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A solution of the equation $f'(x)=\lambda^{2}f(\lambda x)$, $\lambda>1$.

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1. Introduction

The purpose of this paper is to give solutions for the functional-differential equation of advanced type

$$\begin{cases} f'(x)=\lambda^{2}f(\lambda x), & x\in\mathbb{R}=(-\infty,+\infty),\\ f(0)=0, \end{cases} \tag{1.1}$$

where λ is a constant, $\lambda>1$. Our solutions are infinitely differentiable on $\mathbb{R}$. Moreover, if $\lambda\geq 2$, then the solutions are bounded and have arbitrarily large zeros. Our methods give numerical data readily.

Frederickson [1, 2] (1971) investigated functional-differential equations of advanced type

$$f'(x)=af(\lambda x)+\lambda f(x), \tag{1.2}$$

here $\lambda>1$, and proved several properties of solutions. Later, Kato and McLeod [5] (1971) and Kato [4] (1972) studied the asymptotic behaviour of solutions of (1.2).

Frederickson [1] provided a global existence theorem for equations

$$f'(x)=F(f(2x)), \quad x\in\mathbb{R},$$

where F is an odd, continuous function with $F(s)>0$ for $s>0$, by application of the Schauder fixed point theorem. He showed that the absolute value of the solution $|f(x)|$ is periodic for $x\geq 0$. Frederickson [2] also provided a constructive method for solutions for equations

$$f'(z)=af(\lambda z)+bf(z),$$

where $a,\ b \in \mathbb{C}$ and $\lambda > 1$. He further gave solutions in the form of a Diriclet series

$$\varphi(z,\beta) = \sum_{n\in\mathbb{Z}} c_n e^{\beta\lambda^n z}, \quad \Re(\beta z) \le 0,$$

where β is allowed to vary as a parameter. In the case of $b = 0$ and $\beta = i$, the solution is analytic in the upper half plane $\Im z > 0$, continuous on $\Im z \ge 0$, and the line $\Im z = 0$ is a natural boundary. From his result it follows that our solutions of (1.1) cannot be real analytic.

Ivanov, Kitamura, Kusano and Shevelo [3] (1982) investigated the higher order functional-differential equations of the form

$$f^{(n)}(x) = p(x)F(f(g(x))), \tag{1.3}$$

where p, F and g satisfy appropriate conditions. Kusano [6] (1984) also investigated the functional differential equation

$$f^{(n)}(x) = p(x)f(g(x)) \tag{1.4}$$

where n is even, $p : [0,\infty) \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ are continuous, $p(t) > 0$, $g(t)$ is nondecreasing and $\lim_{t\to\infty} g(t) = \infty$. They [3, 6] gave sufficient conditions that the solutions are oscillatory.

If f is a solution of (1.1), then f is also a solution of the equations

$$f''(x) = \lambda^4\lambda f(\lambda^2 x), \quad x \in \mathbb{R}, \tag{1.5}$$

and

$$f'''(x) = \lambda^6\lambda^3 f(\lambda^3 x), \quad x \in \mathbb{R}. \tag{1.6}$$

However, (1.5) and (1.6) don't satisfy the sufficient conditions in [3, 6].

Recently, the author [9] constructed solutions of (1.1) with $\lambda = 2$ by using a little different method from this paper.

We state the main theorem (Theorem 2.3) and application in next section. We can easly apply the solution for the case $\lambda = 2$ to Friedrichs' molifier theorem and we can rewrite defferential operator. For the proof of main theorem, see [10]. In the third section, we give graphs of solutions for the case of $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 2/3, 5/4$.

In the last section, we will give Mathematica programs.

2. Main Results

First, we state two lemmas. Let

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx, \quad \mathcal{F}^{-1}[f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{ix\xi}\, dx,$$

and

$$\operatorname{sinc} \xi = \begin{cases} \sin(\pi\xi)/(\pi\xi), & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Lemma 2.1. *The product*

$$\prod_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\xi}{2\lambda^k \pi}\right), \quad \xi \in \mathbb{R}$$

is converges pointwise and in $L^1(\mathbb{R})$.

