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COMPARISON OF SEMIALGEBRAIC GROUPS WITH LIE GROUPS AND ALGEBRAIC GROUPS

MYUNG-JUN CHOI
DEPARTMENT OF MATHEMATICS
SEOUL NATIONAL UNIVERSITY

AND DONG YOUP SUH
DEPARTMENT OF MATHEMATICS
KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY

ABSTRACT. In this article, we compare semialgebraic groups with Lie groups and algebraic groups.

1. INTRODUCTION

The class of *semialgebraic sets* in $\mathbb{R}^n$ is the smallest collection of subsets containing all subsets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for a real polynomial $p(x) = p(x_1, \ldots, x_n)$, which is stable under finite union, finite intersection and complement. We impose "euclidian topology" on semialgebraic sets. Note that any semialgebraic set has finitely many connected components. A map $f\colon M \to N$ between semialgebraic sets M $(\subset \mathbb{R}^m)$ and N $(\subset \mathbb{R}^n)$ is called a *semialgebraic map* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. Semialgebraic maps need not be continuous with this topology, but we mainly consider continuous semialgebraic maps here.

A semialgebraic set G in $\mathbb{R}^n$ is a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are semialgebraic. Note that being a topological group, the group operations in a semialgebraic group are necessarily continuous.

Since semialgebraic groups are not familiar to many topologists, and not many references for the subject are available, we would like to introduce some of the properties of semialgebraic groups by raising and answering several questions one might ask when he or she first encounters with the notion.

Most likely people might ask whether the class of semialgebraic groups are really different from more familiar classes of groups such as Lie groups or algebraic groups. More precisely we may ask the following questions. Since the base field is the real field $\mathbb{R}$ in this article, every Lie group and algebraic group are real groups. We say that a semialgebraic group G admits a Lie group structure if there exists a Lie group K isomorphic to G as a topological group.

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(Q1) Does every semialgebraic group G admit a Lie group structure? If so, is the Lie group structure unique?

We will see in Corollary 2.6 that the answer to (Q1) is positive.

Note that every algebraic group is automatically a semialgebraic group. Thus we say that a semialgebraic group G admits an algebraic group structure if there exists a (real) algebraic group K isomorphic to G as a semialgebraic group.

(Q1') Similar questions for an algebraic group structure.

Proposition 4.7 shows that the answer to (Q1') is negative.

The converse of (Q1) is not true because any Lie group G with infinitely many connected components can not have a semialgebraic group structure, i.e., G can not be isomorphic to a semialgebraic group as a topological group. On the other hand the converse of (Q1') is obviously true. So the following questions make sense.

(Q2) Which Lie groups admit semialgebraic group structures?

(Q3) For a Lie group admitting a semialgebraic group structure, is the semialgebraic structure unique?

(Q4) Is there a connected Lie group that does not admit a semialgebraic group structure?

We will see in Proposition 4.1 that any compact Lie groups are answers to (Q2). Moreover Proposition 4.3 shows that any connected semisimple Lie groups with finite centers also answer (Q2) and (Q3). However Proposition 4.5 and Example 4.6 show that the answer to (Q3) is negative. On the other hand (Q4) has the positive answer by Example 4.2.

From the answers to the questions, we can see that the class of semialgebraic groups is different from the classes of Lie groups and algebraic groups.

Some of the properties of semialgebraic groups are different from those of Lie groups and algebraic groups. In Lie group theory an abstract subgroup H of a Lie group G is a Lie subgroup if and only if H is closed in G. But Example 3.3 shows that the same is not true for both semialgebraic and algebraic groups. Thus we can ask the following question.

(Q5) For a semialgebraic group G which closed subgroup H is a semialgebraic subgroup of G?

This question is yet to be answered, but a well-known result of algebraic groups shows that if G is a linear group, then every compact subgroup H is an answer to (Q5), see Proposition 3.5.

Another different property of semialgebraic groups from those of Lie groups and algebraic groups is about the existence of a faithful representation. It is well-known that every compact Lie group admits a smooth faithful linear representation. Similarly every compact algebraic group admits an algebraic faithful representations. However Proposition 5.1 shows that there exists a compact semialgebraic group that does not admit a semialgebraic faithful representation, and Proposition 5.2 gives a sufficient condition for the existence of a semialgebraic faithful representation of a compact semialgebraic group.

