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Instability of nondiscrete free subgroups in Lie groups

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Abstract

Consider a nondiscrete free subgroup with two generators in a Lie group.

We study the following question stated by Étienne Ghys: is it always possible to make arbitrarily small perturbation of the generators of the free subgroup in such a way that the new group formed by the perturbed generators be not free? In other terms, is it possible to generate relations by arbitrarily small perturbation of the generators?

We prove the positive answer. We give a survey of related results and open questions.

Contents

1 Introduction and the plan of the paper

1.1 Statement of result and open questions

Let G be a nonsolvable Lie group. It is well-known (see [E]) that almost each (in the sense of the Haar measure) pair of elements $(A, B) \in G \times G$ generates a free subgroup in G. At the same time in the case, when G is semi-simple, there is a neighborhood $U \subset G \times G$ of unity in $G \times G$ where a topologically-generic pair $(A, B) \in U$ generates a dense subgroup: the latter pairs form an open dense subset in U.

The pairs generating groups with relations form a countable union of surfaces (relation surfaces) in $G \times G$. We show that the relation surfaces are dense in U.

The main result of the paper is the following

1.1 Theorem *Any nondiscrete free subgroup with two generators in a nonsolvable Lie group G is unstable. More precisely, consider two elements $A, B \in G$ generating a free subgroup*

$\Gamma = < A, B >$. Let Γ be not discrete. Then there exists a sequence $(A_k, B_k) \to (A, B)$ of pairs converging to (A, B) such that the corresponding groups $< A_k, B_k >$ have relations: there exists a sequence $w_k = w_k(a, b)$ of nontrivial abstract words in symbols a, b and their inverses a^{-1}, b^{-1} such that $w_k(A_k, B_k) = 1$ for all k.

1.2 Remark The lengths of the words w_k constructed in the paper tend to infinity exponentially in k, as $k \to \infty$. If $G = SL_n(\mathbb{R})$, $\overline{< A, B >} = G$ and the pair (A, B) satisfies some additional genericity condition, then the pairs (A_k, B_k) corresponding to the relations w_k tend to (A, B) also exponentially.

1.3 Remark The condition that the subgroup under consideration be nondiscrete is natural: one can provide examples of discrete free subgroups of $PSL_2(\mathbb{C})$ (e.g., the Schottky group) that remain free under any small perturbation of the generators.

The question of instability of nondiscrete free subgroups was stated by É Ghys.

1.4 Definition A real Lie group is said to be essentially compact, if its adjoint action preserves a positive definite scalar product.

1.5 Remark In the case, when the Lie group under consideration is $PSL_2(\mathbb{R})$, the Theorem easily follows from the density of elliptic elements of finite orders in an open domain of $PSL_2(\mathbb{R})$. A similar argument proves Theorem 1.1 in the case of essentially compact Lie group. The case of $PSL_2(\mathbb{C})$ is already nontrivial (in some sense, this is a first nontrivial case). In this case the previous argument cannot be applied, since elliptic elements in $PSL_2(\mathbb{C})$ are nowhere dense. At the same time, there is a short proof of Theorem 1.1 for $PSL_2(\mathbb{C})$ that uses holomorphic motions and quasiconformal mappings. We present it at the end of the paper.

Ghys have also proposed to study approximations of free subgroups by nonfree ones in the following sense. It is well-known that for any $\varepsilon > 0$ and a generic (more precisely, ε-Diophantine) irrational number r there exists a $C > 0$ such that for any irreducible fraction $\frac{m}{n} \in \mathbb{Q}$ one has $|r - \frac{m}{n}| > \frac{C}{n^{2+\varepsilon}}$. This approximation accuracy is optimal in some sense: the continuous fractions give approximations of accuracy no worse than $\frac{1}{n^2}$.

Les us say that a pair of elements of a Lie group is *irrational*, if it generates a free dense subgroup. A pair of elements generating a group with relations will be called *rational*; its *denominator* is the minimal length of relation.

Question 1 (É.Ghys). Given a generic irrational pair of elements in a Lie group. What is the optimal asymptotic accuracy of its approximations by rational pairs, as their denominators tend to infinity?

1.6 Remark The number of (reduced) words $w(a, b)$ of a given length l grows exponentially in l. This motivates the following

Question 2 Is it true that for any irrational pair $(A, B) \in G \times G$ there exist a $c > 0$ and a sequence of rational pairs (A_k, B_k) with denominators l_k such that $dist((A_k, B_k), (A, B)) < e^{-cl_k}$ for all k?

A question (related to the latter one) concerning Diophantine properties of and individual pair $A, B \in SO(3)$ was studied in [KR]. We say that a pair $(A, B) \in SO(3) \times SO(3)$ is *Diophantine* (see [KR]), if there exists a constant D depending on A and B such that for any (reduced) word $w_k = w_k(a, b)$ of length k

$$|w_k(A, B) - 1| > D^{-k}.$$

A.Gamburd, D.Jacobson and P.Sarnak have stated the following question in their joint paper [GJS]: is it true that almost each pair $(A, B) \in SO(3) \times SO(3)$ is Diophantine? V.Kaloshin and I.Rodnianski [KR] proved that almost each pair (A, B) satisfies a weaker inequality with the latter right-hand side replaced by D^{-k^2}.

Question 3 Is there an analogue of Theorem 1.1 for the group of
- germs of one-dimensional real diffeomorphisms?
- germs of one-dimensional conformal diffeomorphisms?
- diffeomorphisms of compact manifold?

