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The identity of various exceptional sets in complex dynamics and the Nevanlinna theory and its application to the potential theory

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This article is an announcement of the preprint [11]. A rational map f is a holomorphic endomorphism of the Riemann sphere $\hat{\mathbb{C}}$.

Notation. Rat denotes the set of all rational endomorphism of $\hat{\mathbb{C}}$. $\hat{\mathbb{C}}$ is identified as the set of all constant functions of $\hat{\mathbb{C}}$.

In the cases f is non-invertible, the Fatou and Julia strategy for studying the complex dynamics $(\hat{\mathbb{C}}, f)$, which treats *forward-images* under iterations, is the separation of $\hat{\mathbb{C}}$ into two completely invariant complementary subsets, one of which is the *Fatou set* $F(f)$, the region of normality of $\{f^k := f^{\circ k}\}$, and the other the *Julia set* $J(f)$. In other words, the restricted dynamical systems $(F(f), f)$ and $(J(f), f)$ are tame and chaotic respectively. Consequently, the dynamical system *around* $J(f)$ has an *almost covering* feature: There exists $E(f) \subset \hat{\mathbb{C}}$ such that for every neighborhood U of a point of $J(f)$, the union of the forward-images of U under iterations covers $\hat{\mathbb{C}} - E(f)$.

Definition 1 (dynamical exceptional set). $E(f)$ is called the *dynamical exceptional set* of f.

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From this almost covering feature, naturally arises the Nevanlinna theoretical study, which treats *preimages* under iterations.

Definition 2 (value distribution). For $f, g \in \mathrm{Rat}$, the *value distribution* $\mu(f, g)$ of f for g is defined by the mass distribution on the $(\deg f + \deg g)$-roots of the equation $f = g$.

The spherical area measure and the chordal distance on $\hat{\mathbb{C}}$ are

$$\sigma(w) = \frac{|dw|}{\pi(1 + |w|^2)^2} \quad \text{and} \quad [z, w] = \frac{|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}$$

respectively. We note that they are normalized as $\sigma(\hat{\mathbb{C}}) = 1$ and $[0, \infty] = 1$.

Definition 3 (dynamical Nevanlinna theory [13]). For $f, g \in \mathrm{Rat}$, the *pointwise proximity function* is defined by

$$(w(g, f))(z) := \log \frac{1}{[g(z), f(z)]} : \hat{\mathbb{C}} \to [0, \infty],$$

and the *mean proximity* by

$$m(g, f) := \int_{\hat{\mathbb{C}}} w(g, f) d\sigma \in [0, \infty).$$

Let $\mathcal{F}$ be a *rational sequence* $\{f_k\}_{k=0}^{\infty} \subset \mathrm{Rat}$ with increasing degrees $\{d_k := \deg f_k\}$. For $g \in \mathrm{Rat}$, the dynamical *Nevanlinna* and *Valiron exceptionalities* are defined by

$$\mathrm{NE}(g; \mathcal{F}) := \liminf_{k \to \infty} \frac{m(g, f_k)}{d_k} \in [0, \infty],$$
$$\mathrm{VE}(g; \mathcal{F}) := \limsup_{k \to \infty} \frac{m(g, f_k)}{d_k} \in [0, \infty]$$

respectively.

From now on, we consider the *iteration* sequence $\{f^k\}_{k=1}^{\infty}$ of a rational map f of degree $d \geq 2$.

Definition 4 (dynamical Nevanlinna and Valiron exceptional sets). The dynamical *Nevanlinna* and *Valiron* exceptional sets of f in $\hat{\mathbb{C}}$ are defined by

$$E_N(f) := \{p \in \hat{\mathbb{C}}; \mathrm{NE}(p; \{f^k\}) > 0\},$$
$$E_V(f) := \{p \in \hat{\mathbb{C}}; \mathrm{VE}(p; \{f^k\}) > 0\}$$

respectively.

We shall use several notions from the geometric measure theory and the potential theory. For the details, see, for example, [3], [10], and [7].

