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Dynamics of entire functions with two singular values

Shunsuke MOROSAWA
Department of Mathematics and Information Science,
Faculty of Science, Kochi University
and
Masahiko TANIGUCHI
Graduate School of Science, Kyoto University

1 Introduction

Form the structural viewpoint, the simplest entire function is a quadratic polynomial. It has two critical points. One is the point at infinity and is always a superattracting fixed point. Hence the behavior of the finite critical point decides the dynamics on the complex plain. In this sense, dynamics of cubic polynomial is decided by two finite critical points. The family of cubic polynomials has been investigated by several authors (e.g. [2], [3] and [4]). In this note, we treat three simplest kinds of entire functions having only two singular values. They are structurally finite entire functions, which are defined by Taniguchi in [9].

Definition

1. Cubic (monic centered) polynomials

$$P_{a,b}(z) = z^3 - 3a^2z + b.$$

The critical points are $\pm a$, and the singular (critical) values are $b \mp 2a^3$. We denote by Poly_3 the family consisting of all cubic polynomials.

2. Simply decorated exponential functions

$$E_{a,b}(z) = a(z+b)e^z \qquad (a \neq 0).$$

The single critical point is $-b-1$, and the singular values are 0 and $-a\exp(-b-1)$. We denote by Dexp_1 the family consisting of all simply decorated exponential functions.

3. Complex error functions
$$C_{a,b}(z) = a\int_0^z e^{-t^2}dt + b \qquad (a \neq 0).$$
The singular values are $\pm aA + b$ with $A^2 = \pi/4$. We denote by Cerf the family consisting of all complex error functions.

Definition We denote by $\mathcal{S}_2$ the union of the families Poly_3, Dexp_1 and Cerf.

The second family has been investigated by Morosawa ([5] and [6]) and others. But the third family seems to have been paid almost no attentions. Recently, Morosawa and Taniguchi obtain some results on complex error functions (see, [10], [7] and [11]).

2 Classification of hyperbolic Fatou components

Let f be a *Speiser function*, i.e. a function having only a finite number of singular values, and ζ be an asymptotic value of f in the Fatou set $F(f)$ of f. Then there exists an asymptotic path for ζ in $F(f)$. The Fatou component containing such a path is called a *prelusive component* of f for ζ. The Fatou component containing a critical point c of f is also called a *prelusive component* for the critical value $f(c)$.

Note that a singular value may have several prelusive components for it. The following Proposition is well-known.

Proposition 1 *The immediate basin for an attracting cyclic contains at least one prelusive component.*

We say that a Speiser function f is *hyperbolic* if the orbit of every singular value of f belongs to basins of attracting periodic points. In the case of hyperbolic entire functions in $\mathcal{S}_2$, there are four kinds of dynamics (cf. [4], [8]).

Definition We say that a hyperbolic function $f \in \mathcal{X}$, where $\mathcal{X} = \text{Poly}_3$, Dexp_1 or Cerf is

1. *of adjacent type* if there exists a prelusive component U for both of singular values and there exists the smallest positive integer p such that
$$f^p(U) \subset U,$$
and we denote the set of all such f with a fixed p by $A_p = A_p(\mathcal{X})$,

2. *of bitransitive type* if there exist different prelusive components U_1 and U_2 and the smallest positive integers p and q such that
$$f^p(U_1) \subset U_2, \qquad f^q(U_2) \subset U_1,$$
and we denote the set of all such f with fixed p and q by $B_{p+q} = B_{p+q}(\mathcal{X})$,

3. *of captured type* if there exist two prelusive components U_1 and U_2 and the smallest non-negative integers p, q, and t such that U_1 is disjoint from $\bigcup_{k=0}^{\infty} f^k(U_2)$, $t \geq 1$ with
$$f^t(U_1) \cap \left(\bigcup_{k=0}^{\infty} f^k(U_2) \right) \neq \emptyset$$
and, $p \geq 0$ and $p+q \geq 1$ with
$$f^{t+p}(U_1) \subset U_2, \qquad f^{p+q}(U_2) \subset U_2,$$
and we denote the set of all such f with fixed p, q, t by $C_{(t)p+q} = C_{(t)p+q}(\mathcal{X})$, and

