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Oscillation and comparison theorems for second-order half-linear differential equations

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1 Introduction

Over the past four decades a great deal of articles have been devoted to the study of oscillation of solutions of half-linear differential equations. For example, those results can be found in [1–6, 9–12]. Especially, it is well-known that all nontrivial solutions of a half-linear differential equation of the form

$$\left(|x'|^{\alpha-1}x'\right)' + \frac{\lambda}{t^{\alpha+1}}|x|^{\alpha-1}x = 0, \qquad t > t_0 \tag{1.1}$$

with $\alpha > 0$, $\lambda > 0$ and $t_0 \geq 0$, are oscillatory if

$$\lambda > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1};$$

otherwise, they are nonoscillatory. This fact means that $(\alpha/(\alpha+1))^{\alpha+1}$ is the lower bound for all nontrivial solutions of (1.1) to be oscillatory. Such a number is generally called the *oscillation constant* (for example, see [7, 8, 13–15]).

Let us add a perturbation to equation (1.1) when λ is the oscillation constant and consider the perturbed half-linear differential equation

$$\left(|x'|^{\alpha-1}x'\right)' + \frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\delta(t)\right\}|x|^{\alpha-1}x = 0, \tag{E_α}$$

where $\delta(t)$ is positive and continuous on some half-line (t_0, ∞). Elbert and Schneider [6] have investigated the asymptotic behaviour of solutions of (E_α). Using their results, we can present the following statements.

Theorem A. *Let $\alpha > 1$. If equation (E_α) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_1) are oscillatory.*

Theorem B. *Let $0 < \alpha < 1$. If equation (E_1) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory.*

It follows from the fact mentioned in the first paragraph and Sturm's comparison theorem for half-linear differential equations that if

$$\liminf_{t\to\infty} \delta(t) > 0, \tag{1.2}$$

then all nontrivial solutions of (E_α) are oscillatory. As to Sturm's separation and comparison theorems, for example, see [5, 11, 12]. On the other hand, if condition (1.2) fails to hold, then there is some possibility that equation (E_α) has a nonoscillatory solution. One of the most interesting case is that $\delta(t) = \lambda/(\log t)^2$ with $\lambda > 0$. In this case, if $\lambda > 1/2$, then all nontrivial solutions of (E_α) are oscillatory; otherwise, they are nonoscillatory (for details, see [6]).

We may regard Theorems A and B as comparison theorems between the linear differential equation

$$x'' + \frac{1}{t^2}\left\{\frac{1}{4} + \frac{1}{2}\delta(t)\right\}x = 0 \tag{E_1}$$

and half-linear differential equations of the form (E_α). Let α and β be positive numbers satisfying $\alpha < 1 < \beta$. Then, combining Theorems A and B, we get the following conclusion: if equation (E_β) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory. A natural question now arises as to whether or not the converse proposition is also true.

The first purpose of this paper is to extend Theorems A and B to a comparison theorem between any two half-linear differential equations. The second purpose is to give an answer to the above question. Our main results are stated as follows:

Theorem 1.1. *Let $0 < \alpha < \beta$. If equation (E_β) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory.*

Remark 1.1. Theorem 1.1 is a generalization of Theorems A and B. To put it precisely, Theorem 1.1 coincides with Theorem A (respectively, Theorem B) when $\alpha = 1$ (respectively, $\beta = 1$).

Theorem 1.2. *Let $0 < \alpha < \beta$. If equation (E_α) has a nontrivial oscillatory solution, then all nontrivial solutions of*

$$\left(|x'|^{\beta-1}x'\right)' + \frac{1}{t^{\beta+1}}\left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + \nu\delta(t)\right\}|x|^{\beta-1}x = 0 \tag{1.3}$$

are oscillatory, where $\nu > (\beta/(\beta+1))^\beta$.

Remark 1.2. It is essential that ν is greater than $(\beta/(\beta+1))^\beta$ in Theorem 1.2. Unfortunately, even if equation (E_α) has a nontrivial oscillatory solution, we cannot judge whether all nontrivial solutions of (E_β) are oscillatory or not.

