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Unstable Periodic Orbits and Chaotic Transitions in a Macro-Economic Model

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Abstract

We analyze a chaotic growth cycle model which represents essential aspects of macro-economic phenomena. Unstable periodic solutions detected from a chaotic attractor of the model are categorized into some hierarchical classes, and relationships between each class of them and characteristics of the attractor are discussed. This approach may be useful to clarify economic laws hidden behind complicated phenomena.

1 Introduction

Little attention has been payed to the point that business cycle models have unstable periodic solutions.[1] Unstable periodic solutions were not generally thought to be solutions of importance in nonlinear dynamics. They were often ignored.

[1]An unstable periodic orbit of a continuous dynamical system is a periodic orbit with at least one eigenvalue whose modulus evaluated in a section vertical to the orbit is greater than unity.

However, studies of unstable periodic solutions to understand complicated chaotic phenomena have received increasing attention in recent years. For example, an unstable periodic solution of Navier-Stokes equation found by Kawahara and Kida (2001) exhibits a coherent structure of wall turbulence obviously. Ishiyama and Saiki (2005a) detected unstable periodic orbits embedded in a chaotic attractor of a generalized Goodwin model[2] and then pointed out a close relation between business cycle models and unstable periodic solutions. In this paper we discuss relationships between a chaotic attractor and unstable periodic solutions in the attractor more deeply than Ishiyama and Saiki (2005a,b) did.

The next section gives the Keynes-Goodwin model already considered in Ishiyama and Saiki (2005a,b), and shows a chaotic attractor of the model for a set of parameters. In section 3 we attempt to understand characteristics of the attractor through classifying unstable periodic orbits shown in the previous papers and much more orbits newly detected. The final section concludes our results.

2 Chaotic behavior of a growth cycle model

Ishiyama and Saiki (2005a,b) pointed out the usefulness of focusing on unstable periodic solutions embedded in the chaotic attractor in case of studying chaotic behavior of a model. Their discussions started from constructing a two-country growth cycle model described below.

$$\frac{du_i}{dt} = 0.5(\frac{0.1}{1-v_i} - 0.48 + \pi_i^e - \alpha)u_i, \tag{1}$$

$$\frac{dv_i}{dt} = (0.1(h_i + 0.7\mu_i(v^* - v_i) - 0.7(1-\delta)(1-u_i)) - (\alpha+\beta))v_i, \tag{2}$$

$$\frac{d\pi_i^e}{dt} = 0.4(\frac{0.1}{1-v_i} - 0.48 - \pi_i^e - \alpha), \tag{3}$$

[2]Goodwin models are continuous dynamical systems to describe economic growth and business cycles generated by class struggle between capitalists and workers. Wolfstetter (1982) first generalized the original model (Goodwin (1967)) to a model with dissipative structure by introducing a government taking a Keynesian fiscal policy. Ishiyama and Saiki (2005a,b) studied dynamical properties of a two-country Keynes-Goodwin model with interactions among capitalists, workers and governments in particular in case capital mobility between countries caused chaotic phenomena.

$$h_i = 1.5(1-u_i)^5 + 3.5(u_j - u_i)^3, \tag{4}$$

$i, j = 1, 2$ $(i \neq j)$,
where h_i as a function of u_1 and u_2 determines the effect of capital share rates in both country on the investment demand in country i. The model's main variables u_i, v_i and π_i^e $(i = 1, 2)$ denote the i-th country's labor share rate, employment ratio and expected inflation respectively. Parameters α, β and δ are the rate of technical progress, the population growth rate, and the income tax rate respectively. Parameters of fiscal policy, μ_1 and μ_2 are different. The relation $\mu_2 > \mu_1 > 0$ means the government of country 2 takes more positive fiscal policy. The equilibrium employment ratio v^* is determined by conditions $1/u_i \cdot du_i/dt = 0$ and $d\pi_i^e/dt = 0$ $(i = 1, 2)$. The function h_i contains a term of mutual actions between countries. For an economically meaningful parameter setting the trajectory starting from almost every point reaches an attractor as depicted in Fig. 1. Our numerical calculations show that the attractor is bounded and the first Lyapunov exponent of it is positive. Hence it is a chaotic attractor.

