数理解析研究所講究録 1447 巻 2005 年 39-45

A NOTE ON INDICES THEOREMS

FILIPPO BRACCI

These are the notes for the talk *Splittings, comfortably embedded subvarieties and index theorems* I gave at the RIMS Symposium on *Topological and geometrical methods of complex differential equations* in Kyoto, 19-23 January 2004. I wish to sincerely thank prof. Shishikura and prof. Ito for the invitation.

These notes contain some well known facts (maybe with a new interpretation) and statements of new results which will be proved in a forthcoming paper.

1. WHAT IS AN INDEX THEOREM?

Let X be a n-dimensional complex variety and let $\varphi \in H^\bullet(X)$ be a (nonzero) element of its cohomology. Often it is not possible to "calculate" such an element directly. It is then important when one can calculate such element using tools like differential geometry or complex analysis. For instance the Chern classes of a vector bundle on X can be calculated using the Chern-Weil theory of connections, provided X is nonsingular.

In applications however it is important to know the image of $P(\varphi) \in H_{2n-\bullet}$ where P denotes the Poincaré homomorphism (isomorphism if X is nonsingular).

Suppose that S is an analytic subset of X and let $U = X \setminus S$. Look at the cohomological exact sequence

$$\ldots \longrightarrow H^\bullet(M,U) \longrightarrow H^\bullet(M) \longrightarrow H^\bullet(U) \longrightarrow \ldots$$

and assume that $H^\bullet(M) \ni \varphi \mapsto 0 \in H^\bullet(U)$. Therefore there exists a lifting $\hat{\varphi} \in H^\bullet(M,U)$ of φ in the relative cohomology. This lifting is not unique in general. Anyhow, by the Alexander homomorphism (isomorphism if S is nonsingular) $A : H^\bullet(M,U) \to H_{2n-\bullet}(S)$ we have the following commuting diagram:

$$\begin{array}{ccc} H^\bullet(M,U) & \longrightarrow & H^\bullet(M) \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ H_{2n-\bullet}(S) & \xrightarrow{i_*} & H_{2n-\bullet}(M) \end{array} \tag{1.1}$$

therefore we have the following formula, which can be called an "index theorem":

$$P(\varphi) = i_*(\hat{\varphi}).$$

In particular if $\bullet = 2n$ and S is a finite set of points, denoting by Res$(\hat{\varphi}, p) \in \mathbb{C}$ the "residue" at $p \in S$, we have

$$\int_M \varphi = \sum_{p \in S} \text{Res}(\hat{\varphi}, p). \tag{1.2}$$

Typical examples of this situation appears when $\varphi = c_n(TM)$ (the top Chern class) and then the left-hand side of (1.2) is just the Euler characteristic $\chi(M)$ of M. An example is the classical Poincaré-Hopf theorem.

However, an "index theorem" as in 1.1 is not very useful. To make it useful one needs

1. A "good reason" for $\varphi \mapsto 0$ and thus a "good" lifting $\hat{\varphi}$.
2. Explicit calculations of $i_*(A(\hat{\varphi}))$.

Both these problems are interesting and many papers have been written on that, see, *e.g.* [21].

In these notes we look at the first point, therefore we examine the question of when "$\varphi \mapsto 0$".

2. Holomorphic action and Bott vanishing

Let M be a n-dimensional complex manifold and V a holomorphic vector bundle on M. We say that there is a *holomorphic action* on V in the sense of Bott (and Lehmann-Suwa) provided $F \subset TM$ is an *involutive* subbundle and there exists a $\mathbb{C}$-bilinear map $\theta : \mathcal{C}^\infty(F) \times \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V)$ such that

1. $\theta([u, v], s) = \theta(u, \theta(v, s)) - \theta(v, \theta(u, s))$ for $u, v \in \mathcal{C}^\infty(F)$ and $s \in \mathcal{C}^\infty(V)$;
2. $\theta(hu, s) = h\theta(u, s)$ for $h \in \mathcal{C}^\infty$, $u \in \mathcal{C}^\infty(F)$ and $s \in \mathcal{C}^\infty(V)$;
3. $\theta(u, hs) = h\theta(u, s) + u(h)s$ for $h \in \mathcal{C}^\infty$, $u \in \mathcal{C}^\infty(F)$ and $s \in \mathcal{C}^\infty(V)$;
4. $\theta(u, s) \in \mathcal{V}$ for $u \in \mathcal{F}$ and $s \in \mathcal{V}$.

