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Properties of limit sets of Teichmüller modular groups

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1 Introduction

For a Riemann surface R, we consider the reduced Teichmüller modular group $\mathrm{Mod}^{\#}(R)$, which is a group of automorphisms on the reduced Teichmüller space $T^{\#}(R)$. If R is of analytically finite type, $\mathrm{Mod}^{\#}(R)$ and $T^{\#}(R)$ are nothing but the ordinary Teichmüller modular group $\mathrm{Mod}(R)$ and the ordinary Teichmüller space $T(R)$, respectively. In this case, $T^{\#}(R)$ is finite dimensional, and the action of $\mathrm{Mod}^{\#}(R)$ on $T^{\#}(R)$ is properly discontinuous. On the other hand, if R is of topologically infinite type, $T^{\#}(R)$ is infinite dimensional and is not locally compact. It is different from the case of finite type that the action of $\mathrm{Mod}^{\#}(R)$ is not necessarily properly discontinuous. On the basis of this fact, we introduced the notions of the limit set and the region of discontinuity for the Teichmüller modular group, which were defined analogously to the theory of Kleinian groups acting on the Riemann sphere.

Definition 1.1 ([1]) For a subgroup G of $\mathrm{Mod}^{\#}(R)$, we define $\Lambda(G)$ as the set of points $p \in T^{\#}(R)$ for which there exists a sequence $\{\chi_n\}$ of distinct elements of G satisfying $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$. We also define $\Omega(G)$ as the complement of $\Lambda(G)$. We call $\Lambda(G)$ the *limit set* of G and $\Omega(G)$ the *region of discontinuity* of G.

Proposition 1.2 ([1]) *Let G be a subgroup of* $\mathrm{Mod}^{\#}(R)$. *Then G acts on $\Omega(G)$ properly discontinuously. Namely, for any point $p \in \Omega(G)$, there exists a neighborhood U of p such that the set $\{\chi \in G \mid \chi(U) \cap U \neq \emptyset\}$ consists of only finitely many elements.*

Furthermore, in [1], we showed that limit sets and regions of discontinuity for Teichmüller modular groups satisfy similar properties to those of limit sets and regions of discontinuity for Kleinian groups, and proposed some problems. In the next section, we shall explain these properties and answers of the problems. In section 3, we observe properties of limit sets which are different from those for Kleinian groups.

2 Properties of limit sets

It is easy to see that for any subgroup $G \subset \mathrm{Mod}^{\#}(R)$, the limit set $\Lambda(G)$ is closed and G-invariant. To state other properties of limit sets, we classify the points in the limit set.

Definition 2.1 ([1]) In a subgroup G of $\mathrm{Mod}^{\#}(R)$, the *stabilizer* of a point $p \in T^{\#}(R)$ is defined by $\mathrm{Stab}_G(p) = \{\chi \in G \mid \chi(p) = p\}$.

We define $\Lambda_0(G)$ as the set of points $p \in \Lambda(G)$ for which there exists a sequence $\{\chi_n\}$ of distinct elements of G satisfying $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$ and $\chi_n(p) \neq p$ for all n, and $\Lambda_\infty(G)$ as the set of points $p \in \Lambda(G)$ such that $\mathrm{Stab}_G(p)$ consists of infinitely many elements. Furthermore, we divide $\Lambda_\infty(G)$ into two disjoint subsets, $\Lambda^1_\infty(G)$ and $\Lambda^2_\infty(G)$. The $\Lambda^1_\infty(G)$ is the set of points $p \in \Lambda_\infty(G)$ such that there exists an element in $\mathrm{Stab}_G(p)$ with infinite order, and the $\Lambda^2_\infty(G)$ is the set of points $p \in \Lambda_\infty(G)$ such that all elements in $\mathrm{Stab}_G(p)$ are of finite order.