The latter question concerning conformal germs is related to study of one-dimensional holomorphic foliations on $\mathbb{CP}^2$ with isolated singularities. A generic vector field on $\mathbb{C}^2$ defines a foliation on $\mathbb{CP}^2$ with invariant line L at infinity; the line L contains a finite number of singularities $a_1, \dots, a_n \in L$.

Let $\Delta \subset \mathbb{CP}^2$ be a transversal cross-section to L, a be its point of intersection with L. A circuit in L around each a_i with base point at a defines a germ of conformal holonomy mapping $h_i : (\Delta, a) \to (\Delta, a)$. Let us choose the circuit around each a_i so that the n circuits generate the fundamental group of $L \setminus \{a_1, \dots, a_n\}$. The *monodromy group* is the group generated by the holonomy germs h_i.

Yu.S.Ilyashenko and A.S.Pyartli have proved [IP] that for a generic polynomial vector field of degree greater than two the monodromy group is free. "Generic" means "lying outside at most a countable union of analytic surfaces".

Question 3. Does there exist an open (or open and dense) subset U in the space of polynomial vector fields of fixed degree (say, greater than two) such that the monodromy group of each vector field from U is free?

1.2 Historical remarks

The famous Tits' alternative [T] says that any subgroup of linear group satisfies one of the two following incompatible statements:
- either it is solvable up-to-finite, i.e., contains a solvable subgroup of a finite index;
- or it contains a free subgroup with two generators.

Any dense subgroup of a semisimple real Lie group satisfies the second statement: it contains a free subgroup with two generators.

The question of possibility to choose the latter free subgroup to be dense was stated in [CG] and studied in [BZ] and [CG]. É.Ghys and Y.Carrière [CG] have proved the positive answer in a particular case. E.Breuillard and T.Gelander [BZ] have proved the positive answer in the general case.

1.3 Motivation and proof of Theorem 1.1 modulo technical details

Let G be the Lie group under consideration, $n = dimG$. Without loss of generality we assume that the group $< A, B >$ is dense (passing to a Lie subgroup).

We prove Theorem 1.1 for a connected simple (real or complex) Lie group (that is not essentially compact in the real case). Theorem 1.1 in the general case then follows easily by using the classical decomposition theorems for Lie algebras: any Lie algebra splits into a semidirect product of a semisimple Lie algebra and a solvable one (the maximal solvable ideal); any semisimple Lie algebra is a direct product of simple ones, see [VO].

In the present note we give a proof of Theorem 1.1 only in the case, when G is a connected complex simple Lie group.

Consider the abstract words $w(a, b)$ as functions in $(a, b) \in G \times G$ with values in G. Theorem 1.1 says that (A, B) is a limit point of relations, see the following Definition.

1.7 Definition Let G be a Lie group, $\gamma_1(s), \gamma_2(s), \dots \in G$ be a countable collection of families of its elements depending on a (finite-dimensional) parameter s. We say that a parameter value s_0 is a *limit point of relations* in the group family $\Gamma(s) =< \gamma_1(s), \gamma_2(s), \dots >$, if there exist a sequence $w_k(\gamma_i)$ of abstract words (of finite lengths) in symbols γ_i, γ_i^{-1} $(i = 1, 2, \dots)$ and a sequence $s_k \to s_0$ such that $w_k(\gamma_i(s_k)) = 1$ for any $k = 1, 2, \dots$, but $w_k(\gamma_i(s)) \not\equiv 1$.

For any $\delta > 0$ and $Q = (Q_1, \dots, Q_n) \in \mathbb{C}^n$ denote $I_\delta(Q)$ the δ- polydisc centered at Q.

To prove the Theorem, we construct a n- parametric deformation $a(S)$, $b(S)$ of the elements A, B, $S = (s_1, \dots, s_n)$, $s_i \in \mathbb{C}$, $|s_i| < 1$, such that the initial parameter value $S^0 = (s_{01}, \dots, s_{0n})$, $A = a(S^0)$, $B = b(S^0)$, is a limit point of relations in the groups $< a(S), b(S) >$. The deformation will be chosen to satisfy a certain genericity condition. For any word $w(a, b)$ denote

$$W(S) = w(a(S), b(S)).$$

To show that S^0 is a limit point of relations, for appropriate $\delta > 0$ we construct a sequence of words $w_k(a, b)$ such that the images under $W_k : S \mapsto w_k(a(S), b(S))$ of small polydiscs $I_{\varepsilon_k}(S^0)$, $\varepsilon_k = O(\frac{1}{k}) \to 0$, contain one and the same polydisc $I_\delta(1)$:

$$W_k(I_{\varepsilon_k}(S^0)) \Supset I_\delta(1), \text{ see Fig.1.} \tag{1.1}$$

This implies that the equation $W_k(S) = 1$ has a solution $S^k \in I_{\varepsilon_k}(S^0)$. One has $S^k \to S^0$, as $k \to \infty$, since $\varepsilon_k \to 0$.

Motivation of the proof of Theorem 1.1. By density, we can always construct a sequence of words w_k so that $w_k(A, B) \to 1$. In the case, when A and B are close to unity, an explicit way to do this is to take w_k to be a sequence u_k of appropriate successive commutators.

To achieve inclusion (1.1), we have to guarantee that the derivative in S of $W_k(S)$ at S^0 tend to infinity. On the other hand, the previous commutators u_k converge exponentially to unity and have exponentially decreasing derivatives in the parameters of the pair (A, B).