If there is a holomorphic action of F of rank r on V, there exists a connection ∇ for V such that for any symmetric homogeneous polynomial φ of degree $d > n - r$ it follows

$$\varphi(\nabla) = 0. \tag{2.1}$$

This last equation is known as *Bott vanishing theorem*. In particular one has $c_t(V) = 0$ for $t > 2(m - r)$.

Notice that if M has (complex) dimension 1 and $V = L$ is a line bundle on M then θ defines itself a holomorphic connection for L and from Bianchi identity one obtains that the curvature of such a connection is identically zero on M. In particular $c_1(L) = 0$.

Given a complex vector bundle V on M, in general one cannot hope to have a holomorphic action on V on all of M. Usually (see next section) there exists an analytic set S such that on $M \setminus S$ there exists a holomorphic action on V. Therefore one has the Bott vanishing outside S. Using compact supported forms as in [4] or Čech-de Rham cohomology as in [17] (see also [18] and [21]) one can define an element of the cohomology of M vanishing on $M \setminus S$ and then having an index theorem as said before.

3. WHEN ARE THERE HOLOMORPHIC ACTION?

We provide three examples and later a general principles.

1. **Holomorphic action given by foliations**. If $\mathcal{F}$ is a one dimensional holomorphic foliation on a 2 dimensional manifold leaving a curve S invariant then we have the following index theorem

$$S \cdot S = \sum_{p \in \mathrm{Sing}(\mathcal{F}) \cup \mathrm{Sing}(S)} \mathrm{Res}(\mathcal{F}, S; p).$$

The previous formula is known as the Camacho-Sad index theorem and it is due to Camacho and Sad [11] for the case S is nonsingular, Lins Neto [20] and Suwa (see [21]) in case S is singular.

The fact that S is $\mathcal{F}$ invariant allows to define, outside $\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Sing}(S)$, a holomorphic action of $T\mathcal{F}$ on the line bundle L_S associated to the divisor S in M. The holomorphic action is given by

$$\theta(f|_S, s) = \pi([f, \tilde{s}]|_S)$$

for $f \in \mathcal{F}$ and $s \in \mathcal{O}_S(L_S)$ such that $\tilde{s} \in \mathcal{O}_M(TM)$ and $\pi(\tilde{s}|_S) = s$, where, since $L_S = N_S$ the normal bundle of S in M outside $\mathrm{Sing}(S)$, $\pi : \mathcal{O}_M(TM) \otimes_{\mathcal{O}_M} \mathcal{O}_S \to \mathcal{O}_S(L_S)$ is the projection. Such a theorem has been generalized to higher dimension (see [17], [18]).

2. **Holomorphic action given by diffeomorphisms**. Assume $f : M \to M$ is holomorphic and $S \subset M$ is a reduced, globally irreducible hypersurface. Suppose that $f|_S = I_S$.

As an example of this picture one can think to the blow up of a point in $\mathbb{C}^n$ and a germ of biholomorphism fixing such a point and tangent to the identity there.

Suppose for the moment that S is smooth. Let N_S be the normal bundle to S in M. One can consider the morphism induced by $df - I$ from N_S to $TM|_S$. However it might happen that such a morphism is identically zero. Thus one should take "higher order differentials". The way to define it is as follows. Let $p \in S$. For $h \in \mathcal{O}_{M,p}$, let

$$\nu_f(h, S, p) := \max\{T \in \mathbb{N} : h \circ f - h \in \mathcal{I}^T_{S,p}\},$$

and

$$\nu_f(S, p) := \min\{\nu_f(h, S, p) : h \in \mathcal{O}_{M,p}\}.$$

The number $\nu_f(S, p)$ is independent of p and we simply denote it by ν_f. We say that f is *tangential* to S provided

$$\min\{\nu_f(h, S, p) : h \in \mathcal{I}_{S,p}\} > \nu_f.$$

In dynamics (see [2]) non-tangential mappings are easily studied, therefore, form this point of view, one can look only at tangential mappings.

