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Kneser's property in C^1-norm for ordinary differential equations

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Let D be an open subset of $\mathbf{R}\times\mathbf{R}^n$. We consider an initial value problem

(1) $$x'=f(t,x), \quad x(0)=\xi,$$

where the the prime denotes the differentiation with respect to t, $(0,\xi)\in D$ and $f:D\to\mathbf{R}^n$ is continuous. H. Kneser proved the following theorem (see Theorem 4.1, p.15 in [1]).

Theorem (Kneser). For every $(0,\xi)\in D$, a set

$$\{x(\tau)\,;\,x \text{ is a solution of (1)}\}$$

is compact and connected in $\mathbf{R}^n$ when $\tau>0$ is sufficiently small.

For simplicity, we assume that $D=[0,1]\times\mathbf{R}^n$ and that f is bounded and continuous. Namely, we suppose that there exists a positive constant M satisying

(2) $$|f(t,x)|\leq M \qquad \text{for } (t,x)\in[0,1]\times\mathbf{R}^n,$$

where $|\cdot|$ denotes any norm in $\mathbf{R}^n$. In this case, the above theorem is reduced to the following theorem.

Theorem 1. For every $\xi\in\mathbf{R}^n$, a set

$$\{x(1)\,;\,x \text{ is a solution of (1)}\}$$

is compact and connected in $\mathbf{R}^n$.

For any $a,\ b\in\mathbf{R}$ with $a<b$, let $C[a,b]$ denote the Banach space of all $\mathbf{R}^n$-valued continuous mappings on $[a,b]$ with the norm $\|\cdot\|$ defined by $\|x\|=\sup_{a\leq t\leq b}|x(t)|$. Similarly, we denote by $C^1[a,b]$ the Banach space of all $\mathbf{R}^n$-valued continuously differentiable mappings on $[a,b]$ with the norm $\|\cdot\|_1$ defined by $\|x\|_1=\max\{\|x\|,\|x'\|\}$.

It is well known that Theorem 1 is extended to the following theorem.

Theorem 2. A set

(3) $$K:=\{x\,;\,x \text{ is a solution of (1)}\}$$

is compact and connected in $C[0,1]$ for every $\xi\in\mathbf{R}^n$.

Since the set K given in (3) is included in $C^1[0,1]$, it might be natural to discuss the property of the set K in the topology of $C^1[0,1]$. In this article, we shall introduce the following theorem.

Theorem 3. The set K given in (3) is compact and connected in $C^1[0,1]$ for every $\xi \in \mathbf{R}^n$.

Proof. First we shall show that K is compact in $C^1[0,1]$. Let $\{x_k\}$ be any sequence in K. It follows from (2) that $|x_k'(t)| \le M$ for $0 \le t \le 1$, and hense $\{x_k\}$ is equicontinuous and uniformly bounded on $[0,1]$ because $x_k(0) = \xi$. Then we may assume, by Ascoli-Arzelà's theorem, that $\{x_k\}$ converges to some x in $C[0,1]$ by taking a subsequence if necessary. Since x_k satisfies an equality

$$x_k(t) = \xi + \int_0^t f(s, x_k(s))\, ds,$$

x satisfies that $x(t) = \xi + \int_0^t f(s, x(s))\, ds$, which implies that $x \in K$. Let L be a compact subset of $\mathbf{R}^n$ defined by

$$L = \{x \in \mathbf{R}^n \ ; \ |x| \le |\xi| + M\}. \tag{4}$$

Then $x_k(t) \in L$ for $0 \le t \le 1$. Since f is uniformly continuous on a compact set $[0,1] \times L$, it follows that

$$x_k'(t) = f(t, x_k(t)) \to f(t, x(t)) = x'(t) \quad \text{as} \quad k \to \infty$$

uniformly for $t \in [0,1]$. Therefore, $\{x_k\}$ converges to x in $C^1[0,1]$, which shows that K is compact in $C^1[0,1]$.