It is known that $\{(f^k)^*\sigma/d^k\}$ converges weakly. The limit is also known as the unique maximal entropy measure (see [8] and [9]).

Definition 5 (the maximal entropy measure).

$$\mu_f := \lim_{k\to\infty} \frac{(f^k)^*\sigma}{d^k}.$$

Definition 6 (accumulation and convergence loci). The *accumulation* and *convergence* loci of the averaged value distributions of f in $\hat{\mathbb{C}}$ are defined by

$$A(f) := \{p \in \hat{\mathbb{C}};\ \text{a subsequence of } \{\mu(f^k,p)/d^k\} \text{ converges to } \mu_f\},$$
$$\mathrm{Conv}(f) := \{p \in \hat{\mathbb{C}}; \lim_{k\to\infty} \frac{\mu(f^k,p)}{d^k} = \mu_f\}$$

respectively.

Now we state Main Theorem.

Main Theorem 1 (characterizations of exceptional sets). *For $f \in$ Rat of degree ≥ 2,*

$$\hat{\mathbb{C}} - E_V(f) = \mathrm{Conv}(f) \subset A(f) = \hat{\mathbb{C}} - E_N(f) \subset \hat{\mathbb{C}} - E(f).$$

Independently, known is the following remarkable theorem which was first proved for polynomials by Brolin [1] and later for rational maps by Lyubich [8] and independently by Freire-Lopes-Mañé [5]. See also [2], [6], [4] for the other proofs.

Theorem 1 (convergence of averaged value distributions). *For $f \in$ Rat of degree ≥ 2,*

$$\hat{\mathbb{C}} - E(f) = \mathrm{Conv}(f).$$

Combining them, we have the following.

Main Corollary 1 (All exceptional sets are same.). *For $f \in$ Rat of degree ≥ 2,*

$$E_N(f) = E_V(f) = E(f) = \hat{\mathbb{C}} - \mathrm{Conv}(f) = \hat{\mathbb{C}} - A(f).$$

Remark 1. In [12], Main Corollary 1 has been already implicitly applied to the Siegel-Cremer linearizability problem of rational maps.

The important consequence of Main Corollary is a convergence theorem of the *potentials* of the averaged value distributions.

Definition 7 (spherical potential). For a regular measure μ on $\hat{\mathbb{C}}$, the *potential* is defined by

$$V_\mu := \int_{\hat{\mathbb{C}}} -\log[\cdot, w]\mu(w) : \hat{\mathbb{C}} \to [0, \infty].$$

Remark 2. In the potential theory, the potential is usually defined as $-V_\mu$, but the definition will be more convenient in our study.

The (axiomatic) potential theory implies that when regular measures μ_k converges to μ, then

$$\liminf_{k\to\infty} V_{\mu_k} = V_\mu$$

quasieverywhere on $\hat{\mathbb{C}}$. For the averaged value distributions, we obtain the stronger conclusion.

Main Theorem 2 (convergence theorem of potentials). *Let $f \in$ Rat be of degree $d \geq 2$. If $p \in \hat{\mathbb{C}} - E(f)$ is not a fixed point, then*

$$\liminf_{k\to\infty} V_{\mu(f^k,p)/d^k} = V_{\mu_f} \tag{1}$$

on $\hat{\mathbb{C}}$. Otherwise (1) *holds on $\hat{\mathbb{C}} - \bigcup_{k>0} f^{-k}(p)$.*

We also characterize such points that the potentials actually *converge* there.

Main Theorem 3 (convergence of potentials and pointwise behavior). *Let $f \in$ Rat be of degree $d \geq 2$. For $p \in \hat{\mathbb{C}} - E(f)$ and $q \in \hat{\mathbb{C}}$,*

$$\lim_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) = V_{\mu_f}(q) \tag{2}$$

if and only if

$$\lim_{k\to\infty} \frac{1}{d^k} \log \frac{1}{[p, f^k(q)]} = 0. \tag{3}$$

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