Lemma 2.2. *Let*

$$u = \mathcal{F}^{-1}[U], \quad U(\xi) = \exp\left(-\frac{i\xi}{2(\lambda-1)}\right) \prod_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\xi}{2\lambda^k \pi}\right). \tag{2.1}$$

Then u has the following properties:

$$u \in C^{\infty}(\mathbb{R}),$$
$$u(x) > 0 \text{ for } x \in \left(0, \frac{1}{\lambda-1}\right), \; u(x) = 0 \text{ for } x \notin \left(0, \frac{1}{\lambda-1}\right),$$
$$u(x) = u(1/(\lambda-1) - x),$$
$$\int_{\mathbb{R}} u(x)\, dx = 1,$$

and

$$u'(x) = \lambda^2 u(\lambda x) \quad \text{for} \quad x \in \left[0, \min\left(\frac{1}{\lambda}, \frac{1}{\lambda(\lambda-1)}\right)\right]. \tag{2.2}$$

Let we define the operator $T : L^1 \to L^1$ as follows.

$$Tf(x) = \lambda \left(\chi_{[0,1]} * f\right)(\lambda x), \quad f \in L^1. \tag{2.3}$$

Then the function u in Lemma 2.2 is given by the following equation.

$$u = \lim_{k} T^k \chi_{[0,1]}. \tag{2.4}$$

Secondly, we define sequences $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ as follows:

$$\begin{cases} n_1 = 0, \quad n_2 = 1, \\ n_{2k-1} = 1, \ n_{2k} = 0, \quad \text{if} \quad n_k = 1 \quad (k \geq 2), \\ n_{2k-1} = 0, \ n_{2k} = 1, \quad \text{if} \quad n_k = 0 \quad (k \geq 2), \end{cases} \tag{2.5}$$

and

$$y_k = \sum_{l=1}^{\infty} C_{k,l} \lambda^{l-1}, \quad k = 1, 2, 3, \cdots, \tag{2.6}$$

where $C_{k,l} \in \{0, 1\}$ $(l = 1, 2, 3, \cdots)$ are coefficients of the binary system such that

$$k - 1 = \sum_{l=1}^{\infty} C_{k,l} 2^{l-1}, \quad k = 1, 2, 3, \cdots.$$

Then we have the following relations.

$$\begin{cases} (-1)^{n_{2k-1}} = (-1)^{n_k}, \\ (-1) \cdot (-1)^{n_{2k}} = (-1)^{n_k}, \end{cases} \qquad k = 1, 2, 3, \cdots, \tag{2.7}$$

$$\begin{cases} y_{2k-1}/\lambda = y_k, \\ y_{2k}/\lambda = y_k + 1/\lambda, \end{cases} \qquad k = 1, 2, 3, \cdots, \tag{2.8}$$

and

$$y_k \geq \lambda^j \quad \text{if} \quad k - 1 \geq 2^j, \quad j = 0, 1, 2, \cdots. \tag{2.9}$$

Hence $\lim_{k \to \infty} y_k = \infty$. If $\lambda \geq 2$, then y_k is strictly increasing. For example,

$$\{n_k\}_{k=1}^{\infty} = \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0 \cdots\},$$

$$\{y_k\}_{k=1}^{\infty} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\} \text{ for } \lambda = 2,$$

$$\{y_k\}_{k=1}^{\infty} = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, \cdots\} \text{ for } \lambda = 4,$$

$$\{y_k\}_{k=1}^{\infty} = \left\{0, 1, \frac{3}{2}, \frac{5}{2}, \frac{9}{4}, \frac{13}{4}, \frac{15}{4}, \frac{19}{4}, \frac{27}{8}, \frac{35}{8}, \frac{39}{8}, \frac{47}{8}, \frac{45}{8}, \cdots\right\} \text{ for } \lambda = 3/2.$$

Our main result is the following:

Theorem 2.3. *Let $\lambda > 1$. Then a solution f of (1.1) can be found as*

$$f(x) = \sum_{k=1}^{\infty} (-1)^{n_k} u(x - y_k),$$

where u, $\{n_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are as in (2.1), (2.5) and (2.6), respectively. The solution f is in $C^{\infty}(\mathbb{R})$ and $f(x) = 0$ for $x \leq 0$. If $\lambda \geq 2$, then f is bounded.