Now we shall show that K is connected. Suppose that K is not connected. Then there exist two nonempty compact sets K_1 and K_2 such that $K_1 \cup K_2 = K$ and that $K_1 \cap K_2 = \emptyset$. It is easy to find an open set G in $C^1[0,1]$ satisfying $K_1 \subset G$ and $\overline{G} \cap K_2 = \emptyset$, where $\overline{G}$ denotes the closure of G. Therefore, we obtain that

$$\partial G \cap K = \emptyset, \tag{5}$$

where ∂G denotes the boundary of G. Let x and y be any fixed elements in K_1 and K_2, respectively.

For any fixed small number $\varepsilon > 0$ and a number T satisfying $0 \le T \le 1$, define a mapping $\varphi : [0,1] \to \mathbf{R}^n$ by

$$\varphi(t) = \begin{cases} x(t) & \text{for } 0 \le t \le T \\ x(T) + \int_T^t f(s, x(T))\, ds & \text{for } T \le t \le T + \varepsilon \\ \varphi(T+\varepsilon) + \int_{T+\varepsilon}^t f(s, \varphi(s-\varepsilon))\, ds & \text{for } T + \varepsilon \le t \le 1. \end{cases} \tag{6}$$

It is not difficult to observe that φ belongs to $C^1[0,1]$. We denote the mapping φ

by φ_T. Clearly, φ_T coincides with x when $T = 1$, while φ_T does not depend on x.

We shall show that the correspondence $T \mapsto \varphi_T$ is a continuous mapping from $[0,1]$ into $C^1[0,1]$. Let $T \in [0,1]$ be fixed, and let $\{T_k\}$ be any sequence in $[0,1]$ converging to T. For simplicity, we denote φ_{T_K} and φ_T, respectively, by φ_k and φ. It will be verified that $\{\varphi_k\}$ converges to φ in $C^1[0,1]$ as $k \to \infty$ in the following two cases where $T_k > T$ holds for $k \in \mathbf{N}$ and $T_k < T$ holds for $k \in \mathbf{N}$. Since $\varepsilon > 0$ and $T_k \to T$ as $k \to \infty$, we may assume that

$$(7) \qquad |T_k - T| < \varepsilon \quad \text{for every } k \in \mathbf{N}.$$

(i) In the case where $T_k > T$ holds for $k \in \mathbf{N}$. It follows from (6) that φ_k is expressed as

$$(8) \quad \varphi_k(t) = \begin{cases} x(t) & \text{for } \ 0 \le t \le T_k, \\ x(T_k) + \displaystyle\int_{T_k}^{t} f(s, x(T_k))\, ds & \text{for } \ T_k \le t \le T_k + \varepsilon, \\ \varphi_k(T_k + \varepsilon) + \displaystyle\int_{T_k+\varepsilon}^{t} f(s, \varphi_k(s - \varepsilon))\, ds & \text{for } \ T_k + \varepsilon \le t \le 1. \end{cases}$$

Since $T_k > T$, an equality $\varphi_k(t) = \varphi(t) = x(t)$ holds for $t \in [0,T]$.

We shall observe that

$$(9) \qquad |\varphi_k(t) - \varphi(t)| \le 2M(T_k - T) + \int_T^{T+\varepsilon} |f(t, x(T_k)) - f(t, x(T))|\, ds \quad \text{for } t \in [T, T+\varepsilon]$$

and

$$(10) \qquad |\varphi_k'(t) - \varphi'(t)| \le \sup_{t\in[T,T_k]} |f(t,x(t)) - f(t,x(T))| + \sup_{t\in[T_k,T+\varepsilon]} |f(t,x(T_k)) - f(t,x(T))| \quad \text{for } t \in [T, T+\varepsilon]$$

hold, where M is the positive constant satisfying (2). Here, notice that an inequality $T < T_k < T + \varepsilon$ holds by assumption (7). For any $t \in [T, T_k]$, we have

$$\begin{aligned} \varphi_k(t) - \varphi(t) &= x(T) + \int_T^t f(s, x(s))\, ds - \left\{ x(T) + \int_T^t f(s, x(T))\, ds \right\} \\ &= \int_T^t \{f(s,x(s)) - f(s,x(T))\}\, ds \end{aligned}$$

and hence it follows from (2) that

$$(11) \qquad |\varphi_k(t) - \varphi(t)| \le 2M(T_k - T) \quad \text{for } \ t \in [T, T_k].$$

