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バナッハ空間における無限区間ファジィ境界値問題

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1 Complete Metric Space of Fuzzy Numbers

Denote $I = [0,1]$. The following definition means that a fuzzy number can be identified with a membership function.

Definition 1 *Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by*

$\mathcal{F}_b^{st} = \{\mu : \mathbf{R} \to I$ *satisfying* (i)-(iv) *below*$\}$.

(i) *μ has a unique number $m \in \mathbf{R}$ such that $\mu(m) = 1$ (normality);*

(ii) *$supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ is bounded in $\mathbf{R}$ (bounded support);*

(iii) *μ is strictly fuzzy convex on $supp(\mu)$ as follows:*

(a) *if $supp(\mu) \neq \{m\}$, then*

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$

for $\xi_1, \xi_2 \in supp(\mu)$ with $\xi_1 \neq \xi_2$ and $0 < \lambda < 1$;

(b) *if $supp(\mu) = \{m\}$, then $\mu(m) = 1$ and $\mu(\xi) = 0$ for $\xi \neq m$;*

(iv) *μ is upper semi-continuous on $\mathbf{R}$ (upper semi-continuity).*

It follows that $\mathbf{R} \subset \mathcal{F}_b^{st}$. Because m has a membership function as follows:

$$\mu(m) = 1\ ; \quad \mu(\xi) = 0\ (\xi \neq m) \tag{1.1}$$

Then μ satisfies the above (i)-(iv).

In usual case a fuzzy number x satisfies *fuzzy convex on* $\mathbf{R}$, *i.e.*,

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)] \tag{1.2}$$

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$. Denote α−cut sets by

$$L_\alpha(\mu) = \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}$$

for $\alpha \in I$.

We introduce the following parametric representation of $\mu \in \mathcal{F}_b^{st}$ as

$$x_1(\alpha) = \min L_\alpha(\mu),$$
$$x_2(\alpha) = \max L_\alpha(\mu)$$

for $0 < \alpha \leq 1$ and

$$x_1(0) = \min supp(\mu),$$
$$x_2(0) = \max supp(\mu).$$

Denote by $C(I)$ the set of all the continuous functions on I to $\mathbf{R}$. The following theorem shows a membership function is characterized by x_1, x_2.

Theorem 1 *Denote the left-, right-end points of the α−cut set of $\mu \in \mathcal{F}_b^{st}$ by $x_1(\alpha), x_2(\alpha)$, respectively. Here $x_1, x_2 : I \to \mathbf{R}$. The following properties* (i)-(iii) *hold.*

(i) $x_1, x_2 \in C(I)$;

(ii) $\max_{\alpha\in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha\in I} x_2(\alpha) = x_2(1)$;

(iii) *x_1, x_2 are non-decreasing, non-increasing on I, respectively, as follows :*

(a) *there exists a positive number $c \leq 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c)$ and that $x_1(\alpha) = m = x_2(\alpha)$ for $\alpha \in [c, 1]$;*

(b) *$x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$;*

Conversely, under the above conditions (i) -(iii), *if we denote*

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \tag{1.3}$$

for $\xi \in \mathbf{R}$, then $\mu \in \mathcal{F}_b^{st}$.

Remark 1 *From the above Condition* (i) *a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $\mathbf{R}^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$.*

In what follows we denote $\mu = (x_1, x_2)$ for $\mu \in \mathcal{F}_b^{st}$. The parametric representation of μ is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be an $\mathbf{R}-$valued function. The corresponding binary operation of two fuzzy numbers $x, y \in \mathcal{F}_{\text{b}}^{st}$ to $g(x,y) : \mathcal{F}_{\text{b}}^{st} \times \mathcal{F}_{\text{b}}^{st} \to \mathcal{F}_{\text{b}}^{st}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x,y)}$ of g is as follows:

$$\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))$$

Here $\xi, \xi_1, \xi_2 \in \mathbf{R}$ and μ_x, μ_y are membership functions of x, y, respectively. From the extension principle, it follows that, in case where $g(x,y) = x + y$,

$$\begin{aligned}
&\mu_{x+y}(\xi) \\
&= \max_{\xi = \xi_1 + \xi_2} \min_{i=1,2} (\mu_i(\xi_i)) \\
&= \max\{\alpha \in I : \xi = \xi_1 + \xi_2,\ \xi_i \in L_\alpha(\mu_i), i = 1, 2\} \\
&= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}.
\end{aligned}$$