The starting point of the construction of w_k is the following observation. Take a commutator u_k from the previously mentioned sequence and take its power $u_k^{m_k}$ so that the value $u_k^{m_k}(A, B)$ be distant from unity (uniformly in k) and belong to some fixed neighborhood of unity (independent on k). Then the derivative of the function $u_k^{m_k}(a, b)$ at (A, B) along

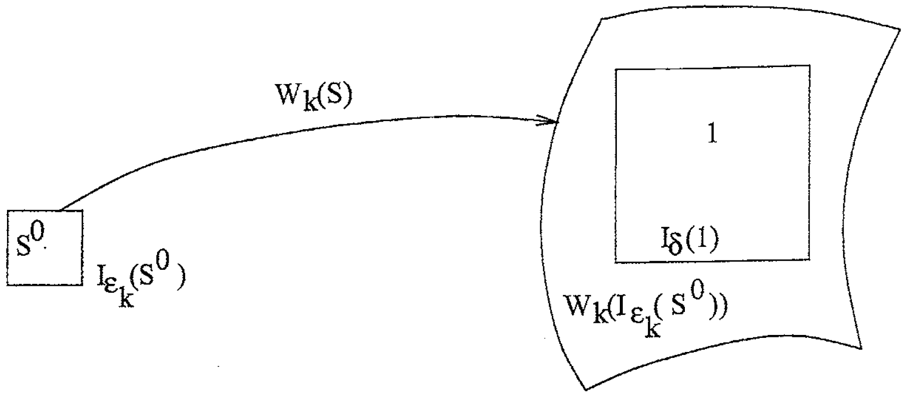

Figure 1:

appropriate direction (independent on k) will be large: it will grow linearly in k, thus, it will tend to infinity. The simplest one-dimensional version (Proposition 1.11) of this statement is formulated and proved below.

In what follows we construct n sequences (the elements of each sequence are numerated by $k = 1, 2, \dots$) of appropriate iterated commutators $w_{ik}(a, b)$ of some words $g_i(a, b)$ (see (1.11)), $i = 1, \dots, n$, so that $w_{ik}(A, B) \to 1$ exponentially, as $k \to \infty$ and the following additional statement holds. For any $k \in \mathbb{N}$ let us choose a collection

$$M_k = (m_{1k}, \dots, m_{nk}), \ m_{ik} \in \mathbb{N},$$

so that the powers $w_{ik}^{m_{ik}}(A, B)$ be distant from unity (uniformly in k). Put

$$\widetilde{w}_{M_k} = w_{1k}^{m_{1k}} \dots w_{nk}^{m_{nk}}.$$

Then the derivative of $\widetilde{W}_{M_k}(S) = \widetilde{w}_{M_k}(a(S), b(S))$ at S^0 in any direction tends to infinity, as $k \to \infty$ (uniformly in the direction).

Afterwards we fix an appropriate word w such that the value $w(A, B)$ satisfies some genericity condition (in particular, $w(A, B) \neq 1$). For each k we choose the powers m_{ik} in the previous product $\widetilde{w}_{M_k}$ in such a way that $\widetilde{w}_{M_k}(A, B)$ provides a best possible approximation for $w(A, B)$. We show then that the equation

$$w(a(S), b(S)) = \widetilde{w}_{M_k}(a(S), b(S))$$

has a solution $S = S^k$, $S^k \to S^0$, so, the words $w_k = w^{-1}\widetilde{w}_{M_k}$ are those we are looking for: $w_k(a(S^k), b(S^k)) = 1$. This will prove Theorem 1.1.

In the next Example we consider a simple family of (abelian) additive subgroups in $\mathbb{R}$. We prove the well-known statement saying that each parameter value is a limit point of relations (Proposition 1.10). The proof of Theorem 1.1 given below uses analogous arguments.

1.8 Example Consider the group $A_0(\mathbb{R})$ of affine automorphisms of $\mathbb{R}$ preserving orientation. This group is generated by multiplications by positive constants and by translations. The subgroup of translations in $\mathbb{A}_0(\mathbb{R})$ is canonically identified with $\mathbb{R}$ and will be denoted by the same symbol $\mathbb{R}$. Define the following family of subgroups $\Gamma(s) \subset \mathbb{R} \subset \mathbb{A}_0(\mathbb{R})$:

$$g(s): x \mapsto sx,\ s > 0,\ t_1 : x \mapsto x + 1,\ g(s), t_1 \in \mathbb{A}_0(\mathbb{R}),\ \ \Gamma(s) =< g(s), t_1 > \cap \mathbb{R}. \qquad (1.2)$$

More generally, for any $u \in \mathbb{R}$ denote

$$t_u : x \mapsto x + u, \quad (t_u \in \mathbb{R} \subset \mathbb{A}_0(\mathbb{R})).$$

1.9 Remark For any $s > 0$ the group $\Gamma(s)$ contains the elements t_{ms^k}, $m \in \mathbb{Z}$, $k \in \mathbb{N} \cup 0$, since it contains $t_{s^k} = g(s)^k \circ t_1 \circ g(s)^{-k}$. In fact, $\Gamma(s) =< t_{ms^k} \mid m, k \in \mathbb{N} \cup 0 >$. The following statement is well-known.