Remark 1.3. From Theorems 1.1 and 1.2, we see that the oscillation constant for equation (E_α) with $\delta(t) = \lambda/(\log t)^2$ is 1/2 for any $\alpha > 0$.

2 Riccati technique

Consider the half-linear differential equation

$$\left(|x'|^{p-1}x'\right)' + \frac{1}{t^{p+1}}\left\{\left(\frac{p}{p+1}\right)^{p+1} + h(t)\right\}|x|^{p-1}x = 0 \tag{2.1}$$

with $p > 0$ a fixed real number, where $h(t)$ is positive and continuous on $(0, \infty)$. Using Riccati's transformation, we prepare some lemmas below. To this end, we denote

$$H_p(\xi) = p\left\{\xi^{(p+1)/p} - \xi + \frac{p^p}{(p+1)^{p+1}}\right\}$$

for $\xi > 0$ and

$$\gamma_p = \left(\frac{p}{p+1}\right)^p.$$

Lemma 2.1. *Let $\xi(s)$ be a positive function on $[s_0, \infty)$ with $s_0 > 0$ satisfying*

$$\dot{\xi}(s) + H_p(\xi(s)) \le 0. \tag{2.2}$$

Then it is nonincreasing and tends to γ_p as $s \to \infty$.

Proof. From

$$H_p(\gamma_p) = p\left\{\left(\frac{p}{p+1}\right)^{p+1} - \left(\frac{p}{p+1}\right)^p + \frac{p^p}{(p+1)^{p+1}}\right\} = 0$$

and

$$\frac{d}{d\xi}H_p(\xi) = (p+1)\xi^{1/p} - p,$$

we see that $H_p(\xi) \ge 0$ for $\xi > 0$ and $H_p(\xi) = 0$ if and only if $\xi = \gamma_p$.

Since $\xi(s)$ is positive for $s \ge s_0$, we have

$$\dot{\xi}(s) \le -H_p(\xi(s)) \le 0$$

by (2.2), namely, $\xi(s)$ is nonincreasing. Hence, there exists a $\mu \ge 0$ such that $\xi(s) \searrow \mu$ as $s \to \infty$. Suppose that $\mu \ne \gamma_p$. If $\mu > \gamma_p$, then $\xi(s) > \mu > (\mu + \gamma_p)/2 > \gamma_p$ for $s \ge s_0$. If $\mu < \gamma_p$, then $\mu < \xi(s) < (\mu + \gamma_p)/2 < \gamma_p$ for s sufficiently large. In either case,

$$\dot{\xi}(s) \le -H_p(\xi(s)) \le -H_p((\mu + \gamma_p)/2) < 0$$

for s sufficiently large, which yields that $\xi(s)$ tends to $-\infty$ as $s \to \infty$. This contradicts the assumption that $\xi(s)$ is positive for $s \ge s_0$. Thus, $\xi(s)$ tends to γ_p as $s \to \infty$. The proof of Lemma 2.1 is complete. □

We next give a sufficient condition for all nontrivial solutions of (2.1) to be nonoscillatory.

Lemma 2.2. *Let $\xi(s)$ be a positive function on $[s_0, \infty)$ with $s_0 > 0$ satisfying*

$$\dot{\xi}(s) + H_p(\xi(s)) + h(e^s) \le 0, \tag{2.3}$$

where h is the function defined in equation (2.1). *Then all nontrivial solutions of* (2.1) *to be nonoscillatory.*

Proof. Define

$$c(s) = -\dot{\xi}(s) - H_p(\xi(s))$$

for $s \ge s_0$. Then we have

$$c(s) \ge h(e^s) \quad \text{for } s \ge s_0. \tag{2.4}$$

Let $u(s)$ be the positive function defined by

$$u(s) = \exp\left(\int_{s_0}^{s} \xi(\sigma)^{1/p} d\sigma\right)$$

for $s \ge s_0$. Then we get

$$\dot{u}(s) = u(s)\xi(s)^{1/p} > 0$$

for $s \ge s_0$, namely,

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^p \quad \text{for } s \ge s_0.$$