3 Unstable periodic orbits on the attractor

We attempt to understand characteristics of the attractor through classifying unstable periodic orbits shown in Ishiyama and Saiki (2005a,b) and more than 500 orbits newly detected. In this section we sort unstable periodic solutions[3] into classes focusing on the labor share rate in country 1 (u_1). Correspondences between the chaotic solution and the classes of periodic solutions are discussed below. It is essential in this context that periodic orbits are to be embedded.[4]

3.1 Simple unstable periodic solutions

The only peroidic solution with one local maximum and one local minimum of u_1 (Fig.2) has been found in the chaotic attractor. This

[3]Unstable periodic solutions are numerically detected by damped-Newton method (Zoldi and Greenside (1998)).
[4]Ishiyama and Saiki (2005b)

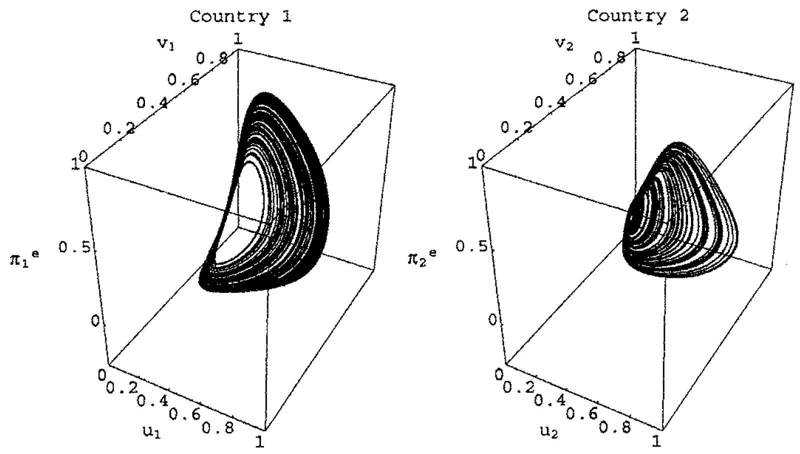

Fig. 1: Projections of a chaotic attractor of the model

The behavior of the model's solution is demonstrated by a numerical simulation. In the simulation parameters are set as $\alpha = 0.02$, $\beta = 0.01$, $\delta = 2/7$, $\mu_1 = 1.25$, $\mu_2 = 6$. The setting is fixed hereafter.
The trajectories projected onto these phase diagrams show how an economy starting from a meaningful point fluctuates after a certain period of time. We can see the outline of the attractor. The first Lyapunov exponent of the attractor is 0.099.

is the simplest periodic solution with the shortest period among unstable periodic solutions we found. It is a representative of business cycles observed in the chaotic attractor. The period of the simplest solution is approximately equal to the length of every cycle[5] observed in the chaotic attractor, where we consider each cycle in chaotic behavior begins from a local maximum of u_1 and ends at the next local maximum of the same variable. In addition Table 1 shows the statistical similarity between this periodic solution and the chaotic solution.

Here we consider a class $\mathrm{UPO_n}$ including the simplest solution. It consists of UPO_k, where UPO_k $(k \geq 1)$ is a set of periodic solutions with $k-1$ times of monotonic expansions of u_1 and the time series of a solution of UPO_k $(k > 1)$ has k local maxima of u_1 (See Fig. 3.). In our calculations, only UPO_1, UPO_2, $\cdots$, UPO_7 are the members of $\mathrm{UPO_n}$ for the parameter setting.

[5]This type of cycles may be caused by class struggle and nonlinear investment activity.

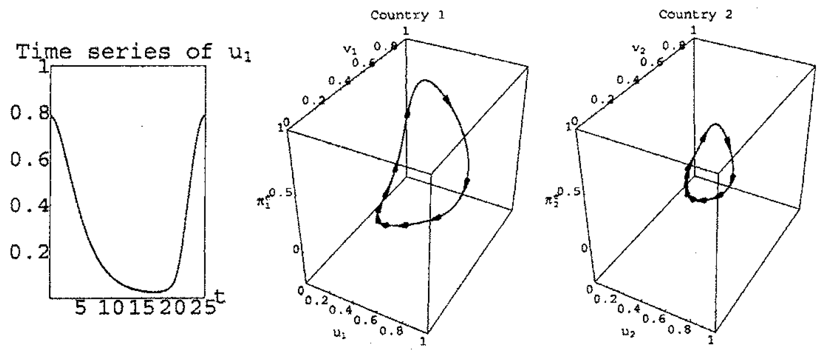

Fig. 2: Time series and phase diagrams of the simplest periodic solution (UPO_1)

Period of the simplest periodic solution is about 25.22.
Arrows on the periodic orbit indicate traveling directions. They also show how an economy typically goes.