Let we define a function space $L^{1,\nu}(\mathbb{R})$.

$$L^{1,\nu} = \{f \in L^1_{loc}; \|f\|_{L^{1,\nu}} < \infty\}$$
$$\|f\|_{L^{1,\nu}} = \sup_{r>0} \frac{1}{r^\nu} \int_{-r}^{+r} |f(x)|dx$$

Theorem 2.4. *The solution f of (1.1) is in $C^\infty \cap L^{1,1/\log_2 \lambda}$.*

Remark 2.1. The solution of (1.1) is tempered distribution.

Remark 2.2. A constant times f is also a solution.

Theorem 2.5. *Let f be the solution in Theorem 2.3 for $\lambda = 2$ and*

$$G_{k,\epsilon}(x) = \left(2^{k(k-1)/2}\epsilon^{k+1}\right)^{-1} (f\chi_{[0,2^k]})(x/\epsilon).$$

If $v \in C^k(\mathbb{R})$ or $v \in L^p_k(\mathbb{R})$ $(k \geq 0,\ 1 \leq p < \infty)$, then

$$\frac{d^k v}{dx^k} = \lim_{\epsilon \to 0} v * G_{k,\epsilon},$$

uniformly on each compact subset in $\mathbb{R}$ or in $L^p(\mathbb{R})$, respectively.

Remark 2.3. $G_{k,\epsilon}$ is in $C^\infty(\mathbb{R})$ with compact support. To prove the theorem we use Friedrichs' molifier $\frac{d^k v}{dx^k} * u_\delta = v * \frac{d^k u_\delta}{dx^k}$, where $u_\delta = u(x/\delta)/\delta$, $\delta > 0$, and u is the function in Lemma 2.2 ($\lambda = 2$).

3. Examples

In this section we give graphs for $\lambda = 4, 3, 2, 31/16, 15/8, 7/4, 3/2, 5/4$.

If $\lambda = 2$, then $\{x > 0 : f(x) = 0\} = \{1, 2, 3, \cdots\}$. If $\lambda > 2$, then $\{x > 0 : f(x) = 0\} = \cup_{k=1}^\infty [y_k + 1/(\lambda - 1), y_{k+1}]$ and its measure is infinity, since $1/(\lambda - 1) < 1 < y_{k+1} - y_k$, $k = 1, 2, 3, \cdots$.