Differentiate $\xi(s)$ to obtain

$$\dot{\xi}(s) = \frac{(\dot{u}(s)^p)^{\cdot} u(s)^p - pu(s)^{p-1}\dot{u}(s)^{p+1}}{u(s)^{2p}} = \frac{(\dot{u}(s)^p)^{\cdot}}{u(s)^p} - p\left(\frac{\dot{u}(s)}{u(s)}\right)^{p+1}$$

for $s \ge s_0$. Hence, we have

$$\begin{aligned} c(s) &= -\frac{(\dot{u}(s)^p)^{\cdot}}{u(s)^p} + p\left(\frac{\dot{u}(s)}{u(s)}\right)^{p+1} - p\left\{\left(\frac{\dot{u}(s)}{u(s)}\right)^{p+1} - \left(\frac{\dot{u}(s)}{u(s)}\right)^{p} + \frac{p^p}{(p+1)^{p+1}}\right\} \\ &= -\frac{(\dot{u}(s)^p)^{\cdot}}{u(s)^p} + p\left(\frac{\dot{u}(s)}{u(s)}\right)^{p} - \left(\frac{p}{p+1}\right)^{p+1}, \end{aligned}$$

and therefore, we see that the positive function $u(s)$ is a nonoscillatory solution of the equation

$$\left(|\dot{u}|^{p-1}\dot{u}\right)^{\cdot} - p|\dot{u}|^{p-1}\dot{u} + \left\{\left(\frac{p}{p+1}\right)^{p+1} + c(s)\right\}|u|^{p-1}u = 0. \tag{2.5}$$

Changing variable $t = e^s$, we can transform equation (2.5) into the equation

$$\left(|x'|^{p-1}x'\right)' + \frac{1}{t^{p+1}}\left\{\left(\frac{p}{p+1}\right)^{p+1} + c(\log t)\right\}|x|^{p-1}x = 0. \tag{2.6}$$

Let $x(t)$ be the solution of (2.6) corresponding to $u(s)$. Then $x(t)$ is positive for $t \ge e^{s_0}$. From (2.4) it follows that

$$c(\log t) \ge h(t) \quad \text{for } t \ge e^{s_0}.$$

Hence, by Sturm's comparison theorem for half-linear differential equations, all nontrivial solutions of (2.1) are nonoscillatory. This completes the proof of Lemma 2.2. □

3 Proof of the main theorems

By means of Lemmas 2.1 and 2.2, we can prove our comparison theorems for half-linear differential equations of the form (E_α).

Proof of Theorem 1.1. By way of contradiction, we suppose that equation (E_β) has an oscillatory solution and equation (E_α) has a nonoscillatory solution $x(t)$. We may assume that $x(t)$ is eventually positive, because the proof of the case that $x(t)$ is eventually negative is carried out in the same way. Hence, there exists a $T > t_0$ such that $x(t) > 0$ for $t \geq T$, and therefore,

$$\left(|x'(t)|^{\alpha-1}x'(t)\right)' = -\frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \gamma_\alpha\delta(t)\right\}|x(t)|^{\alpha-1}x(t) < 0 \tag{3.1}$$

for $t \geq T$. From this we see that $x'(t)$ is also positive for $t \geq T$. In fact, if there exists a $t_1 \geq T$ such that $x'(t_1) \leq 0$, then by (3.1) we have

$$|x'(t)|^{\alpha-1}x'(t) < |x'(t_1)|^{\alpha-1}x'(t_1) \leq 0$$

for $t > t_1$. Hence, we can find a $t_2 > t_1$ such that $x'(t_2) < 0$. By (3.1) again, we obtain

$$|x'(t)|^{\alpha-1}x'(t) \leq |x'(t_2)|^{\alpha-1}x'(t_2) < 0$$

for $t \geq t_2$. We therefore conclude that $x'(t) \leq x'(t_2) < 0$ for $t \geq t_2$, which implies that

$$x(t) \leq x'(t_2)(t - t_2) + x(t_2) \to -\infty$$

as $t \to \infty$. This is a contradiction to the assumption that $x(t)$ is eventually positive.