2. Some basic properties and Lie group structures on semialgebraic groups

Since every semialgebraic set has finitely many connected components, the following proposition is obvious.

Proposition 2.1. *Every semialgebraic group has a finite number of connected components.*

As a consequence any Lie group with infinitely many connected components can not have a semialgebraic group structure.

In semialgebraic category the notion of Nash group appears often. Let M be a smooth manifold of dimension n. A Nash chart is a homeomorphism $\psi\colon U \to S$ where U is an open subset of M and S is an open semialgebraic subset of $\mathbb{R}^n$. Two charts $\psi_i\colon U_i \to S_i$ and $\psi_j\colon U_j \to S_j$ are compatible if $\psi_i(U_i \cap U_j)$ and $\psi_j(U_i \cap U_j)$ are semialgebraic subsets of $\mathbb{R}^n$ and $\psi_j \circ \psi_i^{-1}\colon \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ is smooth and semialgebraic.

Definition 2.2. If there is a finite number of compatible charts ψ_i, $i = 1, \ldots, k$, which cover M, then M is called a *Nash manifold.*

Let $(M, \{\psi_i\})$ and $(N, \{\phi_j\})$ be two Nash manifolds. A *Nash map* $f\colon M \to N$ is a continuous map such that

$$\phi_j \circ f \circ \psi_i^{-1}\colon \psi_i(f^{-1}(V_j) \cap U_i) \to \phi_j(V_j)$$

is smooth and semialgebraic for each i and j, where U_i (resp. V_j) is the domain of ψ_i (resp. ϕ_j).

Definition 2.3. A Nash manifold G is a *Nash group* if G is a group such that the group multiplication and the inversion are Nash maps.

From the definition it is clear that every Nash group is a Lie group. By the well-known results of Gleason [5] and Montgomery-Zippen [7] every locally euclidean topological group has a Lie group structure, and such Lie group structure is unique by Proposition I.3.12 of [2]. Hence every Nash group has a unique Lie group structure.

Remark 2.4. Recall that a semialgebraic group is a topological group from the definition. Hence the group operations are continuous. However in a more general setting, being a topological group is not required in the definition of a semialgebraic group. In this case the group multiplication and the inversion need not be continuous. For example consider the half open interval $G = [0, 1)$ of $\mathbb{R}^1$ equipped with the operation $a * b = a + b - [a + b]$. Here $[\alpha]$ denotes the largest integer less than or equal to α. Then this is a semialgebraic group in the general setting but not a semialgebraic group in our setting because the group operations are not continuous. We call such generalized one a *noncontinous semialgebraic group.*

Proposition 2.5. [10] *Every (noncontinuous) semialgebraic group has a unique Nash group structure.*

As straight forward consequences we have the following corollaries.

Corollary 2.6. *Every (noncontinuous) semialgebraic group has a unique Lie group structure.*

Corollary 2.7. *Every semialgebraic group is locally compact.*

In semialgebraic geometry some people consider a slightly wider class of spaces than semialgebraic sets. A *semialgebraic space* S is obtained from finitely many semialgebraic sets U_i, $i = 1, \dots, n$, by pasting them along open semialgebraic subsets of the semialgebraic sets U_i. Impose the topology on S whose basis consists of the set of all open subset of U_i for all $i = 1, \dots, n$. Nash manifold is an example of semialgebraic space. A map between semialgebaric space is a *semialgebraic map* if its graph is a semialgebraic space.