Furthermore, we have, by (6) and (8), that

$$(12) \qquad \varphi_k'(t) - \varphi'(t) = f(t, x(t)) - f(t, x(T)) \quad \text{for } \ t \in [T, T_k].$$

On the other hand, for $t \in [T_k, T+\varepsilon]$, it follows, respectively, from (6) and (8) that

$$\begin{aligned}\varphi_k(t) &= x(T_k) + \int_{T_k}^{t} f(s, x(T_k))\,ds \\ &= x(T) + \int_{T}^{T_k} f(s, x(s))\,ds + \int_{T_k}^{t} f(s, x(T_k))\,ds\end{aligned}$$

and that

$$\begin{aligned}\varphi(t) &= x(T) + \int_{T}^{t} f(s, x(T))\,ds \\ &= x(T) + \int_{T}^{T_k} f(s, x(T))\,ds + \int_{T_k}^{t} f(s, x(T))\,ds,\end{aligned}$$

and hence, we have

$$|\varphi_k(t) - \varphi(t)| \le 2M(T_k - T) + \int_{T}^{T+\varepsilon} |f(s, x(T_k)) - f(s, x(T))|\,ds. \tag{13}$$

Furthermore, it is clear that the following equality holds.

$$\varphi_k'(t) - \varphi'(t) = f(t, x(T_k)) - f(t, x(T)) \quad \text{for } t \in [T_k, T+\varepsilon]. \tag{14}$$

It then follows from (11) and (13) that (9) holds. Inequality (10) is a direct concequence of (12) and (14). Thus, we obtain, by (9) and (10), that

$$\varphi_k \to \varphi \quad \text{in } C^1[0, T+\epsilon] \quad \text{as } k \to \infty. \tag{15}$$

Now we shall estimate $|\varphi_k(t) - \varphi(t)|$ and $|\varphi_k'(t) - \varphi'(t)|$ on the interval $[T+\varepsilon, 1]$. For any $t \in [T+\varepsilon, T+2\varepsilon]$, it will be verified that the following inequality holds.

$$\begin{aligned}|\varphi_k(t) - \varphi(t)| \le{}& 4M(T_k - T) + \varepsilon \sup_{s\in[T,T+\varepsilon]} |f(s, x(T_k)) - f(s, x(T))| \\ &+ |\varphi_k(T+\varepsilon) - \varphi(T+\varepsilon)| \\ &+ \int_{T+\varepsilon}^{T+2\varepsilon} |f(s, \varphi_k(s-\varepsilon)) - f(s, \varphi(s-\varepsilon))|\,ds.\end{aligned} \tag{16}$$

When $t \in [T+\varepsilon, T_k+\varepsilon]$, it follows from (6) and (8) that

$$\begin{aligned}|\varphi_k(t) - \varphi(t)| \le{}& \int_{T}^{T_k} |f(s, x(s)) - f(s, x(T))|\,ds \\ &+ \int_{T_k}^{T+\varepsilon} |f(s, x(T_k)) - f(s, x(T))|\,ds \\ &+ \int_{T+\varepsilon}^{t} |f(s, x(T_k)) - f(s, \varphi(s-\varepsilon))|\,ds \\ \le{}& 2M(T_k - T) + \int_{T}^{T+\varepsilon} |f(s, x(T_k)) - f(s, x(T))|\,ds \\ &+ \int_{T+\varepsilon}^{T_k+\varepsilon} |f(s, x(T_k)) - f(s, \varphi(s-\varepsilon))|\,ds\end{aligned}$$

$$\leq 4M(T_k - T) + \int_T^{T+\varepsilon} |f(s, x(T_k)) - f(s, x(T))|\, ds$$

$$\leq 4M(T_k - T) + \varepsilon \sup_{s\in[T,T+\varepsilon]} |f(s, x(T_k)) - f(s, x(T))|. \tag{17}$$