Thus we get $x+y = (x_1+y_1, x_2+y_2)$. In the similar way $x - y = (x_1 - y_2, x_2 - y_1)$.

Denote a metric by

$$d_\infty(x,y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_{\text{b}}^{st}$.

Theorem 2 *$\mathcal{F}_{\text{b}}^{st}$ is a complete metric space in $C(I)^2$.*

2 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions μ_x, μ_y of $x, y \in \mathcal{F}_{\text{b}}^{st}$ and $\lambda \in \mathbf{R}$, the following addition and a scalar product are given as follows :

$$\begin{aligned}
\mu_{x+y}(\xi) &= \sup\{\alpha \in [0,1] : \\
&\quad \xi = \xi_1 + \xi_2,\ \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\}, \\
\mu_{\lambda x}(\xi) &= \begin{cases} \mu_x(\xi/\lambda) & (\lambda \neq 0) \\ 0 & (\lambda = 0,\ \xi \neq 0) \\ \sup_{\eta \in \mathbf{R}} \mu_x(\eta) & (\lambda = 0,\ \xi = 0) \end{cases}
\end{aligned}$$

In [5] they introduced the following equivalence relation $(x,y) \sim (u,v)$ for $(x,y), (u,v) \in \mathcal{F}_{\text{b}}^{st} \times \mathcal{F}_{\text{b}}^{st}$, *i.e.,*

$$(x,y) \sim (u,v) \Longleftrightarrow x + v = u + y. \qquad (2.4)$$

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (2.4) means that the following equations hold.

$$x_i + v_i = u_i + y_i \quad (i = 1, 2)$$

Denote an equivalence class by $[x,y] = \{(u,v) \in \mathcal{F}_{\text{b}}^{st} \times \mathcal{F}_{\text{b}}^{st} : (u,v) \sim (x,y)\}$ for $x, y \in \mathcal{F}_{\text{b}}^{st}$ and the set of equivalence classes by

$$\mathcal{F}_{\text{b}}^{st} / \sim = \{[x,y] : x, y \in \mathcal{F}_{\text{b}}^{st}\}$$

such that one of the following cases (i) and (ii) hold:

(i) if $(x,y) \sim (u,v)$, then $[x,y] = [u,v]$;

(ii) if $(x,y) \not\sim (u,v)$, then $[x,y] \cap [u,v] = \emptyset$.

Then $\mathcal{F}_{\text{b}}^{st} / \sim$ is a linear space with the following addition and scalar product

$$[x,y] + [u,v] = [x + u, y + v] \qquad (2.5)$$

$$\lambda[x,y] = \begin{cases} [(\lambda x, \lambda y)] & (\lambda \geq 0) \\ [((-\lambda)y, (-\lambda)x)] & (\lambda < 0) \end{cases} \qquad (2.6)$$

for $\lambda \in \mathbf{R}$ and $[x,y], [u,v] \in \mathcal{F}_{\text{b}}^{st} / \sim$. They denote a norm in $\mathcal{F}_{\text{b}}^{st} / \sim$ by

$$\| [x,y] \| = \sup_{\alpha \in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here d_H is the Hausdorff metric is as follows:

$$\begin{aligned}
&d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)) \\
&= \max(\sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|, \\
&\quad \sup_{\eta \in L_\alpha(\mu_x)} \inf_{\xi \in L_\alpha(\mu_y)} |\xi - \eta|)
\end{aligned}$$

It can be easily seen that $\| [x,y] \| = d_\infty(x,y)$.