Making the change of variable $s = \log t$, we can rewrite equation (E_α) in the form

$$\left(|\dot{u}|^{\alpha-1}\dot{u}\right)^{\cdot} - \alpha|\dot{u}|^{\alpha-1}\dot{u} + \left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \gamma_\alpha\delta(e^s)\right\}|u|^{\alpha-1}u = 0. \tag{3.2}$$

Let $u(s)$ be the solution of (3.2) which corresponds to $x(t)$. Then $u(s) = x(t) > 0$ and $\dot{u}(s) = tx'(t) > 0$ for $s \geq \log T$. Define

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^\alpha$$

and differentiate $\xi(s)$ to obtain

$$\dot{\xi}(s) = \frac{(\dot{u}(s)^\alpha)^{\cdot}}{u(s)^\alpha} - \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^{\alpha+1}.$$

Using (3.2), we have

$$\begin{aligned}\dot{\xi}(s) &= \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} - \gamma_\alpha\delta(e^s) - \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^{\alpha+1} \\ &= -\alpha\left\{\xi(s)^{(\alpha+1)/\alpha} - \xi(s) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right\} - \gamma_\alpha\delta(e^s) \\ &= -H_\alpha(\xi(s)) - \gamma_\alpha\delta(e^s)\end{aligned} \tag{3.3}$$

for $s \geq \log T$.

We here show that there exists an $\varepsilon_0 > 0$ such that

$$\frac{\gamma_\alpha}{\gamma_\beta} H_\beta\left(\frac{\gamma_\beta}{\gamma_\alpha}\xi\right) \leq H_\alpha(\xi) \tag{3.4}$$

for $\gamma_\alpha \leq \xi \leq \gamma_\alpha + \varepsilon_0$. For this purpose, we define

$$F_1(\xi) = H_\alpha(\xi) - \frac{\gamma_\alpha}{\gamma_\beta} H_\beta\left(\frac{\gamma_\beta}{\gamma_\alpha}\xi\right).$$

Then, differentiating $F_1(\xi)$ three times, we obtain

$$\frac{d}{d\xi}F_1(\xi) = (\alpha+1)\xi^{1/\alpha} - \alpha - (\beta+1)\left(\frac{\gamma_\beta}{\gamma_\alpha}\right)^{1/\beta}\xi^{1/\beta} + \beta,$$

$$\frac{d^2}{d\xi^2}F_1(\xi) = \frac{\alpha+1}{\alpha}\xi^{(1-\alpha)/\alpha} - \frac{\beta+1}{\beta}\left(\frac{\gamma_\beta}{\gamma_\alpha}\right)^{1/\beta}\xi^{(1-\beta)/\beta},$$

$$\frac{d^3}{d\xi^3}F_1(\xi) = \frac{1-\alpha^2}{\alpha^2}\xi^{(1-2\alpha)/\alpha} - \frac{1-\beta^2}{\beta^2}\left(\frac{\gamma_\beta}{\gamma_\alpha}\right)^{1/\beta}\xi^{(1-2\beta)/\beta},$$

so that

$$F_1(\gamma_\alpha) = \frac{d}{d\xi}F_1(\xi)\Big|_{\xi=\gamma_\alpha} = \frac{d^2}{d\xi^2}F_1(\xi)\Big|_{\xi=\gamma_\alpha} = 0 \tag{3.5}$$

and

$$\frac{d^3}{d\xi^3}F_1(\xi)\Big|_{\xi=\gamma_\alpha} = \frac{\beta-\alpha}{\alpha\beta}\left(\frac{\alpha+1}{\alpha}\right)^{2\alpha} > 0. \tag{3.6}$$

From (3.6) we can choose an $\varepsilon_0 > 0$ such that

$$\frac{d^3}{d\xi^3}F_1(\xi) > 0 \quad \text{for } \gamma_\alpha \leq \xi \leq \gamma_\alpha + \varepsilon_0.$$

Hence, taking account of this estimation and (3.5), we see that $F_1(\xi) \geq 0$ for $\gamma_\alpha \leq \xi \leq \gamma_\alpha + \varepsilon_0$, as required.