When $t \in [T_k + \varepsilon, T + 2\varepsilon]$, φ_k and φ are expressed, respectively, as

$$\varphi_k(t) = \varphi_k(T_k + \varepsilon) - \varphi_k(T + \varepsilon) + \varphi_k(T + \varepsilon) + \int_{T_k+\varepsilon}^t f(s, \varphi_k(s - \varepsilon))\, ds$$

$$= \int_{T+\varepsilon}^{T_k+\varepsilon} f(s, x(T_k))\, ds + \varphi_k(T + \varepsilon) + \int_{T_k+\varepsilon}^t f(s, \varphi_k(s - \varepsilon))\, ds$$

and

$$\varphi(t) = \varphi(T + \varepsilon) + \int_{T+\varepsilon}^{T_k+\varepsilon} f(s, \varphi(s - \varepsilon))\, ds + \int_{T_k+\varepsilon}^t f(s, \varphi(s - \varepsilon))\, ds.$$

Therefore, we have, for $t \in [T_k + \varepsilon, T + 2\varepsilon]$,

$$\begin{aligned}|\varphi_k(t) - \varphi(t)| &\leq |\varphi_k(T + \varepsilon) - \varphi(T + \varepsilon)| \\ &\quad + \int_{T+\varepsilon}^{T_k+\varepsilon} |f(s, x(T_k)) - f(s, \varphi(s - \varepsilon))|\, ds \\ &\quad + \int_{T_k+\varepsilon}^t |f(s, \varphi_k(s - \varepsilon)) - f(s, \varphi(s - \varepsilon))|\, ds \\ &\leq |\varphi_k(T + \varepsilon) - \varphi(T + \varepsilon)| + 2M(T_k - T) \\ &\quad + \int_{T+\varepsilon}^{T+2\varepsilon} |f(s, \varphi_k(s - \varepsilon)) - f(s, \varphi(s - \varepsilon))|\, ds.\end{aligned}$$

It then follows from this inequality and (17) that (16) holds for $t \in [T + \varepsilon, T + 2\varepsilon]$. Thus, we have $|\varphi_k(t) - \varphi(t)| \to 0$ as $k \to \infty$ uniformly on $[T + \varepsilon, T + 2\varepsilon]$ because of (15) and the uniform continuity of f on $[0, 1] \times L$.

We have to confirm that $|\varphi_k'(t) - \varphi'(t)| \to 0$ as $k \to \infty$ uniformly on $[T+\varepsilon, T+2\varepsilon]$. For $t \in [T + \varepsilon, T_k + \varepsilon]$, it follows that

$$\varphi_k'(t) - \varphi'(t) = f(t, x(T_k)) - f(t, \varphi(t - \varepsilon)). \tag{18}$$

Since $\varphi(t - \varepsilon) = x(T) + \int_T^{t-\varepsilon} f(s, x(T))\, ds$ and $T \leq t - \varepsilon \leq T_k$ hold, we have

$$|x(T_k) - \varphi(t - \varepsilon)| \leq |x(T_k) - x(T)| + M(T_k - T) \text{ for } t \in [T + \varepsilon, T_k + \varepsilon]. \tag{19}$$

For $t \in [T_k + \varepsilon, T + 2\varepsilon]$, we have evidently that

$$\varphi_k'(t) - \varphi'(t) = f(t, \varphi_k(t - \varepsilon)) - f(t, \varphi(t - \varepsilon)). \tag{20}$$

It follows from (18), (19) and (20) that $|\varphi_k'(t) - \varphi'(t)| \to 0$ as $k \to 0$ uniformly for $t \in [T + \varepsilon, T + 2\varepsilon]$. Therefore, we obtain that $\varphi_k \to \varphi$ in $C^1[0, T + 2\varepsilon]$ as $k \to \infty$. Repeating this procedure, we get that $\varphi_k \to \varphi$ in $C^1[0, 1]$ as $k \to \infty$.