4. *of disjoint type* if there exist two prelusive components U_1 and U_2 and the smallest positive integers p and q such that
$$f^p(U_1) \subset U_1, \qquad f^q(U_2) \subset U_2,$$
and
$$\bigcup_{k=0}^{\infty} f^k(U_1) \bigcap \bigcup_{k=0}^{\infty} f^k(U_2) = \emptyset,$$
and we denote that the set of all such f with fixed p and q by $D_{p,q} = D_{p,q}(\mathcal{X})$.

Bergweiler [1] pointed out the following.

Theorem 2 *If a forward invariant Fatou component U of a Speiser function f contains all the singular values of f, then U is completely invariant.*

Corollary 1 *The set $C_{(1)0+1} = C_{(1)0+1}(\mathcal{X})$, where $\mathcal{X} = \mathrm{Poly}_3$, Dexp_1 or Cerf, is empty.*

Next we show a criterion for a hyperbolic function being of capture type.

Theorem 3 *Let f be a hyperbolic function belonging to Poly_3 or Dexp_1. Assume f has a superattracting fixed point ζ_1. Let ζ_2 be another singular value. If there exists some $N > 0$ satisfying $f^N(\zeta_2) = \zeta_1$, then f is of capture type.*

3 Examples

Example 1 *Let*

$$E_{-1,-1}(z) = -(z-1)e^z.$$

Then $E_{-1,-1}$ is hyperbolic and belongs to $C_{(1)0+2}$.

Verification. The function has a critical point 0 and a singular value 0. Since $f(0) = 1$ and $f(1) = 0$, it has a superattracting cycle with period two. Hence it is hyperbolic. By using a graphical analysis, we can find a fixed point $x \in (0,\ 1)$ and it is easy to see that it is repelling. Hence there exists a point in $(-\infty,\ 0)$ which is mapped on x. Since the Fatou set of $E_{-1,-1}(z)$ is symmetric with respect to the real axis, the prelusive component for 0 does not contain 0.

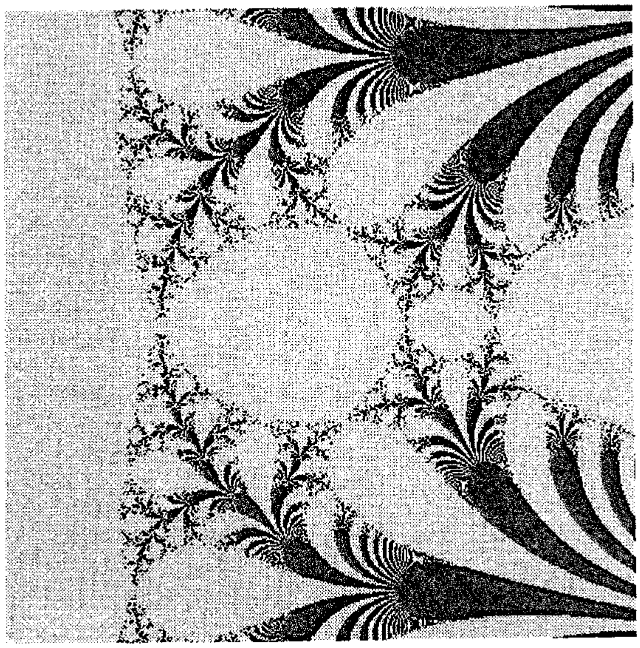

Figure 1: The Julia set of $E_{-1,-1}$. The range shown is $|\Re z| < 2.4$ and $|\Im z| < 2.4$.

Example 2 *Assume a and b satisfy*

$$\begin{aligned} ab &= -b-1 \\ be^{b+1} &= -1. \end{aligned}$$

Then $E_{a,b}$ is hyperbolic and belongs to $C_{(2)0+1}$. In this case, the prelusive component for the asymptotic value is captured by the prelusive component for the critical value.