1.10 Proposition *Let $\Gamma(s) \subset \mathbb{R}$ be the subgroup family from (1.2). Each parameter value $s \in \mathbb{R}_+$ is a limit point of relations in the group family $\Gamma(s)$. The same statement holds for nonzero complex parameter values s and the corresponding family $\Gamma(s) \subset \mathbb{C}$ of subgroups in the complex affine group.*

In the proof of Proposition 1.10 and Theorem 1.1 we use the following

1.11 Proposition *Let $0 < s_0 < 1$, $0 < \delta < u$. For any $k \in \mathbb{N}$ let $m_k \in \mathbb{N}$ be the number chosen so that $m_k s_0^k$ provides a best possible approximation for u:*

$$|m_k s_0^k - u| < s_0^k. \text{ Put } \psi_k(s) = m_k s^k. \qquad (1.3)$$

There exists a sequence $\varepsilon_k > 0$, $\varepsilon_k = O(\frac{1}{k}) \to 0$, such that the ψ_k- image of the ε_k- neighborhood of s_0 contains the closed δ- neighborhood of u. The same statement holds true for any $s_0, u \in \mathbb{C}^* = \mathbb{C} \setminus 0$, $\delta > 0$ such that $|s_0| < 1$, $|u| > \delta$ and $m_k \in \mathbb{N}$ are chosen so that $m_k |s_0|^k$ be a best possible approximation of $|u|$: $||m_k|s_0|^k - |u|| < |s_0|^k$.

Proof To avoid the details, we give the proof in the real case only: the proof in the complex case is analogous. By definition,

$$\psi_k'(s) = ks^{-1}\psi_k(s), \text{ hence, } \psi_k'(s) \to \infty, \text{ as } k \to \infty, \text{ if } s^{-1},\ \psi_k(s) \text{ are distant from } 0.$$

The closed δ- neighborhood of u is disjoint from 0. By definition, the function $s(\psi_k)$ inverse to ψ_k is uniformly bounded on this neighborhood, thus, s^{-1} is uniformly bounded from below. This together with the previous formula implies that the derivative $(\psi_k'(s))^{-1}$ of the inverse function is $O(\frac{1}{k})$ on the same neighborhood. This implies the Proposition. □

Proof of Proposition 1.10. Fix a $s_0 > 0$ and let us prove that it is a limit point of relations in $\Gamma(s)$. Without loss of generality we suppose that $s_0 \neq 1$ (one can achieve this by perturbation of s_0) and $0 < s_0 < 1$ (one can achieve this by changing $g(s_0)$ to its inverse). Proposition 1.11 applied to $u = 1$ implies that there exists a sequence $m_k \in \mathbb{N}$, $m_k \to \infty$, as $k \to \infty$, such that the equation

$$m_k s^k = 1$$

has a solution $s_k \to s_0$, as $k \to \infty$. By definition, the parameter values $s = s_k$ correspond to the (nonidentical) relations $t_{s^k}^{m_k} = t_1$ in $\Gamma(s)$. This proves Proposition 1.10. □

1.12 Remark The previous values s_k corresponding to the latter relations have asymptotics $s_k - s_0 = O(\frac{1}{k})$. In the real case $s_k \to s_0$ exponentially: $s_k - s_0 = O(\frac{s_0^k}{k})$.

For the proof of Theorem 1.1 we construct a deformation $(a(S), b(S))$ depending on a n-dimensional parameter $S = (s_1, \dots, s_n)$, $s_i \in \mathbb{C}$, and auxiliary words $w_{ik}(a,b)$, $i = 1, \dots, n$, $k \in \mathbb{N}$, satisfying the following statements.

1.13 Lemma *Let G be a complex simple Lie group of dimension n. Then a generic pair of elements $A, B \in G$ generating a dense subgroup admits a deformation $(a(S), b(S))$ depending on a complex parameter $S = (s_1, \dots, s_n)$, $0 < |s_i| < 1$, that satisfies the following statement. Let S^0 be the parameter value corresponding to the initial pair: $(A, B) = (a(S^0), b(S^0))$. There exist words $w_{ik}(a,b)$, $i = 1, \dots, n$, $k \in \mathbb{N}$, and a basis $v_1, \dots, v_n$ of the Lie algebra T_1G such that*

$$w_{ik}(a(S), b(S)) = \exp(s_i^k v_{ik}(S)), \;\; v_{ik}(S) = v_i + o(1), \;\; as\ k \to \infty, \;\; S \to S^0, \tag{1.4}$$

where the vector function $v_{ik}(S)$ in the previous exponent has uniformly bounded (in k) derivatives in S in some neighborhood of S^0 (independent on k) and converges uniformly on the same neighborhood to an analytic vector function.

The Lemma is proved in the next two Subsections.

Proof of Theorem 1.1. Consider the family $(a(S), b(S))$ and the vectors v_i from the Lemma. The latters form a base, hence, the mapping $(t_1, \dots, t_n) \mapsto \exp(t_1 v_1), \dots, \exp(t_n v_n)$ is a 1-to-1 mapping of a neighborhood of 0 in the complex t- space onto a neighborhood $V \subset G$ of unity. This defines a coordinate system on V. The group generated by A and B is dense, hence, there exists an abstract word $w(a,b)$ such that $w(A,B) \in V$, hence,

$$w(A,B) = \exp(\tau_1 v_1) \dots \exp(\tau_n v_n), \;\; \tau_i \in \mathbb{C}. \tag{1.5}$$

We choose (and fix) the word w so that in addition $\tau_i \neq 0$ for each $i = 1, \dots, n$ (this is possible by density). Then

