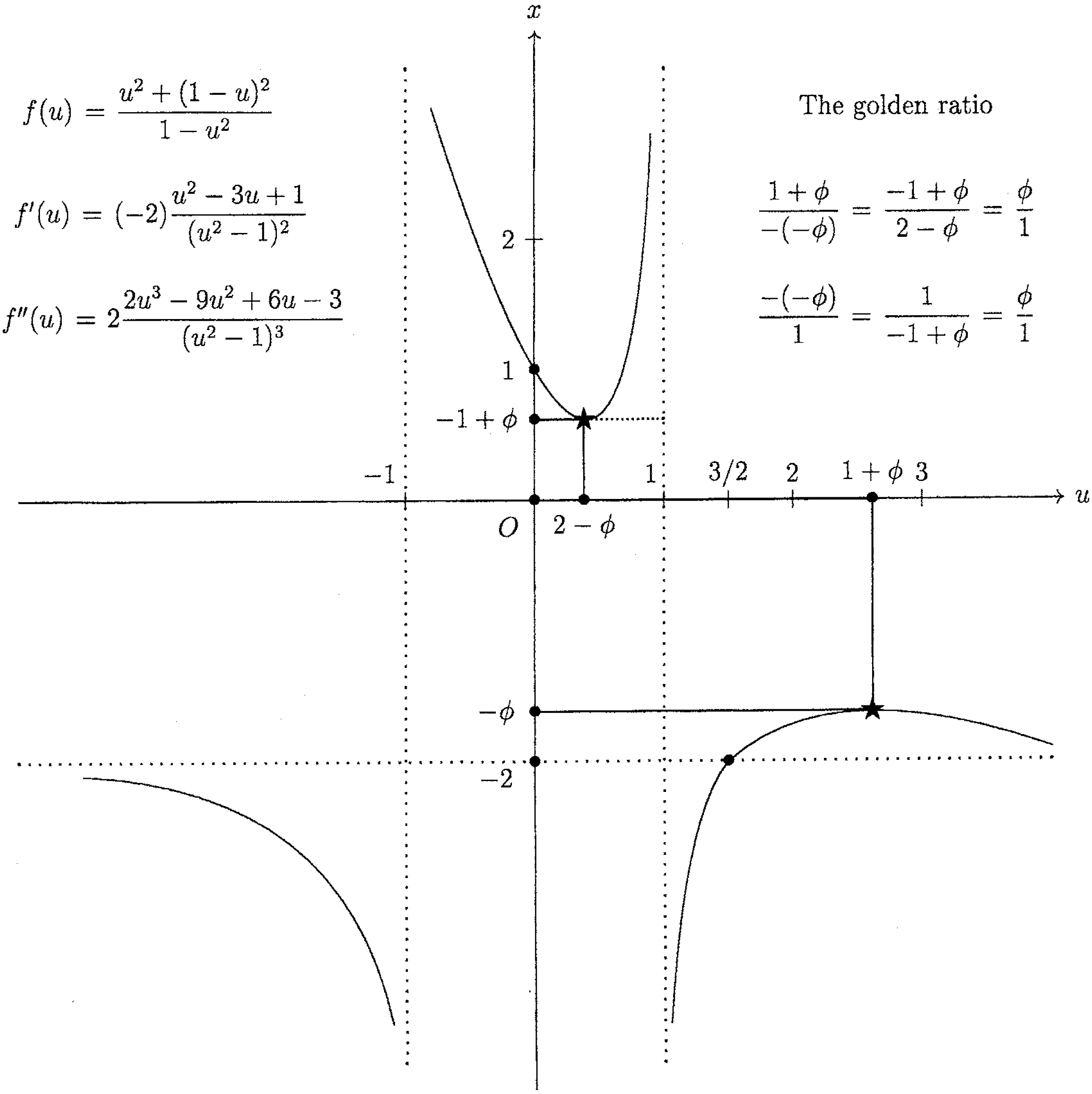


Fig.4 Curve $x = f(u)$ has dual golden extremum points ★

We have the equality

$$\begin{aligned} f(u) &< -2 \quad \text{on} \quad (-\infty, -1) \\ f(u) &\geq -1+\phi \quad \text{on} \quad (-1, 1) \\ f(u) &\leq -\phi \quad \text{on} \quad (1, \infty). \end{aligned}$$

The first equality attains iff $\hat{u} = 2 - \phi$, and the second equality attains iff $u^* = 1 + \phi$.

## 5.2 Two-variable optimization problems

Let us now consider how the one-variable function

$$x = f(u) = \frac{u^2 + (1-u)^2}{1-u^2}$$

generates two-variable optimization problems $(\mathrm{m}_3)$, $(\mathrm{m}_4)$ which constitute the preceding couple $(\mathrm{M}_2)$.

Generally speaking, the following three techniques preserve equivalence as optimization problem.

1. A strictly monotone transformation between criteria keeps the optimum point invariant.
2. One maximization leads to the other minimization [2] (Inverse Theorem, Reverse Theorem and Duality Theorem [3–6], Principle of Reciprocity [8])
3. Under homogeneity, Constraint $g(x, y) = c$ may be replaced with Constraint $g(x, y) = 1$.

We separate the optimization of $f(u)$ over $R^1$ into minimization on the open interval $(-1, 1)$ and maximization on its complement $(-\infty, -1) \cup (1, \infty)$ as follows.

$$(\mathrm{O}_3) \quad \text{minimize} \quad \frac{u^2 + (1-u)^2}{1-u^2} \quad \text{subject to} \quad \text{(i)} \ |u| < 1$$

$$\Longleftrightarrow \quad \min \quad \frac{\left(\frac{y}{x}\right)^2 + \left(1 - \frac{y}{x}\right)^2}{1 - \left(\frac{y}{x}\right)^2} \quad \text{s.t.} \quad \text{(i)} \ \left|\frac{y}{x}\right| < 1$$

$$\Longleftrightarrow \quad \min \quad \frac{y^2 + (y-x)^2}{x^2 - y^2} \quad \text{s.t.} \quad \text{(i)} \ y^2 < x^2$$

$$\Longleftrightarrow \quad (\mathrm{m}_3) \quad \min \quad y^2 + (y-x)^2 \quad \text{s.t.} \quad \text{(i)}' \ x^2 - y^2 = 1,$$

$$
\begin{array}{llll}
(\mathrm{O})_4 & \text{Maximize} \quad \dfrac{u^2+(1-u)^2}{1-u^2} & \text{subject to} & \text{(i)} \ |u|>1 \\
\Longleftrightarrow & \text{Max} \quad \dfrac{\left(\frac{y}{x}\right)^2+\left(1-\frac{y}{x}\right)^2}{1-\left(\frac{y}{x}\right)^2} & \text{s.t.} & \text{(i)} \ \left|\dfrac{y}{x}\right|>1 \\
\Longleftrightarrow & \text{Max} \quad \dfrac{-y^2-(y-x)^2}{-x^2+y^2} & \text{s.t.} & \text{(i)} \ x^2<y^2 \\
\Longleftrightarrow \ (\mathrm{m}_4) & \text{Max} \quad -y^2-(y-x)^2 & \text{s.t.} & \text{(i)}' \ -x^2+y^2=1
\end{array}
$$

where $\Longleftrightarrow$ means equivalence between optimization problems.

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