Because of (3.3), Lemma 2.1 is available for $p = \alpha$ and $s_0 = \log T$, and therefore, there exists an $s_1 > s_0$ such that

$$\gamma_\alpha \leq \xi(s) \leq \gamma_\alpha + \varepsilon_0$$

for $s \geq s_1$. Hence, together with (3.3) and (3.4), we get

$$\dot{\xi}(s) + \frac{\gamma_\alpha}{\gamma_\beta} H_\beta\left(\frac{\gamma_\beta}{\gamma_\alpha}\xi(s)\right) + \gamma_\alpha\delta(e^s) \leq 0$$

for $s \geq s_1$. Let $\eta(s) = \gamma_\beta\xi(s)/\gamma_\alpha$. Then we see that $\eta(s)$ satisfies

$$\dot{\eta}(s) + H_\beta(\eta(s)) + \gamma_\beta\delta(e^s) \leq 0$$

for $s \geq s_1$. Hence, from Lemma 2.2 with $p = \beta$ and $h(e^s) = \gamma_\beta \delta(e^s)$ we conclude that all nontrivial solutions of (E_β) are nonoscillatory. This contradicts the assumption that equation (E_β) has an oscillatory solution. Thus, we have completed the proof of Theorem 1.1. □

Proof of Theorem 1.2. Suppose to the contrary that equation (E_α) has an oscillatory solution and equation (1.3) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. Let $T > t_0$ be a number satisfying $x(t) > 0$ for $t \geq T$. From the same manner as in the proof of Theorem 1.1, we see that $x'(t)$ is also positive for $t \geq T$.

By putting $t = e^s$, equation (1.3) becomes

$$\left(|\dot{u}|^{\beta-1}\dot{u}\right)^{\cdot} - \beta|\dot{u}|^{\beta-1}\dot{u} + \left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + (\gamma_\beta + \varepsilon)\delta(e^s)\right\}|u|^{\beta-1}u = 0$$

for some $\varepsilon > 0$, where $u(s) = x(e^s) = x(t)$. Define

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^\beta,$$

which is positive for $s \geq \log T$. A simple calculation shows that

$$\dot{\xi}(s) = -H_\beta(\xi(s)) - (\gamma_\beta + \varepsilon)\delta(e^s) \tag{3.7}$$

for $s \geq \log T$. Hence, it follows from Lemma 2.1 with $p = \beta$ and $s_0 = \log T$ that

$$\xi(s) \searrow \gamma_\beta \quad \text{as} \quad s \to \infty. \tag{3.8}$$

Let

$$c = \frac{\gamma_\beta + \varepsilon}{\gamma_\alpha} \quad \text{and} \quad \eta(s) = \frac{\xi(s) + \varepsilon}{c}.$$

Then, from (3.7) and (3.8) it turns out that

$$\dot{\eta}(s) + \frac{1}{c}H_\beta(c\eta(s) - \varepsilon) + \gamma_\alpha\delta(e^s) = 0 \tag{3.9}$$

for $s \geq s_0$ and

$$\eta(s) \searrow \gamma_\alpha \quad \text{as} \quad s \to \infty, \tag{3.10}$$

respectively.