$w(a(S), b(S)) = \exp(t_1(S) v_1) \dots \exp(t_n(S) v_n)$, $t_i(S^0) = \tau_i$, $t_i(S)$ are analytic in S. Let

$$S^0 = (s_{01}, \dots, s_{0n}).$$

For any $i = 1, \dots, n$ and any k (large enough) let $m_{ik} \in \mathbb{N}$ be the minimal number such that $|\tau_i| < m_{ik} |s_{0i}|^k$ (in other terms, such that $m_{ik}|s_{0i}|^k$ be a best approximation for $|\tau_i|$). Put

$$M_k = (m_{1k}, \dots, m_{nk}), \;\; \widetilde{w}_{M_k} = w_{1k}^{m_{1k}} \dots w_{nk}^{m_{nk}}.$$ Then by (1.4),

$$\widetilde{W}_{M_k}(S) = \widetilde{w}_{M_k}(a(S), b(S)) = \exp(m_{1k} s_1^k v_{1k}(S)) \dots \exp(m_{nk} s_n^k v_{nk}(S)). \tag{1.6}$$

There exist a $\delta > 0$ and a sequence $\varepsilon_k > 0$, $\varepsilon_k = O(\frac{1}{k}) \to 0$, such that

$$\widetilde{W}_{M_k}(I_{\varepsilon_k}(S^0)) \Supset I_\delta(w(A,B)). \tag{1.7}$$

Indeed, this would be true if the function $\widetilde{W}_{M_k}(S)$ be given by (1.6) where $v_{ik}(S) = v_i \equiv const$ be constant vectors independent on k (then the coordinate t_i of the value $\widetilde{W}_{M_k}(S)$ will be equal to $m_{ik}s_i^k$). This follows from Proposition 1.11 and linear independence of the v_i' s. Taking variable vectors $v_{ik}(S)$ corresponds to taking a variable coordinate system, where the coordinates of the corresponding values $\widetilde{W}_{M_k}(S)$ are given by the same formula $m_{ik}s_i^k$. The variable coordinate systems are well defined and have uniformly bounded derivatives in S in some neighborhood of S^0 (independent on k), as do $v_{ik}(S)$ (Lemma 1.13). Inclusion (1.7) persists under a variation of the coordinate system with uniformly bounded derivative (for appropriate $\delta > 0$ independent on k). This proves (1.7).

By (1.7), the equation

$$\widetilde{w}_{M_k}(a(S), b(S)) = w(a(S), b(S))$$

has a solution $S^k \to S^0$, as in the proof of Proposition 1.10. Thus, we have found the sequence of words

$$w_k(a,b) = w^{-1}\widetilde{w}_{M_k}, \ w_k(a(S^k), b(S^k)) = 1.$$

The words w_k are nontrivial, since their values at $(a(S), b(S))$ are not equal to 1 identically in S: their derivatives in S at S^0 are large and tend to infinity, as $k \to \infty$ (as those of $\widetilde{w}_{M_k}(a(S), b(S))$; the derivative of $w^{-1}(A(S), B(S))$ at S^0 is independent on k). Theorem 1.1 is proved modulo Lemma 1.13. □

1.4 The iterated commutators w_{ik}. The sketch of the proof of Lemma 1.13

To define the deformation and the parameter S from Lemma 1.13, let us introduce the following notation. Recall that the group G acts on itself by conjugations: $g : h \mapsto ghg^{-1}$. The unity is fixed. The derivative at unity of the action of g is a linear operator denoted

$$Ad_g : T_1G \to T_1G \text{ and called the } \textit{adjoint action of } g.$$

(If G is a matrix group, then the adjoint action is also defined by matrix conjugation: for any $h \in T_1G$ one has $Ad_g(h) = ghg^{-1}$.) For a generic element $g \in G$ put

$$s(g) = \text{the eigenvalue of } Ad_g - Id \text{ with the maximal module; denote } v_g \text{ the eigenvector.} \tag{1.8}$$

1.14 Proposition *Let G be a complex simple Lie group. For a generic element $g \in G$ the previous value $s(g)$ is uniquely defined.*

For the proof of the Lemma we show that if A, B are generic elements, then there exist n words $g_i(a,b)$, $i = 1, \dots, n$, such that the values $s_i = s(g_i(a,b))$ (as functions in variable elements $a, b \in G$) are independent: their rank at $a = A$, $b = B$ is equal to n (see Lemma 1.15 below). This is the main technical part of the proof of Lemma 1.13. We show that one can achieve that $0 < |s_{0i}| = |s_i(A,B)| < 1$ and the vectors

$$v_i = v_{g_i(A,B)} \text{ be linearly independent.}$$

Take arbitrary n- parametric deformation of A, B such that the previous functions s_i (considered now as functions in the parameter of deformation) are independent. Then one can put $S = (s_1, \dots, s_n)$ to be the parameter of deformation. As it is shown below, the deformation $(A(S), B(S))$ thus constructed is a one we are looking for.

1.15 Lemma (Main Technical Lemma). *Let G be a complex simple Lie group, $dimG = n$. Then there exist n abstract words $g_i(a,b)$, $i = 1,\dots,n$, satisfying the following statements.*

1) The values $s_i(a,b) = s(g_i(a,b))$ are well-defined for a generic pair (a,b).

2) The functions $s_i(a,b)$ have rank n on an open and dense set of pairs $(a,b) \in G \times G$.

Addendum to Lemma 1.15. *Let G be a complex simple Lie group, $dimG = n$, $\varepsilon > 0$, $A_1,\dots,A_n \in G$. Let $A, B \in G$ be a pair of elements generating a dense subgroup. Then the words $g_i(a,b)$ from Lemma 1.15 can be chosen so that $dist(g_i(A,B), A_i) < \varepsilon$.*

Lemma 1.15 is proved in the next Subsection. The proof of the Addendum is omitted here: it will be presented in the complete version of the paper. Roughly speaking, the Addendum follows from the results of the next Subsection.