Let assume that f is tangential to S. In a local chart with coordinates $\{z_1, \dots, z_n\}$ assume that S is given by $z_1 = 0$. Then consider the (local) foliation defined by

$$\tilde{X}_f := \sum_{j=1}^{n} \frac{z_j \circ f - z_j}{z_1^{\nu_f}} \frac{\partial}{\partial z_j}.$$

This local foliation depends of course on the local coordinates chosen, but one can show ([2]) that once restricted to S one has a "canonical" section $X_f : N_S^{\otimes \nu_f} \to TS$ (if f is non-tangential the image is just in $TM|_S$). Nonetheless, using such local foliations one can define a holomorphic action similar to that in the previous examples, and then getting a residue theorem as the previous one (see [2] for details and generalizations to higher codimensional and singular cases)

3. Variation. The holomorphic action, now known as "variation" was introduced in [16] and later generalized in [19]. Let $\mathcal{F}$ be a holomorphic foliation, $\mathcal{Q} = \mathcal{O}(TM)/\mathcal{F}$ be the quotient sheaf called the "normal sheaf" to $\mathcal{F}$. Let S be a leaf of $\mathcal{F}$. Outside the singularities of $\mathcal{F} \otimes \mathcal{O}_S$ one has a natural action of $\mathcal{F}$ on $\mathcal{Q}$ defined similarly to that of the first example.

The principle underlying the previous examples has been generalized in [7]. Referring the reader to such a paper for details, we briefly sketch the idea.

Let M be a complex n-dimensional manifold, $S \subset M$ a subvariety, $\mathcal{F}$ an involutive, coherent subsheaf of $\mathcal{O}(TS)$ which is a foliation on the nonsingular part of S. Let $\mathcal{E}$ be a coherent $\mathcal{O}_S$-submodule of $\mathcal{O}(TS)$ (involutiveness is not required). It is possible to define a $\mathcal{O}_M$-morphism

$$\chi : \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2 \to \mathcal{O}(TM) \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2$$

Such a morphism is injective if S is locally complete intersection.

Now let $v \in \mathcal{O}(TM)$. We say that v is *tangentially vanishing at the first order with respect to* $\mathcal{E}$ if the image of v into $\mathcal{O}(TM) \otimes \mathcal{O}_S$ is zero and if $w \in \mathcal{O}(TM) \otimes \mathcal{I}_S$ is the unique preimage of v then $\pi(w) \in \chi(\mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{I}_S/\mathcal{I}_S^2)$, where $\pi : \mathcal{O}(TM) \otimes \mathcal{I}_S \to \mathcal{O}(TM) \otimes \mathcal{I}_S/\mathcal{I}_S^2$.

Let $\{U_\alpha\}$ be an open covering of M and assume that for all α is defined an involutive $\mathcal{O}_M|_{U_\alpha}$-module $\mathcal{G}_\alpha$. We say that $\{U_\alpha, \mathcal{G}_\alpha\}$ is a *first order tangency extension* of $\mathcal{F}$ with respect to $\mathcal{E}$ if

1. $\mathcal{G}_\alpha \otimes \mathcal{O}_S = \mathcal{F}|_{U_\alpha}$.
2. Let $p \in S$. For any $f_\alpha \in \mathcal{G}_{\alpha,p}$ and $f_\beta \in \mathcal{G}_{\beta,p}$ such that $f_\alpha|_S = f_\beta|_S$ then $f_\alpha - f_\beta$ is tangentially vanishing at the first order with respect to $\mathcal{E}$.

Theorem 3.1. *Let M be a complex manifold, $S \subset M$ a submanifold. Let $\mathcal{F}$ be a nonsingular foliation on S and let $F \subset TS$ be the associated bundle. Let $L \subset TS$ be a (possibly non-involutive) subbundle such that $[L, F] \subseteq L$. If $\mathcal{F}$ admits a first order tangency extension with respect to $\mathcal{O}(L)$ then there is a holomorphic action of F on $TM|_S/L$.*

Notice that if $L = TS$ then one has the "Camacho-Sad action" of the first two examples, while if $L = F$ one has variation.

4. Dropping tangency

As one can see, there are two main hypotheses in the previous theorem for holomorphic actions. The first one is about *injectivity* of $\mathcal{F}$. the second one is about *tangency* of $\mathcal{F}$ to S.

These two hypotheses are of very different nature. To drop the first hypothesis one can try to consider the "minimum involutive" extension of $\mathcal{F}$ (in case it is not involutive), but this generates bigger singularities, not easy to control. We do not know whether it is possible to get "genuine" holomorphic action in this case.