As another generalization of the notion of semialgebraic group, one might define a semialgebraic group to be a topological group which is a semialgebraic space with a semialgebraic multiplication and inversion. Let us call this generalized semialgebraic group an *extended semialgebraic group.* Since Nash group is an extended semialgebraic group, Proposition 2.5 implies that every noncontinuous semialgebraic group is semialgebraically isomorphic to an extended semialgebraic group. Note that every semialgebraic space satisfies the T_1-condition, i.e., every point is closed. Moreover we can see that every T_1 topological group satisfies T_3-condition, i.e., T_1 and regular. By Robson [11] every T_3 semialgebraic space can be semialgebraically embedded in some $\mathbb{R}^n$. Therefore every extended semialgebraic group is semialgebraically isomorphic to a semialgebraic group, and vice versa. Similarly we can see that every noncontinuous semialgebraic group is semialgebraically isomorphic to a semialgebraic group, and vice versa. So are Nash groups and semialgebraic groups.

Definition 2.8. A *semialgebraic homomorphism* $f\colon G \to H$ between two semialgebraic groups G and H is a semialgebric map which is at the same time an abstract group homomorphism. A *Nash homomorphism* between two Nash groups is defined similarly.

We do not require a semialgebraic (Nash) homomorphism to be a continuous map because it is automatically achieved as in the following proposition.

Proposition 2.9. *Any semialgebraic homomorphism $f\colon G \to H$ is always continuous. Therefore f is a Nash homomorphism with the unique Nash group structures on G and H.*

Proof. By the triviality of a semialgebraic map (see Theorem 9.3.2 of [1]) we can find an open subset of G on which f is continuous. By translating this open set by elements of G, we can see that f is continuous everywhere. By Proposition I.3.2 of [2], f is a smooth homomorphism. Hence f is a Nash homomorphism. □

3. On semialgebraic subgroups

Definition 3.1. An injective semialgebraic homomorphism $f\colon H \to G$ is called a *semialgebraic subgroup* of G.

In Lie group theory an injective Lie group homomorphism $f\colon H \to G$ needs not be an embedding, e.g., $f\colon \mathbb{Z} \to S^1$, $n \mapsto e^{in}$. However in semialgebraic category we do not have to be careful about embedability of a semialgebraic subgroup as the following proposition says.

Proposition 3.2. *If $f\colon H \to G$ is a semialgebraic subgroup of G, then f is an embedding, and hence $f(H)$ is closed in G.*

Proof. Since f is a semialgebraic map, $f(H)$ is a semialgebraic subset of G by Tarski-Seidenberg principle, see [1, Proposition 2.2.7]. Moreover since f is an injective homomorphism, $f(H)$ with the group multiplication of G makes $f(H)$ a semialgebraic group. Thus by Proposition 2.9 both $f\colon H \to f(H)$ and $f^{-1}\colon f(H) \to H$ are continous homomorphisms. So $f\colon H \to G$ is an embedding, i.e., $f(H)$ is a submanifold of G, when viewed as a Lie group. But in Lie group theory an abstract subgroup K of a Lie group L is a submanifold of L if and only if K is closed in L, see Theorem I.3.11 in [2]. So $f(H)$ is closed in G. □

By Proposition 3.2 we may consider a semialgebraic subgroup of G to be a semialgebraic subset H of G which forms a semialgebraic group with the group multiplication of G. Indeed, if H is a semialgebaric subset of a semialgebraic group $(G, \cdot)$ such that $(H, \cdot)$ forms a semialgebraic group, then the inclusion $i\colon H \to G$ is clearly an injective semialgebraic group homomorphism. In this case, H is automatically a closed subset of G, and also a submanifold of G.

As is mentioned in the proof of Proposition 3.2, every abstract subgroup H of a Lie group G is an embedded Lie subgroup of G (hence a submanifold) if and only if H is closed in G. However the same is not true for semialgebraic groups or algebraic groups as the following example shows.

Example 3.3. Let $G = GL_2(\mathbb{R})$ and

$$H = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{\sqrt{2}t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Then H is a closed subgroup of a semialgebraic group G. However H with the matrix multiplication as the group operation is not a semialgebraic group. Therefore H is not a semialgebraic subgroup of G. Note that this does not necessarily mean that H does not have a semialgebraic group structure, because

$$f\colon H \to (\mathbb{R}, +), \qquad \begin{pmatrix} e^t & 0 \\ 0 & e^{\sqrt{2}t} \end{pmatrix} \mapsto t$$

is a Lie group isomorphism, and $(\mathbb{R}, +)$ is clearly a semialgebraic group. So H has a semialgebraic group structure.