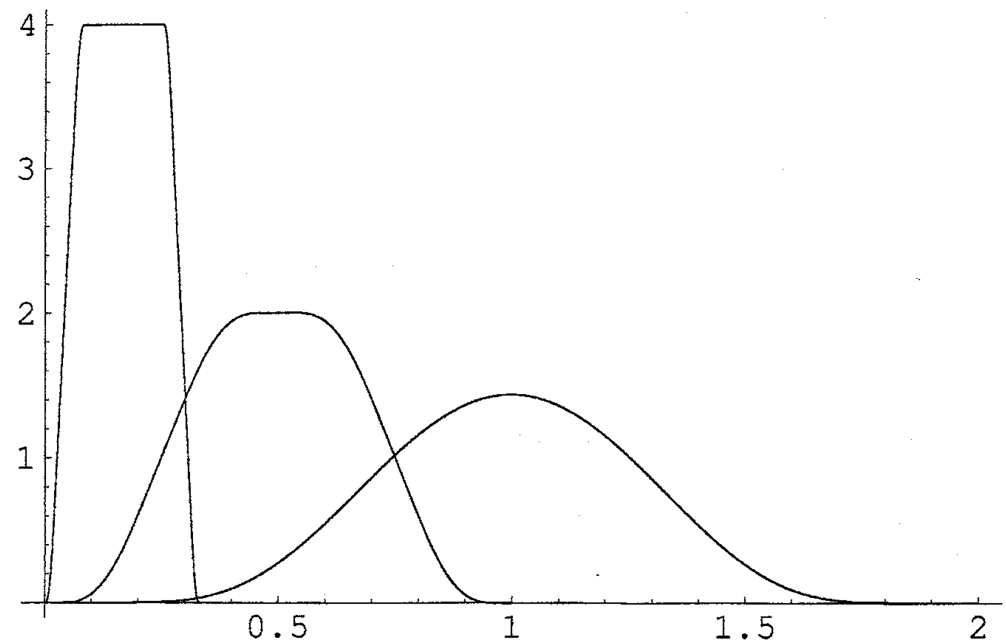

FIGURE 1. u ($\lambda = 4, 2, 3/2$)

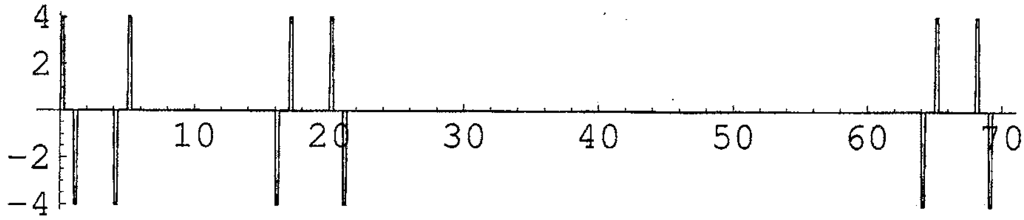

FIGURE 2. $f'(x) = 4^2 f(4x)$

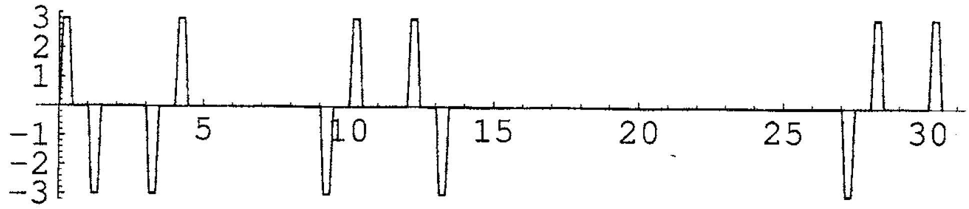

FIGURE 3. $f'(x) = 3^2 f(3x)$

4. PROGRAM OF U(X)

Mathematica program (Part 1): The function u [by using u_{n+1}=T(u_{n})]