Verification. The first equation implies $-b-1$ is a superattracting fixed point. The second implies $E_{a,b}(0) = -b-1$. From Theorem 3, we obtain the claim.

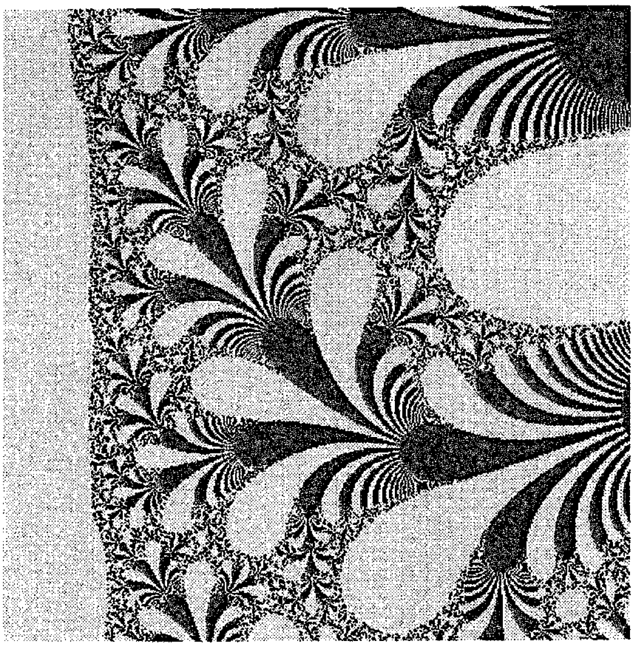

Figure 2: The Julia set of $E_{a,b}$, where $a = -0.952639\cdots + i0.114414\cdots$ and $b = -3.08884\cdots + i7.46149\cdots$ which satisfy the equations in Example 2. It has a superattracting fixed point $2.08884\cdots - i7.46149\cdots$. The range shown is $-7 < \Re z < 3$ and $-8 < \Im z < 2$.

Example 3 *Assume a and b satisfy*

$$\begin{aligned} a &= \frac{1}{1+2b} \\ \frac{1}{1+2b} - 2b\exp\left(\frac{3}{2}+b\right) &= 0. \end{aligned}$$

Then $E_{a,b}$ is hyperbolic and belongs to $C_{(2)0+1}$. In this case, the prelusive component for the critical value is captured by the prelusive component for the asymptotic value.

Verification. The first equation implies $1/2$ is an attracting fixed point. The second implies $E_{a,b}(-b-1) = 0$.