Table 1: Mean values of variables of the simplest periodic solution and the chaotic solution

	$\bar{u}_1$	$\bar{v}_1$	$\bar{\pi}_1^e$	$\bar{u}_2$	$\bar{v}_2$	$\bar{\pi}_2^e$	period
UPO_1	0.265	0.694	0.000	0.264	0.772	0.000	25.22
Chaos	0.247	0.709	0.000	0.226	0.781	0.000	

Table 1 shows statistics of two solutions, that is, the simplest unstable periodic solution embedded in the chaotic attractor and the chaotic solution representing complicated phenomena. Respectively $\bar{u}_i$, $\bar{v}_i$ and $\bar{\pi}_i^e$ are mean values of u_i, v_i and π_i^e $(i = 1, 2)$.

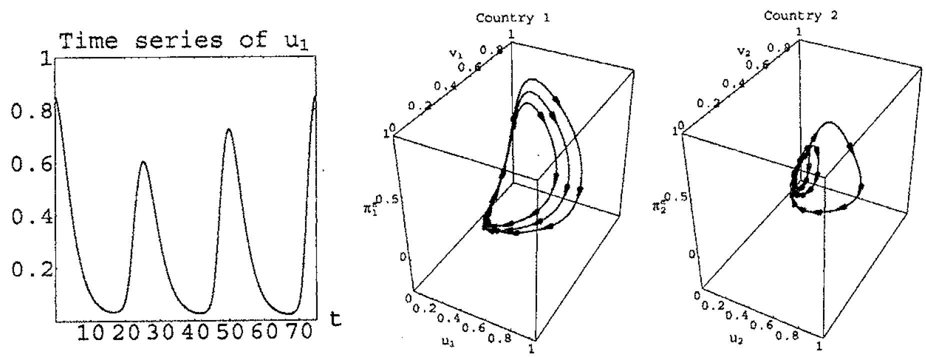

Fig. 3: Time series and phase diagrams of an example of UPO$_n$ (UPO_3)

The time series of a solution of UPO_3 has three local maxima of u_1. Each local maximum of u_1 is higher than the previous local maximum except one. The time series starting from maximum of u_1 represents a typical growth pattern of the model.
Arrows on the periodic orbit indicate traveling directions. They are put with a certain time interval. Some arrows gathering mean gradual fluctuation continues.

3.2 Complicated unstable periodic solutions

Except solutions of $\mathrm{UPO_{m,n}}$, unstable periodic solutions have transitions among patterns as dynamical properties, where each pattern is a series of expanding oscillations like UPO_n and it is called regime n in Ishiyama and Saiki (2005a,b). We name the transition from UPO_m type pattern (regime m) to UPO_n type pattern (regime n) transition $m \to n$. $\mathrm{UPO_{m,n}}$ is a class of periodic solutions which contains transition $m \to n$ and transition $n \to m$, while $\mathrm{UPO_{l,m,n}}$ is a class of periodic solutions consisting of transitions $l \to m$, $m \to n$ and $n \to l$. Fig. 4 gives examples of solutions of these classes.

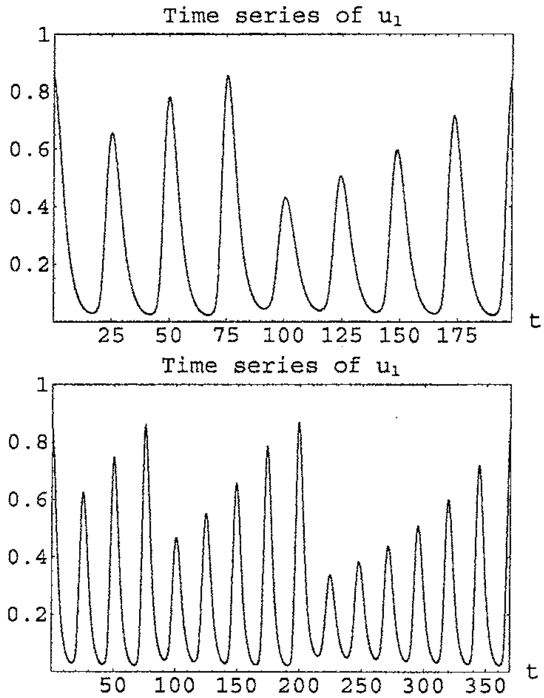

Fig. 4: Time series of examples of $\mathrm{UPO_{m,n}}$ ($UPO_{3,5}$) and $\mathrm{UPO_{l,m,n}}$ ($UPO_{3,5,7}$)