To show that there exists an $\varepsilon_0 > 0$ such that

$$H_\alpha(\eta) \leq \frac{1}{c}H_\beta(c\eta - \varepsilon) \tag{3.11}$$

for $\gamma_\alpha \leq \eta \leq \gamma_\alpha + \varepsilon_0$, we define

$$F_2(\eta) = \frac{1}{c}H_\beta(c\eta - \varepsilon) - H_\alpha(\eta).$$

Differentiating $F_2(\eta)$ twice, we have

$$\frac{d}{d\eta}F_2(\eta) = (\beta+1)(c\eta-\varepsilon)^{1/\beta} - \beta - (\alpha+1)\eta^{1/\alpha} + \alpha,$$

$$\frac{d^2}{d\eta^2}F_2(\eta) = \frac{c(\beta+1)}{\beta}(c\eta-\varepsilon)^{(1-\beta)/\beta} - \frac{\alpha+1}{\alpha}\eta^{(1-\alpha)/\alpha},$$

so that

$$F_2(\gamma_\alpha) = \frac{d}{d\xi}F_2(\eta)\Big|_{\eta=\gamma_\alpha} = 0$$

and

$$\frac{d^2}{d\xi^2}F_2(\eta)\Big|_{\eta=\gamma_\alpha} = \frac{\varepsilon}{\gamma_\alpha\gamma_\beta} > 0.$$

Hence, we can select an $\varepsilon_0 > 0$ such that

$$\frac{d^2}{d\xi^2}F_2(\eta) > 0 \quad \text{for } \gamma_\alpha \le \eta \le \gamma_\alpha + \varepsilon_0,$$

and therefore, $F_2(\eta) \ge 0$ for $\gamma_\alpha \le \eta \le \gamma_\alpha + \varepsilon_0$. Thus, the inequality (3.11) is shown.

By (3.10), there exists an $s_1 > s_0$ such that

$$\gamma_\alpha \le \eta(s) \le \gamma_\alpha + \varepsilon_0$$

for $s \ge s_1$. Hence, together with (3.9) and (3.11), we have

$$\dot\eta(s) + H_\alpha(\eta(s)) + \gamma_\alpha\delta(e^s) \le 0$$

for $s \ge s_1$. Using Lemma 2.2 with $p = \alpha$ and $h(e^s) = \gamma_\alpha\delta(e^s)$, we see that all nontrivial solutions of (E_α) are nonoscillatory. This is a contradiction to the assumption that equation (E_α) has an oscillatory solution. We have thus proved Theorem 1.2. □

4 Discussion and another comparison theorem

Let us now look at Theorem 1.2 from a different angle. To this end, we consider the more general half-linear differential equation

$$\left(|x'|^{\alpha-1}x'\right)' + a(t)|x|^{\alpha-1}x = 0, \tag{4.1}$$

where $\alpha > 0$ and $a(t)$ is positive and continuous on (t_0,∞) for some $t_0 \ge 0$. Then, we can guarantee that all solutions of (4.1) are continuable in the future. Hence, it is worth while to discuss whether solutions of (4.1) are oscillatory or not.

The Hille-Wintner comparison theorem has been widely studied by many authors. For example, Kusano and Yoshida [9] presented the following comparison theorem of Hille-Wintner type for half-linear differential equations (see also [10]).

Theorem C. *Consider*

$$\left(|x'|^{\alpha-1}x'\right)' + b(t)|x|^{\alpha-1}x = 0, \tag{4.2}$$

where $b(t)$ *is positive and continuous on* (t_0, ∞). *Suppose that*

$$\int_t^\infty a(s)ds \le \int_t^\infty b(s)ds$$

for all sufficiently large t. *If all nontrivial solutions of* (4.1) *are oscillatory, then those of* (4.2) *are also oscillatory.*

We can regard the number α in equations (4.1) and (4.2) as a positive parameter. In Theorem C, needless to say, the parameter α is fixed and the integral of the coefficient $a(t)$ is compared with that of the coefficient $b(t)$. Let us fix the coefficient $a(t)$ and move the parameter α to the contrary. Then we have another comparison theorem for half-linear differential equations.