(ii) In the case where $T_k < T$ holds for $k \in \mathbf{N}$. When $t \in [0, T_k]$, we have that $\varphi_k(t) = \varphi(t)$ holds. For $t \in [T_k, T]$, it follows from (2), (6) and (8) that

$$|\varphi_k(t) - \varphi(t)| \le \int_{T_k}^{t} |f(s, x(T_k)) - f(s, x(s))|\, ds \le 2M(T - T_k).$$

Therefore, $\{\varphi_k\}$ converges to φ uniformly on $[0, T]$. Furthermore, for $t \in [T_k, T]$, we have that

$$\varphi_k'(t) - \varphi'(t) = f(t, x(T_k)) - f(t, x(t))$$

and that $|x(T_k) - x(t)| \le M(t - T_k) \le M(T - T_k) \to 0$ as $k \to \infty$. Therefore, it follows that $\{\varphi_k'\}$ converges to φ' uniformly on $[0, T]$, and hence we obtain that

$$\varphi_k \to \varphi \ \text{ in } \ C^1[0, T] \ \text{ as } \ k \to \infty. \tag{21}$$

Now we shall show that, for $t \in [T, T + \varepsilon]$,

$$|\varphi_k(t) - \varphi(t)| \le 4M(T - T_k) + \varepsilon \sup_{s \in [T, T+\varepsilon]} |f(s, x(T_k)) - f(s, x(T))|. \tag{22}$$

For $t \in [T, T_k + \varepsilon]$, φ_k and φ are expressed, respectively, as

$$\varphi_k(t) = x(T_k) + \int_{T_k}^{T} f(s, x(T_k))\, ds + \int_{T}^{t} f(s, x(T_k))\, ds$$

and

$$\varphi(t) = x(T_k) + \int_{T_k}^{T} f(s, x(s))\, ds + \int_{T}^{t} f(s, x(T))\, ds,$$

which imply that

$$\begin{aligned} |\varphi_k(t) - \varphi(t)| &\le 2M(T - T_k) + \int_{T}^{t} |f(s, x(T_k)) - f(s, x(T))|\, ds \\ &\le 2M(T - T_k) + \varepsilon \sup_{s \in [T, T+\varepsilon]} |f(s, x(T_k)) - f(s, x(T))|. \end{aligned} \tag{23}$$

For $t \in [T_k + \varepsilon, T + \varepsilon]$, φ_k and φ are expressed, respectively, as

$$\begin{aligned} \varphi_k(t) = x(T_k) &+ \int_{T_k}^{T} f(s, x(T_k))\, ds + \int_{T}^{T_k+\varepsilon} f(s, x(T_k))\, ds \\ &+ \int_{T_k+\varepsilon}^{t} f(s, \varphi_k(s - \varepsilon))\, ds \end{aligned}$$

and

$$\begin{aligned} \varphi(t) = x(T_k) &+ \int_{T_k}^{T} f(s, x(s))\, ds + \int_{T}^{T_k+\varepsilon} f(s, x(T))\, ds \\ &+ \int_{T_k+\varepsilon}^{t} f(s, x(T))\, ds, \end{aligned}$$

which imply that

$$|\varphi_k(t) - \varphi(t)| \le \int_{T_k}^{T} |f(s, x(T_k)) - f(s, x(s))|\, ds$$

$$
\begin{aligned}
&+\int_{T}^{T_k+\varepsilon} |f(s,x(T_k)) - f(s,x(T))|\,ds \\
&+\int_{T_k+\varepsilon}^{t} |f(s,\varphi_k(s-\varepsilon)) - f(s,x(T))|\,ds \\
\le\ & 2M(T-T_k) + \int_{T}^{T_k+\varepsilon} |f(s,x(T_k)) - f(s,x(T))|\,ds \\
&+\int_{T_k+\varepsilon}^{T+\varepsilon} |f(s,\varphi_k(s-\varepsilon)) - f(s,x(T))|\,ds \\
\le\ & 4M(T-T_k) + \int_{T}^{T+\varepsilon} |f(s,x(T_k)) - f(s,x(T))|\,ds \\
\le\ & 4M(T-T_k) + \varepsilon \sup_{s\in[T,T+\varepsilon]} |f(s,x(T_k)) - f(s,x(T))|.
\end{aligned}
$$