1.16 Proposition *Let G be a complex simple Lie group, $n = dimG$. For a generic collection of n elements $A_1,\dots,A_n \in G$ the corresponding adjoint action eigenvectors v_{A_i} (see (1.8)) are well-defined and linearly independent.*

1.17 Corollary *Let G be a complex simple Lie group. Let $(A,B) \in G \times G$ be a given pair generating a dense subgroup in G. If the pair (A,B) is generic, then there exist words $g_i(a,b)$, $i = 1,\dots,n$, such that the functions $s_i(a,b) = s(g_i(a,b))$ have rank n in some neighborhood of (A,B) and in addition*

3) the vectors $v_i = v_{g_i(A,B)}$ are linearly independent;

4) $0 < |s_i(A,B)| < 1$.

The Corollary follows from the Addendum applied to a generic collection $A_1,\dots,A_n \in G$ of elements close to unity.

Using the words g_i from the Corollary (we suppose that (A,B) is generic), we construct the words w_{ik} that satisfy the statement of Lemma 1.13. To do this, let us introduce the following notation.

For any $g \in G$ define the commutator mapping $\phi_g : G \to G$:

$$\phi_g(h) = ghg^{-1}h^{-1}. \tag{1.9}$$

We use the following properties of the commutator mapping.

1.18 Proposition *Let G be a Lie group, $g \in G$. The derivative at unity of the mapping ϕ_g is equal to $Ad_g - Id$. In particular, its eigenvalue with maximal module is equal to $s(g)$; the corresponding eigenvector is v_g.*

1.19 Corollary *Let G be a Lie group, $g \in G$ be an element (close to unity) such that $s(g)$ be uniquely defined and $|s(g)| < 1$. Then the mapping ϕ_g is contracting to unity. It has the unique weakest contracting direction v_g and a unique strong contracting invariant hypersurface Σ_g (passing through the unity) transversal to v_g, see Fig.2. For any $x \in G \setminus \Sigma_g$ close to unity*

$$\phi_g^k(x) = \exp(s^k(g)(cv_g + o(1))), \text{ as } k \to \infty, \; c = c(x,g) \text{ is independent on } k. \tag{1.10}$$

The derivative of the latter $o(1) = o(1)(x,g)$ in the parameters of (x,g) is uniformly bounded (in k) on some neighborhood of (x,g) (independent on k) and tends to zero on the latter neighborhood, as $k \to \infty$.

The dynamics of ϕ_g

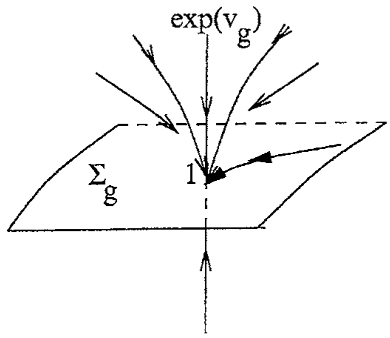

Figure 2:

Proof of Lemma 1.13 modulo Lemma 1.15 and its Addendum. Let g_i be the words from Corollary 1.17. Consider the strong contracting hypersurfaces of $\phi_{g_i(A,B)}$. Let $h(a,b)$ be a word such that its value $h(A,B)$ lies outside the previous hypersurfaces and is attracted to unity under each mapping $\phi_{g_i(A,B)}$, $i = 1, \ldots, n$. Put

$$w_{i0} = h, \ \ w_{ik} = g_i w_{i(k-1)} g_i^{-1} w_{i(k-1)}^{-1}. \tag{1.11}$$

The words w_{ik} defined by recurrent formula (1.11) satisfy the statements of Lemma 1.13. Indeed, the asymptotics (1.4) holds with $v_i = v_{g_i(A,B)}$ by definition and (1.10) (one has to normalize the previous vectors in appropriate way so that the corresponding constants c from (1.10) be equal to 1). Lemma 1.13 is proved. □

1.5 Independent eigenvalues. Proof of Lemma 1.15

Denote Ω_0 the space of all the nontrivial abstract words $w(a,b)$ in symbols a, b, a^{-1}, b^{-1}.

Consider each abstract word $w(a,b)$ as a function $G \times G \to G$. Let us choose an eigenvalue $\lambda_w = \lambda_w(a,b)$ of $Ad_{w(a,b)}$ and consider it as a (multivalued) function in (a,b). Let us choose a λ_w for each w.

1.20 Definition Any collection

$$\{\lambda_w\}_{w \in \Omega_0}$$

is called a *complete system of eigenvalues* (associated to words from Ω_0).

1.21 Remark For any finite collection of words $w_1, \ldots, w_k$ and any system of eigenvalue functions $\lambda_{w_i}(a,b)$ associated to them the rank of the functions $\lambda_{w_i}(a,b)$ is constant on a connected open dense subset in $G \times G$. This implies that the rank of a complete system of eigenvalues is constant on an intersection of open dense subsets in $G \times G$. The corresponding ranks will be simply called *the ranks of systems of eigenvalues*. (The rank of an infinite collection of functions is defined to be the maximal rank of a finite subcollection.)