Proposition 2.2 ([1]) (i) *For a subgroup G of $\mathrm{Mod}^{\#}(R)$, the set $\Lambda(G) - \Lambda^2_\infty(G)$ does not have an isolated point.* (ii) *If $\Lambda(G) - \Lambda^2_\infty(G)$ is not empty, then the limit set $\Lambda(G)$ is an uncountable set.*

Remark 2.3 The limit set $\Lambda(\Gamma)$ of a non-elementary Kleinian group Γ is a perfect set, which is proved by using the fact that the orbit $\Gamma(p)$ of a point $p \in \Lambda(\Gamma)$ under the action of Γ is dense in $\Lambda(\Gamma)$ (see [6, Theorem 2.4]). However, for a subgroup $G \subset \mathrm{Mod}^{\#}(R)$, the orbit $G(p)$ of $p \in \Lambda(G)$ is not dense in $\Lambda(G)$, in general (see Example 3.7). Thus the proof of Proposition 2.2 is completely different from that for Kleinian groups.

The following proposition is the answer of the first problem in [1, Problem 1].

Proposition 2.4 ([5]) *For a subgroup G of $\mathrm{Mod}^{\#}(R)$, the set $\Lambda_0(G)$ is dense in $\Lambda(G) - \Lambda^2_\infty(G)$.*

For a Kleinian group Γ, the set of accumulation points of $\Gamma(p)$ for a point p in the region of discontinuity of Γ is coincident with the limit set $\Lambda(\Gamma)$, and hence $\Lambda(\Gamma)$ is nowhere dense. On the other hand, for a subgroup $G \subset \mathrm{Mod}^{\#}(R)$, the orbit $G(p)$ of the point $p \in \Omega(G)$ does not have an accumulation point. However, in [1, Problem 2] and [1, Problem 3], we conjectured that for any Riemann surface R, the region of discontinuity $\Omega(\mathrm{Mod}^{\#}(R))$ is connected and the limit set $\Lambda(\mathrm{Mod}^{\#}(R))$ is nowhere dense in $T^{\#}(R)$ unless $\Omega(\mathrm{Mod}^{\#}(R))$ is empty. Under a certain assumption, these conjectures are true. To state the assumption, we make a couple of definitions given in terms of hyperbolic geometry of Riemann surfaces.

Definition 2.5 For a constant $M > 0$, we define R_M to be the set of points $p \in R$ for which there exists a non-trivial simple closed curve passing through p with hyperbolic length less than M. The set R_ϵ is called the *ϵ-thin part* of R if ϵ (> 0) is smaller than the Margulis constant (see [6, p.56]). Further, a connected component of the ϵ-thin part corresponding to a puncture is called the *cusp neighborhood*.

Definition 2.6 ([1], [2]) We say that a subdomain $R' \subseteq R$ satisfies the *lower bound condition* in R if there exists a constant $\epsilon > 0$ such that the $R_\epsilon \cap R'$ consists of either cusp neighborhoods or neighborhoods of geodesics which are homotopic to boundary components. We also say that R' satisfies the *upper bound condition* in R if there exist a constant $M > 0$ and a connected component U of $R_M \cap R'$ such that the homomorphism of $\pi_1(U)$ to $\pi_1(R')$ induced by the inclusion map of U into R' is surjective.

Now we state the results.

Proposition 2.7 ([5]) *Let R be a Riemann surface satisfying the lower and upper bound conditions. Then the region of discontinuity $\Omega(\mathrm{Mod}^\#(R))$ is connected, and the limit set $\Lambda(\mathrm{Mod}^\#(R))$ is nowhere dense in $T^\#(R)$.*

3 Examples

In this section, we observe some properties of limit sets of Teichmüller modular groups which are different from those for Kleinian groups. First, we recall some results which were proved in other papers.

The following proposition gives a sufficient condition for the limit set to coincide with the whole Teichmüller space.

Proposition 3.1 ([1]) *If R does not satisfy the lower bound condition, then $T(R) = \Lambda(\mathrm{Mod}^\#(R))$.*

For readers' convenience, we explain the proof of Proposition 3.1.