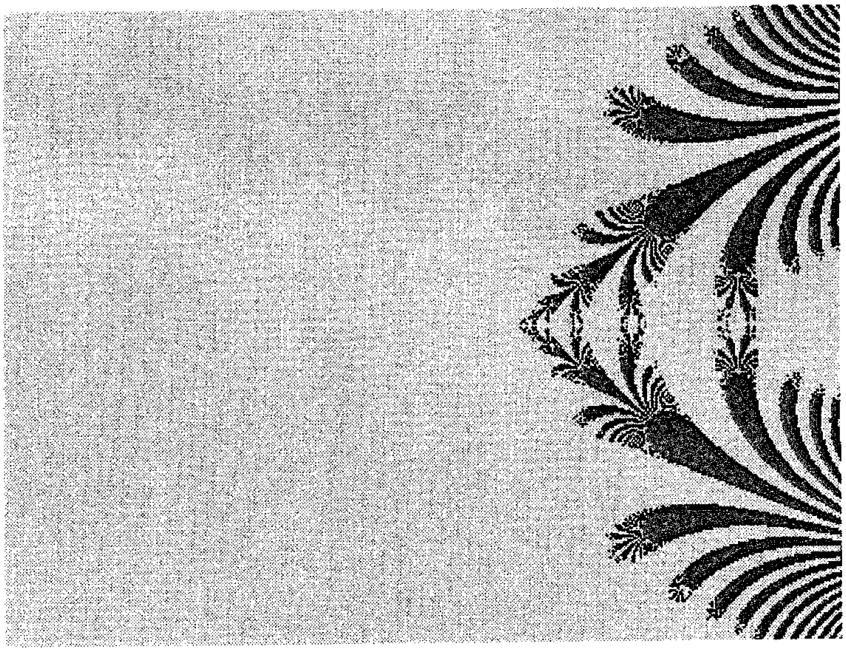

Figure 3: The Julia set of $E_{a,b}$, where $a = -0.0605096\cdots$ and $b = -5.51185\cdots$ which satisfy the equations in Example 3. It has an attracting fixed point 1/2. The range shown is $-2 < \Re z < 6$ and $|\Im z| < 3$.

Example 4 *Let*

$$C_{a,b}(z) = a \int_0^z e^{-w^2} dw + b,$$

with $a = 1.41055 + i1, 23448$ *and* $b = -0.121077 - i0.8811$. *Then it belongs to* $C_{(2)0+1}$.

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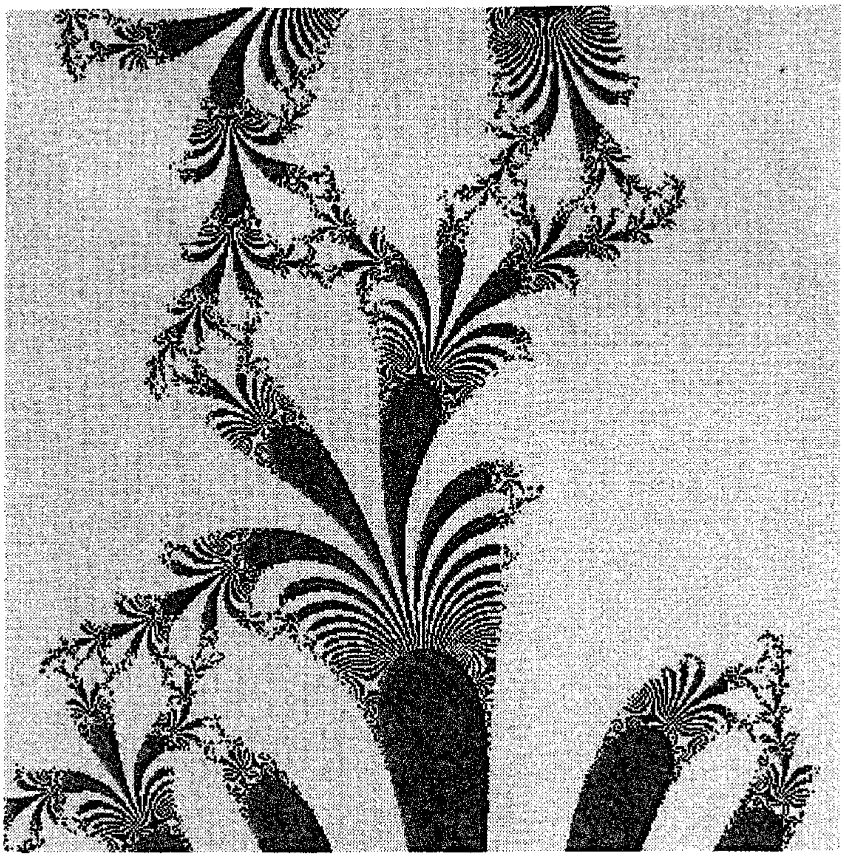

Figure 4: The Julia set of $C_{a,b}$. It has an attracting fixed point $0.8910492\cdots - i0.0372985\cdots$. Its asymptotic values are $a_1 = 1.12899\cdots - i0.2129294\cdots$ and $a_2 = -1.371144\cdots - i1.975129\cdots$. The immediate basin is the prelusive component for a_1 and $C^2_{a,b}(a_2)$ is contained in it. The range shown is $-2.3 < \Re z < 1.7$ and $-2.4 < \Im z < 1.6$.

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