Lemma 1.15 is implied by the following more general

1.22 Lemma *The rank of any complete system of eigenvalues is equal to n.*

We will consider also *generalized* $\mathbb{C}$- *($\mathbb{R}$- or $\mathbb{Q}$-) words*

$$w(a,b) = a^{t_1} b^{t_2} \dots a^{t_k}, \ t_i \in \mathbb{C}, \mathbb{R}, \mathbb{Q} \text{ respectively,}$$

whose symbols are powers of a and b; the number of the symbols is finite. Denote $\Omega(\mathbb{C})$, $\Omega(\mathbb{R})$, $\Omega(\mathbb{Q})$ the corresponding spaces of all the generalized words with exponents respectively in $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$.

Let $(a,b) \in G \times G$ be a given pair, $w(a,b)$ be a given generalized word (we consider it as a multivalued analytic function in (a,b) with values in G). Let $\lambda_w(a,b)$ be some eigenvalue of $Ad_{w(a,b)}$. The value $\lambda_w(a,b)$ depends analytically on (a,b) and the exponents of w. This yields a (multivalued) analytic family λ_w of (multivalued) analytic functions in (a,b) depending on the parameters t_i, which are the exponents of w. (The branching points vary with t_i.) The family λ_w thus obtained is called an *analytic eigenvalue family* (the length of the word is not fixed).

1.23 Proposition *The rank of any complete system of eigenvalues associated to words in Ω_0 is no less than the rank of appropriate analytic eigenvalue family associated to the words from $\Omega(\mathbb{C})$.*

Proof Replacing symbols a, b by their degree 2^N roots transforms the words from Ω_0 to those from $\Omega(\mathbb{Q})$ with rational exponents $t_i \in 2^{-N}\mathbb{Z}$ (denote Ω_N the space of the latter words). Passing to the roots is a locally 1-to-1 transformation at a neighborhood of a generic pair $(a,b) \in G \times G$. This implies that for any $N \in \mathbb{N}$ the rank of a complete eigenvalue system corresponding to words from Ω_0 is equal to the rank of another eigenvalue system corresponding to the words from Ω_N. Passing to the limit, as $N \to \infty$, yields an analytic eigenvalue family associated to words from $\Omega(\mathbb{R})$ whose rank is no greater than that of the initial system of eigenvalues. This follows from a Baire category argument (in the space of exponents). The new eigenvalue family extends to the complex parameter space of complex exponents without increasing the rank (the latter statement follows from analyticity). This proves the Proposition. □

1.24 Lemma *The rank of any analytic eigenvalue family associated to the words from $\Omega_{\mathbb{C}}$ is equal to n.*

Lemma 1.24 is proved below. Together with Proposition 1.23, it implies Lemma 1.22.

For any $(a,b) \in G \times G$ denote $Conj(a,b)$ the conjugacy class of the pair (a,b) (with respect to the action of G on $G \times G$ by conjugations).

1.25 Proposition *Let G be a connected complex simple Lie group, $n = dimG$. The dimension of conjugacy class of a generic pair $(a,b) \in G \times G$ is equal to n. (Hence, its codimension is also equal to n.)*

1.26 Remark Each eigenvalue of any (may be generalized) word $w(a,b)$ is constant on each conjugacy class. Thus, the rank of any system of eigenvalues of words is no greater than n.

Lemma 1.24 is implied by the following

1.27 Lemma *A generic pair $(A,B) \in G \times G$ satisfies the following statement. Let $v \in T_{(A,B)}(G \times G)$ be a vector such that there exists an analytic eigenvalue family $\lambda_w(a,b)$ depending on $w \in \Omega(\mathbb{C})$ such that $\frac{d\lambda_w}{dv}(A,B) = 0$ for all w. Then the vector v is tangent to $Conj(a,b)$.*

Proof By analyticity, it suffices to prove the statement of the Lemma for a generic pair (a,b) generating a dense subgroup.

1.28 Proposition *Let G be a connected complex simple Lie group, $a,b \in G$ be elements generating a dense subgroup. Then each element $g \in G$ can be represented as a value $w(a,b)$ of appropriate generalized word $w \in \Omega(\mathbb{C})$.*

When we move (a,b) in the direction v, the values $w(a,b)$ are moved in the direction of the derivatives of $w(a,b)$ along v. In what follows we prove that this motion is given by an infinitesimal local automorphism of G. A classical theorem (see [VO]) says that each local automorphism close to identity of a simple Lie group is given by a conjugation with some element $g \in G$. This together with the previous statement implies that v is tangent to the conjugacy class of (a,b). To show that the previous motion is given by an infinitesimal automorphism, let us introduce the following notations.

Given $a,b \in G$, $v \in T_{(a,b)}(G \times G)$, denote

$$\Omega(a,b) = \{w \in \Omega(\mathbb{C}) |\ w(a,b) = 1\},\ U(a,b) = \{\frac{dw(a,b)}{dv} |\ w \in \Omega(a,b)\} \subset T_1 G. \qquad (1.12)$$

1.29 Proposition *The space $U(a,b)$ is an ideal in the Lie algebra T_1G.*

The Proposition follows from definition.