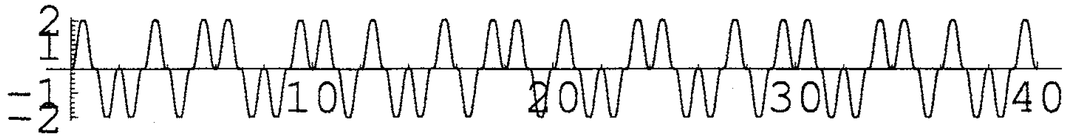

FIGURE 4. $f'(x) = 2^2 f(2x)$

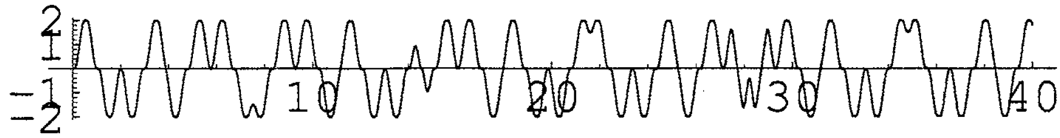

FIGURE 5. $f'(x) = (31/16)^2 f(31x/16)$

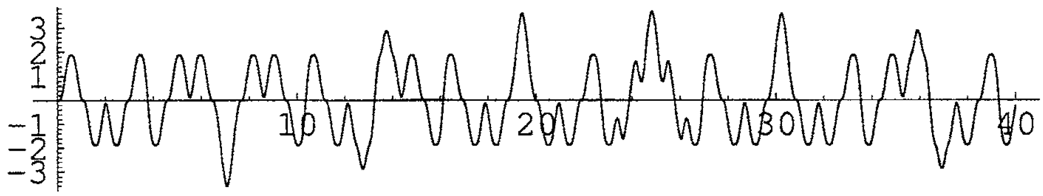

FIGURE 6. $f'(x) = (15/8)^2 f(15x/8)$

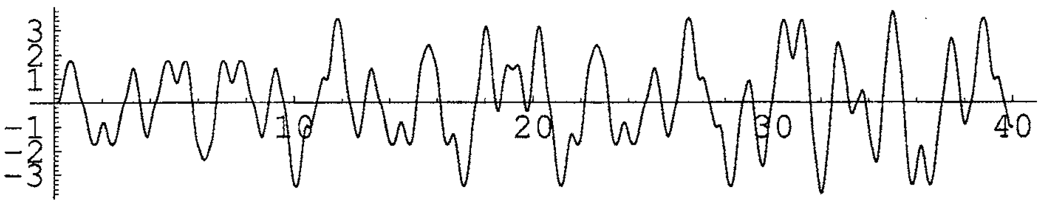

FIGURE 7. $f'(x) = (7/4)^2 f(7x/4)$

```
* Setting lambda (1<lam<9)
In[1]: lam = 1.75;


* Calculation of the data ; u_{0}, ... , u_{50}
In[2]: udata[0]=
         Table[If[0 < i - 10000 =< 1000, lam - 1, 0], {i, 1, 20000}];
In[3]: Timing[ Do[udata[k]= Table[If[1 =< j - 10000 =< 1000,
         lam * Sum[udata[k = 1][[i + 10000]],
           {i,Round[lam * (j - 10000)]- Round[(lam - 1)* 1000]+ 1,
```