It follows from this inequality and (23) that (22) holds. Therefore, we have that

(24) $$\varphi_k(t) \to \varphi(t) \quad \text{uniformly for} \quad t \in [T, T+\varepsilon] \quad \text{as } k \to \infty.$$

On the interval $[T, T+\varepsilon]$, we have

$$
\varphi_k'(t) - \varphi'(t) = \begin{cases} f(t,x(T_k)) - f(t,x(T)) & \text{for } \ t \in [T, T_k+\varepsilon], \\ \\ f(t,\varphi_k(t-\varepsilon)) - f(t,x(T)) & \text{for } \ t \in [T_k+\varepsilon, T+\varepsilon]. \end{cases}
$$

For $t \in [T_k+\varepsilon, T+\varepsilon]$, notice that $\varphi_k(t-\varepsilon)$ is expressed as

$$\varphi_k(t-\varepsilon) = x(T_k) + \int_{T_k}^{t-\varepsilon} f(s,x(T_k))\,ds,$$

and hence, we have

$$|\varphi_k(t-\varepsilon) - x(T)| \le |x(T_k) - x(T)| + \int_{T_k}^{t-\varepsilon} |f(s,x(T_k))|\,ds \le 2M(T-T_k).$$

Therefore, we obtain that $\varphi_k'(t) - \varphi'(t) \to 0$ as $k \to \infty$ uniformly for $t \in [T, T+\varepsilon]$. Which, together with (21) and (23), implies that

(25) $$\varphi_k \to \varphi \ \text{ in } \ C^1[0, T+\varepsilon] \ \text{ as } \ k \to \infty.$$

For $t \in [T+\varepsilon, T+2\varepsilon]$, φ_k and φ are, respectively, expressed as

$$\varphi_k(t) = \varphi_k(T_k+\varepsilon) + \int_{T_k+\varepsilon}^{T+\varepsilon} f(s,\varphi_k(s-\varepsilon))\,ds + \int_{T+\varepsilon}^{t} f(s,\varphi_k(s-\varepsilon))\,ds$$

and

$$\varphi(t) = \varphi(T_k+\varepsilon) + \int_{T_k+\varepsilon}^{T+\varepsilon} f(s,x(T))\,ds + \int_{T+\varepsilon}^{t} f(s,\varphi(s-\varepsilon))\,ds,$$

and hence, it follows that

$$
\begin{aligned}
|\varphi_k(t) - \varphi(t)| \le\ & |\varphi_k(T_k+\varepsilon) - \varphi(T_k+\varepsilon)| \\
&+ \int_{T_k+\varepsilon}^{T+\varepsilon} |f(s,\varphi_k(s-\varepsilon)) - f(s,x(T))|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{T+\varepsilon}^{t} |f(s,\varphi_k(s-\varepsilon)) - f(s,\varphi(s-\varepsilon))|\,ds \\
&\le |\varphi_k(T_k+\varepsilon) - \varphi(T_k+\varepsilon)| + 2M(T-T_k) \\
&\quad + \int_{T+\varepsilon}^{T+2\varepsilon} |f(s,\varphi_k(s-\varepsilon)) - f(s,\varphi(s-\varepsilon))|\,ds \\
&\le |\varphi_k(T_k+\varepsilon) - \varphi(T_k+\varepsilon)| + 2M(T-T_k) \qquad (26)\\
&\quad + \varepsilon \sup_{s\in[T+\varepsilon,T+2\varepsilon]} |f(s,\varphi_k(s-\varepsilon)) - f(s,\varphi(s-\varepsilon))|.
\end{aligned}
$$

We note, by (25), that

$$\varphi_k(s-\varepsilon) \to \varphi(s-\varepsilon) \quad \text{uniformly for } s \in [T+\varepsilon, T+2\varepsilon] \text{ as } k \to \infty,$$

which shows, by (26), that

$$|\varphi_k(t) - \varphi(t)| \to 0 \text{ as } k \to \infty \text{ uniformly for } t \in [T+\varepsilon, T+2\varepsilon].$$