Recall that the Lie algebra T_1G is simple, hence, it contains no nontrivial ideal. Therefore, by the previous Proposition, either $U(a,b) = 0$, or $U(a,b) = T_1G$. The latter case is impossible: it would contradict vanishing of the derivatives along v of the eigenvalues λ_w. Thus,

$$U(a,b) = 0.$$

This implies that if two words $w_1, w_2 \in \Omega(\mathbb{C})$ take the same value at (A,B), then they have the same derivative along v. Hence, the derivatives of $w(a,b)$ form a well-defined vector field on G. The field is analytic, since it depends analytically on the exponents of the words. Therefore, it defines an infinitesimal local analytic diffeomorphism of G. The latter is an infinitesimal automorphism, since the previous motion of words respects multiplication (by definition). This together with the previous discussion proves the Lemma. The proof of Lemmas 1.27, 1.24 and 1.15 is completed. □

2 A short proof of Theorem 1.1 for $G = PSL_2(\mathbb{C})$

Let $A,B \in PSL_2(\mathbb{C})$ generate a dense free group. We prove Theorem 1.1 by contradiction. Suppose there is a neighborhood $V \subset PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ of the pair (A,B) such that each pair $(a,b) \in V$ generates a free subgroup. Thus, each word $w(a,b)$ is a holomorphic function in $(a, b) \in PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ with values in $PSL_2(\mathbb{C})$; distinct words define holomorphic functions with disjoint graphs over V. Using holomorphic motion of fixed points of the elements $w(a, b) \in PSL_2(\mathbb{C})$, we construct a nonstandard measurable almost complex structure on $\overline{\mathbb{C}}$ invariant under the action of $< A, B >$ (and hence, under the action of the whole group $PSL_2(\mathbb{C})$ by density). But the only measurable almost complex structure preserved under the action of $PSL_2(\mathbb{C})$ on $\overline{\mathbb{C}}$ is the standard complex structure - a contradiction.

2.1 Remark The author's initial proof of Theorem 1.1 in the case, when $G = PSL_2(\mathbb{C})$, followed a similar scheme (using holomorphic motion of fixed points) but was longer than the one presented below. The final quasiconformal mapping argument, which simplified the proof essentially, is due to Étienne Ghys.

Recall that an element $b \in PSL_2(\mathbb{C})$ is called *elliptic*, if its action on $\overline{\mathbb{C}}$ is conjugated to a rotation. It is called *hyperbolic*, if it has two fixed points: one attracting and the other one repelling. Otherwise it is *parabolic*, i.e., has a unique fixed point and is conjugated to the translation.

The liberty assumption implies that the elements $w(a, b) \in PSL_2(\mathbb{C})$ are hyperbolic whenever $(a, b) \in V$: in other terms, the multipliers of their fixed points have modules different from 1. Indeed, their fixed points are holomorphic functions in $(a, b) \in V$ (may be multivalued with possible double branchings corresponding to the parabolic elements $w(a, b)$), and so are the multipliers of the fixed points. Suppose the multiplier $\mu(a, b)$ of a fixed point of $w(a, b)$ has a unit module. Then one can find points (a', b') arbitrarily close to (a, b) where $\mu(a', b')$ is equal to nontrivial roots of unity (the holomorphic mappings are open). This means that $w(a', b') \in PSL_2(\mathbb{C})$ is a nontrivial element of finite order - a contradiction to the liberty assumption.

Thus, each element $w(a, b) \in PSL_2(\mathbb{C})$, $(a, b) \in V$, is hyperbolic, hence, its fixed points are analytic functions in (a, b). The graphs of the fixed point functions are disjoint. Indeed, otherwise, if two distinct hyperbolic elements of $PSL_2(\mathbb{C})$ have one common fixed point, then their commutator is parabolic: the latter fixed point is its unique fixed point. This contradicts the hyperbolicity of the commutator.

In other words, the previous fixed point families form a holomorphic motion over V of the fixed points of the elements $w(A, B)$. The latter elements are dense by assumption, hence, so are their fixed points. The previous holomorphic motion extends up to a holomorphic motion of the Riemann sphere: this means that one can extend the collection of graphs of fixed points of $w(a, b)$ up to filling-in the product $V \times \overline{\mathbb{C}}$ by disjoint graphs of holomorphic functions on V with values in $\overline{\mathbb{C}}$. This follows immediately from density and an elementary normality argument (e.g., a version of Montel's theorem, see [L]).

2.2 Remark The well-known Slodkowski theorem [S] says that any holomorphic motion in $D \times \overline{\mathbb{C}}$ of any subset of the Riemann sphere over unit disc D extends up to a holomorphic motion of the whole Riemann sphere. Here we do not use this theorem in full generality.

It is well-known (see, e.g., [ST]) that any holomorphic motion has a quasiconformal holonomy. More precisely, in our case this means the following. For any $(a, b) \in V$ consider the mapping $h_{a,b} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined to send the fixed points of $w(a, b)$ to those of $w(A, B)$. The mapping $h_{a,b}$ extends up to a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ (depending holomorphically on the parameters (a, b)). The quasiconformal homeomorphism $h_{a,b}$ transforms the standard complex structure on $\overline{\mathbb{C}}$ to a measurable almost complex (denoted by $\sigma(a,b)$). It follows from construction that $\sigma(a,b)$ is invariant under the group $< A, B >$, and hence, under $PSL_2(\mathbb{C})$ by density. Now to prove the Theorem, it suffices to show that for a generic pair (a,b) the almost complex structure $\sigma(a,b)$ is not standard.

For any $(a,b) \in V$ the elements a and b are hyperbolic with distinct fixed points; the latters form a quadruple denoted $Q(a,b)$ of points in $\overline{\mathbb{C}}$. If the cross-ratios of two quadruples $Q(a,b)$ and $Q(A,B)$ are distinct, then the quasiconformal homeomorphism $h_{a,b}$, which sends $Q(a,b)$ to $Q(A,B)$, is not conformal; hence, $\sigma(a,b)$ is not standard. This together with the discussion at the beginning of the Section proves Theorem 1.1.

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