As for the hypothesis on "tangency", one can start with $\mathcal{F} \subset \mathcal{O}(TM) \otimes \mathcal{O}_S$, instead of $\mathcal{F} \subset \mathcal{O}(TS)$. In some case, depending on the geometric embedding of (the nonsingular part of) S into M, it is still possible to have a holomorphic action.

The condition we have is known as "comfortably embedded" submanifold and it was introduced in [2]. Such a condition generalizes the ones introduced in [12] and [13]. In [3] we develop and give details of what follows.

A subvariety is said to be *comfortably embedded* whenever its nonsingular part is so (as usual, on singularities one patches by means of the Čech-de Rham or compact supported cohomology theory). Thus we only look at nonsingular submanifold S of M.

First of all we need a way to project to the tangent bundle of S, TS. The first condition is thus that S being *splitting* into M. This means that the exact sequence of $\mathcal{O}_S$-modules:

$$0 \to \mathcal{I}_S/\mathcal{I}_{S^2} \to \Omega_{M,S} \to \Omega_S \to 0$$

splits (here $\Omega_{M,S} = \Omega_M \otimes \mathcal{O}_S$ and Ω_M is the sheaf of holomorphic differentials on M). This condition is equivalent to the following ones:

1. The Grothendieck-Atiyah map $\delta : H^0(S, \mathrm{Hom}(N_S, N_S)) \to H^1(S, \mathrm{Hom}(N_S, T_S))$ is such that $\delta(id) = 0$.
2. There exists an atlas $\{U_\alpha, (z^1_\alpha, \dots, z^n_\alpha\}$ such that $S \cap U_\alpha = \{z^1_\alpha = \dots = z^m_\alpha = 0\}$ and $\frac{\partial z^p_\beta}{\partial z^r_\alpha} \in \mathcal{I}_S$ for $p = m+1, \dots, n$ and $r = 1, \dots, m$.
3. There exists $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}^2_S$ which lifts the natural map $\mathcal{O}_M/\mathcal{I}^2_S \to \mathcal{O}_S$.
4. There is a first order infinitesimal retraction from the first infinitesimal neighborhood $S(1)$ of S to S.
5. The sequence of sheaves of rings
$$0 \to \mathcal{I}_S/\mathcal{I}^2_S \to \mathcal{O}_M/\mathcal{I}^2_S \to \mathcal{O}_S \to 0$$
splits (and this allows to give a structure of $\mathcal{O}_S$-module to $\mathcal{O}_M/\mathcal{I}^2_S$).
6. The first infinitesimal neighborhood $S(1)$ of S in M is isomorphic to the first infinitesimal neighborhood $S_N(1)$ of S in N_S.

Let $\sigma : TM|_S \to TS$ be a splitting morphism. Assume that $\mathcal{F}$ be a one-dimensional foliation in M. Consider $\mathcal{F}^\sigma := \sigma(\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S)$. In the most cases $\mathcal{F}^\sigma$ is *faithful*, that is $\mathrm{Sing}(\mathcal{F}^\sigma) \neq S$.

If $\mathcal{F}^\sigma$ has a first order extension with respect to TS then we say that S is *comfortably embedded* into M. If S is splitting into M, the condition of being comfortably embedded is equivalent to the following:

1. There exists an atlas $\{U_\alpha, (z^1_\alpha, \ldots, z^n_\alpha\}$ such that $S \cap U_\alpha = \{z^1_\alpha = \ldots = z^m_o = 0\}$ and $\frac{\partial^2 z^r_\alpha}{\partial z^s_\beta \partial z^t_\beta} \in \mathcal{I}_S$ for $r, s, t = 1, \ldots, m$.
2. In an atlas as before, let $h_{\alpha\beta} := \frac{1}{2} \sum \frac{\partial z^u_\beta}{\partial z^u_\alpha} \frac{\partial^2 z^r_\alpha}{\partial z^s_\beta \partial z^t_\beta} |_S \partial_{\beta,r} \otimes \omega^s_\beta \otimes \omega^t_\beta$. Then $\{h_{\alpha\beta}\}$ defines a class $[h] \in H^1(S, N_S \otimes N^*_S \otimes N^*_S)$ and $[h] = 0$.
3. The sequence of sheaves of rings
$$0 \to \mathcal{I}^2_S/\mathcal{I}^3_S \to \mathcal{I}_S/\mathcal{I}^3_S \to \mathcal{I}_S/\mathcal{I}^2_S \to 0,$$
splits.
4. The second infinitesimal neighborhood of S in N_S, $S_N(2)$ is isomorphic to the analytic space $(S, \mathcal{O}_M/\mathcal{I}^2_S \oplus \mathcal{I}^2_S/\mathcal{I}^3_S)$.