Sketch of the proof of Proposition 3.1. By the assumption, there exists a sequence $\{c_{n*}\}$ of simple closed geodesics on R such that c_{n*} are not freely homotopic to boundary components and satisfy $\ell(c_{n*}) \to 0$ $(n \to \infty)$. Let $[h_n]$ be an element of $\mathrm{Mod}^\#(R)$ that is the Dehn twist along c_n for each n. We can take a representative h_n so that $\lim_{n\to\infty} K(h_n) = 1$. Hence, for $p_0 = [R, id]$, we have $\lim_{n\to\infty} d_T([h_n](p_0), p_0) = 0$, which means that $p_0 \in \Lambda(\mathrm{Mod}^\#(R))$. Let $p = [S, f]$ an arbitrary point in $T^\#(R)$. By Lemma 3.2 below, we have $\ell(f(c_n)_*) \to 0$. Thus we can apply the same argument as above also to p. ∎

Lemma 3.2 ([7]) *Let f be a quasiconformal map on R onto another Riemann surface, and c a non-trivial simple closed curve on R. Then the inequality $\ell(f(c)_*) \leq K(f)\ell(c_*)$ holds. Here c_* and $f(c)_*$ are geodesics which are homotopic to c and $f(c)$, respectively.*

Next we state conditions for limit sets to be empty.

Proposition 3.3 ([2]) *Let R be a Riemann surface, and R' a subdomain of R satisfying the lower and upper bound conditions. Suppose that R' and $R-R'$ have non-abelian fundamental groups. Let G be the set of elements $[g] \in \mathrm{Mod}^{\#}(R)$ such that $g(c)$ is homotopic to c for all curves c on $R - R'$. Then $\Lambda(G) = \emptyset$.*

Proposition 3.4 ([2]) *Let R be a Riemann surface satisfying the lower and upper bound conditions, and G a subgroup of $\mathrm{Mod}^{\#}(R)$ satisfying the following: there exist compact subsets C_1 and C_2 on R such that, for every $[g_0] \in G$, $g(C_1) \cap C_2 \neq \emptyset$ for all quasiconformal maps g in $[g_0]$. Then $\Lambda(G) = \emptyset$.*

Using the results above, we give examples which show that limit sets of Teichmüller modular groups have different properties from those of Kleinian groups.

For a Kleinian group Γ, the limit set is coincident with the closure of the set of all loxodromic fixed points for Γ (see [6, Theorem 2.4]). Analogously to this fact, in [1, Problem 1], we proposed the problem that the closure $\overline{\Lambda_\infty(G)}$ of $\Lambda_\infty(G)$ is coincident with $\Lambda(G)$ for a subgroup $G \subset \mathrm{Mod}^{\#}(R)$. However, this is not true, in general.

Example 3.5 There exist a Riemann surface R such that $\overline{\Lambda_\infty(\mathrm{Mod}^{\#}(R))}$ is a proper subset of $\Lambda_\infty(\mathrm{Mod}^{\#}(R))$. Indeed, let R be a Riemann surface with only one puncture such that it does not satisfy the lower bound condition. Then the set of conformal automorphisms of R consists of only finitely many elements, and hence $\Lambda_\infty(\mathrm{Mod}^{\#}(R)) = \emptyset$. On the other hand, $\Lambda(\mathrm{Mod}^{\#}(R)) = T^{\#}(R)$ by Proposition 3.1.

For a normal subgroup Γ' of a non-elementary Kleinian group Γ, the limit set $\Lambda(\Gamma')$ of Γ' is coincident with that of Γ (see [6, Lemma 2.22]). However, for normal subgroups of Teichmüller modular groups, this is not true.

Example 3.6 There exists a Riemann surface R and two subgroups G_1 and G_2 of $\mathrm{Mod}^{\#}(R)$ such that G_2 is a normal subgroup of G_1, whereas the limit sets $\Lambda(G_1)$ and $\Lambda(G_2)$ do not coincide.

Indeed, let R be a Riemann surface that does not satisfy the lower bound condition, and R' a compact subset of R with non-abelian fundamental group. Let G_1 be the set of elements $[g] \in \mathrm{Mod}^{\#}(R)$ such that $g(R') = R'$. Since G_1 contains the Dehn twists along simple closed geodesics on R whose lengths tend to 0, we have $\Lambda(G_1) = \Lambda(\mathrm{Mod}^{\#}(R)) = T^{\#}(R)$ by the proof of Proposition 3.1. On the other hand, let G_2 be the set of elements $[g] \in G_1$ such that $g(c)$ is homotopic to c for all curves c on $R - R'$. Then G_2 is a normal subgroup of G_1. However, $\Lambda(G_2) = \emptyset$ by Proposition 3.3.