Note that $\| [x,y] \| = 0$ in $\mathcal{F}_{\text{b}}^{st} / \sim$ if and only if $x = y$ in $\mathcal{F}_{\text{b}}^{st}$.

3 Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $\mathcal{F}_{\text{b}}^{st}$ has an induced Banach space.

Theorem 3 *Let S be a bounded closed subset in $\mathcal{F}_{\text{b}}^{st}$. Assume that S contains any segments of $x, y \in S$, i.e., $\lambda x + (1 - \lambda) y \in S$ for $\lambda \in I$. Let V be an into continuous mapping on S. Assume that the closure $cl(V(S))$ is compact in $\mathcal{F}_{\text{b}}^{st}$. Then V has at least one fixed point x in S, i.e., $V(x) = x$.*

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

Theorem 4 *Let $\mathcal{F}$ be a complete metric space with a metric d. Assume that $\mathcal{F}$ is closed under addition and scalar product, and that $d(\lambda x, 0) =$*

$|\lambda|d(x,0)$ *for the scalar product* λx *and* $\lambda \in \mathbf{R}, x \in \mathcal{F}$. *Denote* $X = \{[x,0] : x, 0 \in \mathcal{F}\}$. *Here* $[x,y]$ *for* $x, y \in \mathcal{F}$ *are equivalence classes of* (2.4) *and* 0 *is the origin. Then* X *is a Banach space concerning addition* (2.5), *scalar product* (2.6) *and norm* $\| [x,0] \| = d(x,0)$ *for* $[x,0] \in X$.

Moreover let S *be a bounded closed subset in* $\mathcal{F}$. *Assume that* S *contains any segments of* $x, y \in S$ *in the same meaning of Theorem* 3. *Let* V *be an into continuous mapping on* S. *Assume that the closure* $cl(V(S))$ *is compact in* $\mathcal{F}$. *Then* V *has at least one fixed point in* S.

4 FBVP on Infinite Intervals

In this section we deal with the following FBVP on an infinite interval:

$$\frac{dx}{dt} = p(t)x + f(t,x), \quad x(\infty) = c \qquad (4.7)$$

Here $p : \mathbf{R}_+ \to \mathcal{F}_b^{st}$, $f : \mathbf{R}_+ \times \mathcal{F}_b^{st} \to \mathcal{F}_b^{st}$ are continuous functions. Let denote $\mathbf{R}_+ = [0,\infty)$ and $c \in \mathcal{F}_b^{st}$. The following assumptions play important roles in considering the existence of solutions of (4.7).

Assumption.

(A1) Assume that there exists a $K > 0$ such that

$$\int_0^\infty d(p(s),0)ds = K < \infty;$$

(A2) There exist positive real numbers a, r, R and integrable function $m : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$d(f(t,x),0) \leq m(t) \text{ for } (t,x) \in \mathbf{R}_+ \times S_1;$$
$$\int_0^\infty m(s)ds \leq rR;$$
$$[R + N_p(a + \| L \| R)]K < 1.$$

Here

$$S_1 = \{x \in \mathcal{F}_b^{st} : d(x,0) \leq \min(ar, r)\}$$

and N_p is independent on the function p.
$L : C_r^{\lim} \to \mathcal{F}_b^{st}$ is a linear operator as $L(x) = x(\infty)$ and

$$C_r^{\lim} = \{x \in C(\mathbf{R}_+ : \mathcal{F}_b^{st}) : \exists x(\infty), d(x,0) \leq r\}.$$

(A3) There exists no solution of

$$\frac{dx}{dt} = p(t)x, L(x) = 0$$

except for the zero solution.

We expect the following existence theorem for solutions of FBVP on the infinite interval.

Under assumptions (A1) - (A3) we expect that there exists at least one solution of (4.7) in $C_r^{\lim}$ for any $c \in S_1$ by applying the Schauder's fixed point theorem in $C_r^{\lim}$.

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