Typical examples of comfortably embedded submanifolds are zero sections of vector bundles and blow ups along comfortably embedded submanifolds (for instance Stein submanifolds of some space, or a point).

We have the following result (which can be generalized to several (co)-dimensions):

Theorem 4.1. *Let S be a compact complex (possibly singular) curve into a two dimensional manifold M. Let $\mathcal{F}$ be a foliation in M. Assume that S is comfortably embedded in M (this is always the case if S has a singularity) and let $\sigma : M \to S$ be the splitting. If $\mathcal{F}^\sigma$ is σ-faithful and $\Sigma :=$ Sing$(\mathcal{F}^\sigma) \cup$ Sing(S) then*
$$S \cdot S = \sum_{p \in \Sigma} Res(\sigma, \mathcal{F}, p).$$
Moreover, if $(U, (w_1, w_2))$ is a chart around $p \in \Sigma$ so that $U \cap S = \{l = 0\}$, $dl \wedge dw_2 \neq 0$, $\mathcal{F} = a\frac{\partial}{\partial l} + b\frac{\partial}{\partial w_2}$ on $S \cap U \setminus \{p\}$ then
$$Res(\sigma, \mathcal{F}, p) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{1}{b} \frac{\partial a}{\partial l} dw_2,$$
where Γ is the link of the singularity p in S.

References

1. M. Abate, *The residual index and the dynamics of holomorphic maps tangent to the identity*. Duke Math. J. 107, 1, (2001), 173-207.
2. M. Abate, F. Bracci, F. Tovena, *Index theorems for holomorphic self-maps*. Ann. of Math. (to appear)
3. M. Abate, F. Bracci, F. Tovena, *Index Theorems for subvarieties transversal to a holomorphic foliation*, in preparation.
4. P. Baum and R. Bott, *Singularities of holomorphic foliations*. J. Diff. Geom. 7 (1972), 279-342.
5. R. Bott, *A residue formula for holomorphic vector fields*, J. Diff. Geom. 1 (1967), 311-330.
6. F. Bracci, *The dynamics of holomorphic maps near curves of fixed points*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. II (2003), 493-520.
7. F. Bracci, *First order extensions of holomorphic foliations*. Hokkaido Math. J., (to appear).
8. F. Bracci and T. Suwa, *Residues for singular pairs and dynamics of biholomorphic maps of singular surfaces*. Preprint 2003.

9. F. Bracci and T. Suwa, *Residues for holomorphic foliations of singular pairs*. Adv. Geom. (to appear).
10. F. Bracci and F. Tovena, *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*. Math. Z. 242 (2002), 481-490.
11. C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*. Ann. of Math. (2) 115 (1982), 579-595.
12. C. Camacho, *Dicritical singularities of holomorphic vector fields*. Contemporary Math., 269, (2001), 39-45.
13. C. Camacho, H. Movasati, P. Sad, *Fibered neighborhoods of curves in surfaces*. J. Geom. Anal. 13 (2003), 55-66.
14. H. Dulac, *Solutions d'un système d'équations différentielles dans le voisinage des valeurs singuliè res*. Bull. Soc. Math. France 40, (1912), 324-383.
15. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*. Prentice-Hall 1965.
16. B. Khanedani and T. Suwa, *First variations of holomorphic forms and some applications*, Hokkaido Math. J. 26 (1997), 323-335.
17. D. Lehmann, *Résidus des sous-variétés invariants d'un feuilletage singulier*. Ann. Inst. Fourier, Grenoble, 41, 1 (1991), 211-258.
18. D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*. J. Diff. Geom. 42, 1, (1995), 165-192.
19. D. Lehmann and T. Suwa, *Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties*, International J. Math. 10 (1999), 367-384.
20. A. Lins Neto *Algebraic solutions of polynomial differential equations and foliations in dimension two*. Holomorphic Dynamics (Mexico 1986), Lecture Notes in Math., vol. 1345, Springer-Verlag, 1988, 192-232.
21. T. Suwa *Indices of vector fields and residues of singular holomorphic foliations*. Hermann, Paris, 1998.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA "TOR VERGATA", VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALY.

E-mail address: fbracci@mat.uniroma2.it