As we mentioned in the previous section, for a Kleinian group Γ and for the limit set $\Lambda(\Gamma)$ of Γ, the closure $\overline{\Gamma(p)}$ of the orbit $\Gamma(p)$ of a point $p \in \Lambda(\Gamma)$ under the action of Γ is coincident with $\Lambda(\Gamma)$. However, it is not true in the case of Teichmüller modular groups.

Example 3.7 There exist a Riemann surface R, a subgroup $G \subset \mathrm{Mod}^{\#}(R)$ and a point $p \in \Lambda(G)$ such that the orbit $G(p)$ of p under the action of G is a proper subset of $\Lambda(G)$.

For example, we consider a Riemann surface R that does not satisfy the lower bound condition. Let $\{c_i\}_{i=1}^{\infty}$ a family of mutually disjoint simple closed geodesics on R whose lengths tend to 0, and δ_i the Dehn twist along c_i. Let G be a subgroup of $\mathrm{Mod}^{\#}(R)$ that is the direct product of the infinite cyclic groups $\langle \delta_i \rangle$. Then, by the proof of Proposition 3.1, we see that $\Lambda(G) = \Lambda(\mathrm{Mod}^{\#}(R)) = T^{\#}(R)$. On the other hand, for any point $p \in \Lambda(G)$, the complex structures of accumulation points of $G(p)$ are the same as that of p. Indeed, there exists a compact subset C of R such that every element $[g]$ of G satisfies $g(C) \cap C \neq \emptyset$. Then we may assume that every sequence $\{g_n\}$ of distinct elements of G converges to a quasiconformal automorphism g of R, and we see that $g \in G$ (see [4, Proposition 2]). Thus $[g_n](p) \to [g](p)$ and $G(p)$ is closed, which means that the complex structures of accumulation points of $G(p)$ are the same as that of p. Hence, $G(p)$ is a proper subset of $T^{\#}(R) = \Lambda(G)$.

In Example 3.7, $G(p)$ is not discrete but it is closed, namely, $\overline{G(p)} = G(p)$. However, for a Riemann surface with the lower and upper bound conditions, a different situation occurs. The following proposition was proved by using Proposition 3.4.

Proposition 3.8 ([5]) *Suppose that R satisfies the lower and upper bound conditions. Let G be a subgroup of $\mathrm{Mod}^{\#}(R)$, and p a point in $T^{\#}(R)$. If the orbit $G(p)$ is not a discrete set in $T^{\#}(R)$, then $\overline{G(p)} - G(p) \neq \emptyset$.*

In [3, Example 5], we constructed a Riemann surface R such that it satisfies the lower and upper bound conditions and the orbit $G(p_0)$ of the base point p_0 of $T^{\#}(R)$ is not discrete for $G = \mathrm{Mod}^{\#}(R)$.

References

[1] E. Fujikawa, *Limit sets and regions of discontinuity of Teichmüller modular groups*, Proc. Amer. Math. Soc. **132** (2004), 117–126.

[2] E. Fujikawa, *Modular groups acting on infinite dimensional Teichmüller spaces*, In the tradition of Ahlfors-Bers, III: Proceedings of the 2001 Ahlfors-Bers Colloquium, Contemporary Math., American Mathematical Society, to appear.

[3] E. Fujikawa, H. Shiga and M. Taniguchi, *On the action of the mapping class group for Riemann surfaces of infinite type*, J. Math. Soc. Japan, to appear.

[4] K. Matsuzaki, *The infinite direct product of Dehn twists acting on infinite dimensional Teichmüller spaces*, Kodai Math. J. **26** (2003), 279–287.

[5] K. Matsuzaki, *Dynamics of Teichmüller modular groups and general topology of moduli spaces*, preprint.

[6] K. Matsuzaki and M. Taniguchi, *Hyperbolic Manifolds and Kleinian Groups*, Oxford Science Publications, 1998.

[7] S. A. Wolpert, *The length spectra as moduli for compact Riemann surfaces*, Ann. Math. **109** (1979), 323–351.