Each orbit of these classes is considered as a cyclical series of growth patterns of u_1. The trajectory of an arbitrary solution of $UPO_{3,5}$ has transitions $3 \to 5$ and $5 \to 3$, while on the trajectory of an arbitrary solution of $UPO_{3,5,7}$ we can see transitions $3 \to 5$, $5 \to 7$ and $7 \to 3$.

3.3 Hierarchical structure of solutions

Let us consider correspondences between chaos-transitions and transitions in unstable periodic orbits with respect to the above classes. Chaos-transitions are transitions observed in the chaotic behavior.

The value of the c-th cell in the r-th row in Fig. 5 denotes relative frequency of transition $r \to c$ observed in long time chaotic evolution. Transition $r \to c$ is a chaos-transition, called chaos-transition $r \to c$, if and only if the value of the c-th cell in the r-th row in Fig. 5 is positive. Transitions corresponding to cells bounded by dashed lines and thick lines are represented by solutions of $\mathrm{UPO}_{\mathrm{m,n}}$ and $\mathrm{UPO}_{\mathrm{l,m,n}}$ respectively. Note that these transitions are chaos-transitions frequently observed. Moreover $\mathrm{UPO}_{\mathrm{l,m,n}}$ have not only all transitions of $\mathrm{UPO}_{\mathrm{m,n}}$ but some other chaos-transitions. It suggests that the more complicated periodic solutions than $\mathrm{UPO}_{\mathrm{l,m,n}}$ have transitions related to more sorts of chaos-transitions. In fact a periodic solution of another class has transition $4 \to 2$ corresponding to chaos-transition $4 \to 2$ for example. In addition no transitions other than chaos-transitions are covered with the transitions of any periodic orbits in the attractor.

Transition in the chaotic attractor

To

From		1	2	3	4	5	6	7	8	9
	1	34	197	33	0	0	0	0	0	0
	2	125	517	223	0	0	0	0	0	0
	3	29	51	197	179	92	40	0	0	0
	4	26	42	44	37	33	37	74	0	0
From	5	24	29	27	23	21	22	36	8	0
	6	9	13	18	21	27	22	20	6	0
	7	14	15	41	21	15	16	23	7	0
	8	0	0	5	13	3	0	0	0	0
	9	0	0	0	0	0	0	0	0	0

Fig. 5: Distribution of chaos-transitions and transitions represented by $\mathrm{UPO}_{\mathrm{m,n}}$ and $\mathrm{UPO}_{\mathrm{l,m,n}}$ (Value in each cell is the frequency of a chaos-transition divided by 100.)

The c-th cell in the r-th row with positive number denotes chaos-transition $r \to c$. The cells bounded by dashed lines and thick lines mean chaos-transitions corresponding to transitions covered with $\mathrm{UPO}_{\mathrm{m,n}}$ and $\mathrm{UPO}_{\mathrm{l,m,n}}$ respectively. Note that the existence of UPO_k implies transition $k \to k$ is observable in the chaotic economic growth.

4 Conclusions

We focus on three classes of unstable periodic solutions embedded in the chaotic attractor of a growth cycle model. We study correspondences between these classes and the general chaotic behavior. Typical dynamics and statistical properties of solutions of the model can be represented by the simplest class of periodic solutions. The other classes contain recursive transitions among two or three typical patterns corresponding to transitions observed in the chaotic growth. We have successfully related the presence of such a transition of chaotic economic dynamics generated by the two-country Keynes-Goodwin model to the existence of unstable periodic solutions embedded in the attractor.

Generally infinite number of unstable periodic orbits are embedded in a chaotic attractor. Our results obtained in this paper emphasize usefulness of unstable periodic orbits to study business cycle models.

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