Moreover, we also obtain that

$$\varphi_k'(t) - \varphi'(t) = f(t,\varphi_k(t-\varepsilon)) - f(t,\varphi(t-\varepsilon)) \to 0 \quad \text{as} \quad k \to \infty$$

uniformly for $t \in [T+\varepsilon, T+2\varepsilon]$. These facts, together with (25), imply that

$$\varphi_k \to \varphi \quad \text{in} \quad C^1[0, T+2\varepsilon] \quad \text{as} \quad k \to \infty.$$

Repeating this procedure, we get that $\{\varphi_k\}$ converges to φ in $C^1[0,1]$. Thus, we proved the continuity of the mapping $T \mapsto \varphi_T$.

Similarly to φ_T, we can define a mapping $\psi = \psi_T : [0,1] \to \mathbf{R}^n$ by using y instead of x for the above $\varepsilon > 0$ and T. We note that ψ_T coincides with y when $T = 1$ while ψ_T does not depend on y when $T = 0$. Moreover, the mapping $[0,1] \ni T \mapsto \psi_T \in C^1[0,1]$ is coutinuous. Here, notice that φ_T coincides with ψ_T when $T = 0$. Since $x \in G$ while $y \notin G$, we can choose a T with $0 \le T < 1$ satisfying

$$\varphi_T \in \partial G \quad \text{or} \quad \psi_T \in \partial G.$$

We denote the above T by $T(\varepsilon)$. For any fixed sequence $\{\varepsilon_k\}$ of positive numbers converging to 0, we denote $T(\varepsilon_k)$ by T_k. Moreover, the mappings φ_{T_k} and ψ_{T_k} will be donoted, respectively, by φ_k and ψ_k. We may assume, without loss of generality, that the relation $\varphi_k \in \partial G$ holds for every $k \in \mathbf{N}$. It follows from (6) that φ_k satisfies the following three equalities;

$$\varphi_k(t) = x(t) \quad \text{for} \quad t \in [0, T_k], \tag{27}$$

$$\varphi_k(t) = x(T_k) + \int_{T_k}^{t} f(s, x(T_k))\,ds \quad \text{for} \quad t \in [T_k, T_k+\varepsilon_k], \tag{28}$$

$$\varphi_k(t) = x(T_k+\varepsilon_k) + \int_{T_k+\varepsilon_k}^{t} f(s, \varphi_k(s-\varepsilon_k))\,ds \quad \text{for} \quad t \in [T_k+\varepsilon_k, 1]. \tag{29}$$

Therefore, we have that $|\varphi_k'(t)| \le M$ for $t \in [0,1]$ and that $\varphi_k(0) = \xi$, and hence, by Ascoli-Arzelà's theorem, we may assume that $\{\varphi_k\}$ converges to some $\bar{\varphi}$ in $C[0,1]$ by taking a subsequence if necessary. Furthermore, we may assume that $\{T_k\}$ converges to some T_0 in $[0,1]$.

It is clear from (27) that $\bar{\varphi}(t) = x(t)$ holds for $0 \le t < T_0$. By letting $k \to \infty$ in (28), we have that $\bar{\varphi}(T_0) = x(T_0)$. For any t with $T_0 < t \le 1$, an inequality $T_k < T_k + \varepsilon_k < t$ holdes for large k, it then follows from (29) that

$$\bar{\varphi}(t) = x(T_0) + \int_{T_0}^{t} f(s, \bar{\varphi}(s))\, ds \quad \text{for } T_0 < t \le 1.$$

These facts show that $\bar{\varphi}$ is a solution of (1), namely, $\bar{\varphi} \in K$.