Theorem 4.1. *Consider*

$$\left(|x'|^{\beta-1}x'\right)' + a(t)|x|^{\beta-1}x = 0, \tag{4.3}$$

where $a(t)$ *is the same as in equation* (4.1). *Suppose that* $0 < \alpha < \beta$. *If all nontrivial solutions of* (4.1) *are oscillatory, then those of* (4.3) *are also oscillatory.*

Proof. The proof is by contradiction. We suppose that all nontrivial solutions of (4.1) are oscillatory and equation (4.3) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. As in the proof of Theorem 1.1, we see that $x'(t)$ is also eventually positive.

Define the function $\xi(t)$ by

$$\xi(t) = \left(\frac{x'(t)}{x(t)}\right)^\beta.$$

Then there exists a $T > t_0$ such that $\xi(t) > 0$ and

$$\xi'(t) = -a(t) - \beta\xi(t)^{(\beta+1)/\beta} < 0 \tag{4.4}$$

for $t \ge T$, namely, $\xi(t)$ is decreasing and bounded from below. Hence, we can find a $\mu \ge 0$ such that

$$\xi(t) \searrow \mu \quad \text{as} \quad t \to \infty,$$

and therefore, we have

$$\xi'(t) = -a(t) - \beta\xi(t)^{(\beta+1)/\beta} \le -\beta\mu^{(\beta+1)/\beta}$$

for $t \ge T$. If $\mu > 0$, then $\xi(t)$ has to tend to $-\infty$ as $t \to \infty$. This contradicts the fact that $\xi(t)$ is eventually positive. Thus, $\xi(t)$ tends to zero as $t \to \infty$. From this property of $\xi(t)$ and the assumption that $0 < \alpha < \beta$, we see that there exists a $t_1 > T$ such that

$$\alpha\xi(t)^{(\alpha+1)/\alpha} \le \beta\xi(t)^{(\beta+1)/\beta}$$

for $t \geq t_1$. Hence, together with (4.4), we have

$$\xi'(t) \leq -a(t) - \alpha\xi(t)^{(\alpha+1)/\alpha} \tag{4.5}$$

for $t \geq t_1$.

It is easy to check that the function

$$y(t) = \exp\left(\int_{t_1}^{t} \xi(\tau)^{1/\alpha} d\tau\right)$$

is a nonoscillatory solution of

$$\left(|x'|^{\alpha-1}x'\right)' + b(t)|x|^{\alpha-1}x = 0,$$

where $b(t) = -\xi'(t) - \alpha\xi(t)^{(\alpha+1)/\alpha}$. From (4.5) it follows that $a(t) \leq b(t)$ for $t \geq t_1$. Hence, Sturm's comparison theorem implies that (4.1) also has a nonoscillatory solution. This is a contradiction, thereby completing the proof of Theorem 4.1. □

In the case that

$$t^{\alpha+1}a(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{4.6}$$

for t sufficiently large, we can rewrite equation (4.1) in the form (E_α) with

$$\delta(t) = \left(\frac{\alpha+1}{\alpha}\right)^{\alpha} t^{\alpha+1}a(t) - \frac{\alpha}{\alpha+1} > 0.$$

Suppose that all nontrivial solutions of (4.1) are oscillatory. Then, from Theorem 1.2 we see that all nontrivial solutions of

$$\left(|x'|^{\beta-1}x'\right)' + c(t)|x|^{\beta-1}x = 0$$

with

$$c(t) = \frac{1}{t^{\beta+1}}\left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + \left(\left(\frac{\beta}{\beta+1}\right)^{\beta} + \varepsilon\right)\delta(t)\right\}$$

are oscillatory. Since $0 < \alpha < \beta$, we have

$$\begin{aligned} c(t) &= \frac{1}{t^{\beta+1}}\left\{\left(\frac{\beta}{\beta+1}\right)^{\beta+1} + \left(\left(\frac{\beta}{\beta+1}\right)^{\beta} + \varepsilon\right)\delta(t)\right\} \\ &< \frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\delta(t)\right\} = a(t) \end{aligned}$$

for t sufficiently large. Hence, from Theorem C we conclude that all nontrivial solutions of (4.3) are also oscillatory. This means that Theorem 1.2 is sharper than Theorem 4.1 in the case (4.6).

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