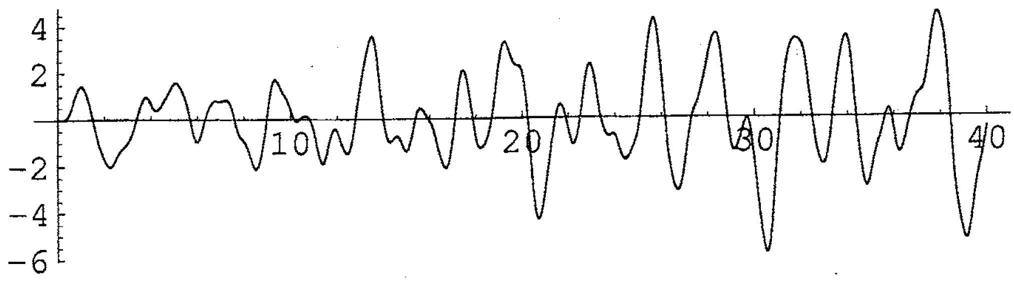

FIGURE 8. $f'(x) = (3/2)^2 f(3x/2)$

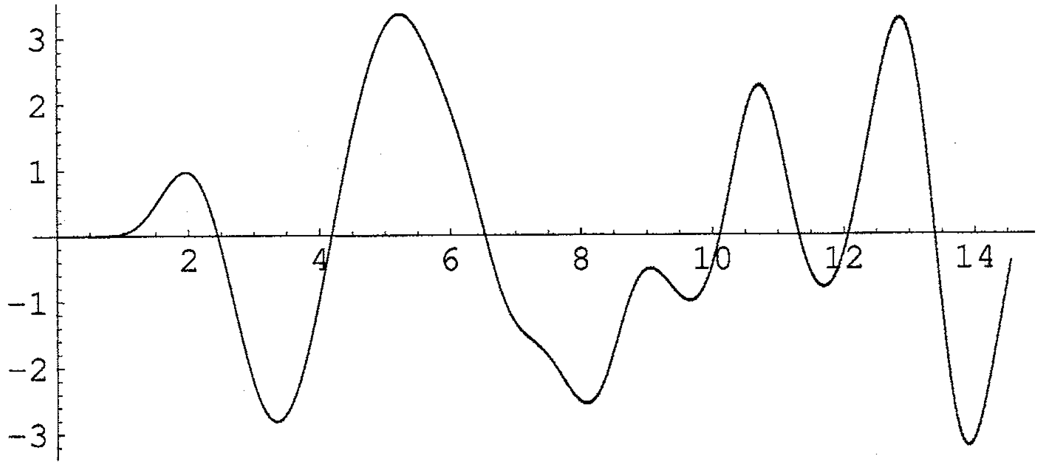

FIGURE 9. $f'(x) = (5/4)^2 f(5x/4)$

```
            Round[lam * (j - 10000)] }]* 0.001/(lam - 1), 0],
              {j, 1, 20000}], {k, 1, 50}]]
Out[3]: {121.14 Second, Null}

* Graph of u_{50}
In[4]: ulist[k_]:= Table[{(i - 10000)* 0.001/(lam - 1),
         Part[udata[k], i]}, {i, 10000, 11000}]
In[5]: ListPlot[ulist[50], PlotJoined -> True,
         PlotRange → {0, 1.1 * lam}]

* Save the data
```

```
In[6]: udata[50]>> c:/mdata/udata7ov4-50
In[7]: ulist[50]>> c:/mdata/ulist7ov4-50
In[8]: Export["c:/mdata/u7ov4.eps",
         ListPlot[ulist[50], PlotJoined -> True,
           PlotRange → {0, 1.1 * lam}]]
```

5. Program of f(x)

Mathematica program (Part 2): The solution f on the interval [0,tau]

```
* Setting lambda (1<lam<9) and tau
In[1]: lam = 1.75; tau = 30;
In[2]: kk = Round[Log[lam, tau]+ 0.5]
Out[2]: 7

* Load the data
In[3]: udata = << c:/mdata/udata7ov4-50;
In[4]: ud = Table[Part[udata, i], {i, 10000, 11000}];

* Sequences m_{k} and y_{k}
In[5]: m[1]= 0; m[2]= 1;
         Do[m[k]= If[Mod[k, 2]==0, Mod[m[k /2]+ 1, 2], m[(k + 1)/2]],
           {k, 3, 2^kk + 1}]
In[6]: Do[b[k, 1]= k - 1; Do[c[k, l]= Mod[b[k, l], 2];
         b[k, l + 1]= (b[k, l]- c[k, l])/2, {l, 1, kk + 1}],
           {k, 1, 2^kk + 1}]
In[7]: Do[y[k]=
         Sum[c[k, l]* lam^(l - 1), {l, 1, kk + 1}], {k, 1, 2^kk + 1}]

* Calculation of the solution
  as the sum of (-1)^{m_{k}}u(x-y_{k}), k=1, 2, ... ,2^kk.
```

```
In[8]: Do[yy[k]= Round[y[k]* 1000 * (lam - 1)], {k, 1, 2^kk + 1}]
In[9]: zz[1]= Table[0, {i, 1, yy[2^kk]}]; Do[z[k]=
         Table[0, {i, 1, yy[k]}], {k, 2, 2^kk}];
           Do[zz[k]= Table[0, {i, 1, yy[2^kk]- yy[k]}], {k, 2, 2^kk}];
In[10]: udy[1]= Join[ud, zz[1]]; Do[udy[k]=
          Join[z[k], ud* (-1)^m[k], zz[k]], {k, 2, 2^kk}]
In[11]: fd = Sum[udy[k], {k, 1, 2^kk}];

* Save the graph of the solution

In[12]: ii=tau*(lam-1)*1000;
In[13]: flist = Table[{i * 0.001/(lam - 1),
          Part[fd, i]}, {i, 1, ii}];
In[14]: Export["c:/mdata/f7ov4.eps",
          ListPlot[flist, PlotJoined -> True, AspectRatio → Automatic]];
```

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