Now we shall show that $\{\varphi_k\}$ converges to $\bar{\varphi}$ in $C^1[0,1]$. For every $k \in \mathbf{N}$, let $\bar{\varphi}_k$ be a mapping defined by

$$\text{(30)} \qquad \bar{\varphi}_k(t) = \begin{cases} \bar{\varphi}(t) & \text{for } 0 \le t \le T_k \\ \bar{\varphi}(T_k) & \text{for } T_k \le t \le T_k + \varepsilon_k \\ \bar{\varphi}(t - \varepsilon_k) & \text{for } T_k + \varepsilon_k \le t \le 1. \end{cases}$$

Then, it is clear that $\bar{\varphi}_k(t) \to \bar{\varphi}(t)$ uniformly for $t \in [0,1]$ as $k \to \infty$. Furthermore, it follows from (27) through (29) that φ_k' satisfies the following eqality

$$\text{(31)} \qquad \varphi_k'(t) = \begin{cases} f(t, \varphi_k(t)) & \text{for } 0 \le t \le T_k \\ f(t, \varphi_k(T_k)) & \text{for } T_k \le t \le T_k + \varepsilon_k \\ f(t, \varphi_k(t - \varepsilon_k)) & \text{for } T_k + \varepsilon_k \le t \le 1. \end{cases}$$

Since $\bar{\varphi}$ is a solution of (1), we have an inequality

$$\text{(32)} \qquad |\varphi_k'(t) - \bar{\varphi}'(t)| \le |\varphi_k'(t) - f(t, \bar{\varphi}_k(t))| + |f(t, \bar{\varphi}_k(t)) - f(t, \bar{\varphi}(t))|.$$

It is clear that the second term of the right hand side in the above tends to 0 as $k \to \infty$. By (30) and (31), we have

$$\varphi_k'(t) - f(t, \bar{\varphi}_k(t)) = \begin{cases} f(t, \varphi_k(t)) - f(t, \bar{\varphi}(t)) & \text{for } 0 \le t \le T_k \\ f(t, \varphi_k(T_k)) - f(t, \bar{\varphi}(T_k)) & \text{for } T_k \le t \le T_k + \varepsilon_k \\ f(t, \varphi_k(t - \varepsilon_k)) - f(t, \bar{\varphi}(t - \varepsilon_k)) & \text{for } T_k + \varepsilon_k \le t \le 1. \end{cases}$$

Since $\{\bar{\varphi}_k\}$ converges to $\bar{\varphi}$ uniformly on $[0,1]$, we can conclude from (32) and the above equality that $\{\bar{\varphi}_k'\}$ converges to $\bar{\varphi}'$ uniformly on $[0,1]$, which assures that $\{\varphi_k\}$ converges to $\bar{\varphi}$ in $C^1[0,1]$. It then follows from the relation $\varphi_k \in \partial G$ and the closedness of ∂G that $\bar{\varphi}$ belongs to ∂G, which contrdicts (5) and the fact that $\bar{\varphi} \in K$. This completes the proof. □

Corollary 1. A set

$$\{(x(1), x'(1))\,;\ x \text{ is a solution of } (1)\}$$

is compact and connected in $\mathbf{R}^{2n}$ for every $\xi \in \mathbf{R}^n$.

Corollary 2. If E is a compact and connected subset of $\mathbf{R}^n$, then a set

$$\{x\,;\ x \text{ is a solution of } (1) \text{ with } \xi \in E\}$$

is compact and connected in $C^1[0,1]$.

Example. An initial value problem

$$x' = 2\sqrt{|x|}, \qquad x(0) = 0 \tag{33}$$

admits two solutions $x_1(t) \equiv 0$ and $x_2(t) = t^2$. It follows from Corollary 1 that a compact and connected set

$$\{(x(1), x'(1)); x \text{ is a solution of } (33)\}$$

contains two points $(x_1(1), x_1'(1)) = (0,0)$ and $(x_2(1), x_2'(1)) = (1,2)$. Therefore (33) admits a solution x satisfying

$$x(1) + x'(1) = 2$$

because the straight line $x + y = 2$ separates two points $(0,0)$ and $(1,2)$.

REFERENCES

[1] Hartman, P., Ordinary Differential Equations, John Wiley and Sons, Inc. 1964.