As is mentioned in Introduction this example makes the question (Q5) reasonable to ask. The authors do not have a general answer to the question, but the well-known result of Proposition 3.5 on compact algebraic group gives a partial answer to the question.

Definition 3.4. A *linear semialgebraic group G* is a semialgebraic group with a semialgebraic faithful representation, i.e., there exists an injective semialgebraic homomorphism $\phi\colon G \to GL_n(\mathbb{R})$ for some n.

Proposition 3.5. *A compact abstract subgroup H of a linear semialgebraic group G is an algebraic group, hence a semialgebraic group.*

Proof. See Theorem 3.4.5 in [8]. □

4. Semialgebraic group structures on Lie groups

Since every compact Lie group G admits a smooth faithful representation $\phi\colon G \to GL_n(\mathbb{R})$, G is smoothly isomorphic to a compact subgroup $\phi(G)$ of $GL_n(\mathbb{R})$. By Proposition 3.5 $\phi(G)$ is an algebraic group. This shows that any compact Lie group admits an algebraic group structure, hence a semialgebraic group structure. Since any smooth isomorphism of two compact algebraic groups is always a polynomial isomorphism (see Theorem 5.2.11 in [8]), the algebraic group structure of a compact Lie group is unique. However this does not necessarily imply that the semialgebraic group structure on a compact Lie group is unique. In fact there is a compact Lie group (the circle group S^1) with nonunique semialgebraic group structures as we will see in Proposition 4.5 and Example 4.6. We thus have the following proposition.

Proposition 4.1. *Every compact Lie group admits a (not necessarily unique) semialgebraic group structure.*

We now consider general Lie groups which are not necessarily compact. Note that every Lie group decomposes into a solvable group and a semisimple group. Namely, for a Lie group G the quotient group of G by the largest connected solvable normal subgroup is semisimple. Remember that a Lie group is semisimple if and only if it has no normal abelian subgroup of dimension > 0. For solvable Lie groups, there is an example which does not admit a semialgebraic group structure as in Example 4.2.

Example 4.2. [9] Let G be a subgroup of $GL_3(\mathbb{R})$ defined by

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^\alpha & v \\ 0 & 0 & 1 \end{pmatrix} : t > 0, u, v \in \mathbb{R} \right\}$$

for a positive irrational number α. Then G is a solvable group that does not admit a semialgebraic group structure.

Note that the circle group is solvable but it has nonunique semialgebraic group structures as is mentioned above.

For semisimple Lie groups we have the following proposition.

Proposition 4.3. [3] *Every connected semisimple Lie group G with the finite center has a unique semisimple group structure.*

Note that any semisimple Lie group has the discrete center. Since the center of a semialgebraic group is a semialgebraic subgroup, any semisimple semialgebraic group must have the finite center. So having finite center is a necessary condition for a semisimple Lie group to have a semialgebraic group structure. However the authors do not know the existence of any semisimple Lie group with the infinite center.

Example 4.4. (See p139 of [8]) Let $SO_{k,l}$ with $k, l > 0$ be the group of unimodular matrices corresponding to linear operators preserving a nondegenrate quadratic form of signature (k, l). Then $SO_{k,l}$ is a nonconnected semisimple Lie group. In fact $SO_{k,l}$ is an irreducible algebraic group. Therefore $SO_{k,l}$ has the finite center. Let G be the identity component of $SO_{k,l}$. Then G is a semisimple semialgebraic group. Note that G is not an algebraic group.

The above example shows that not all semialgebraic groups are algebraic.

The following result of Madden and Stanton shows the existence of nonunique semialgebraic group structures on the circle group S^1.

Proposition 4.5. [6] *There are infinitely many different Nash group structures on the Lie group S^1. Some of them are not Nash embedable in any $\mathbb{R}^n$ as a Nash manifold.*

Since the class of Nash group and that of semialgebraic groups are equivalent as is mentioned in Section 2 (above Definition 2.8), Proposition 4.5 tells us that S^1 has infinitely many different semialgebraic group structures.

The following example exhibits a concrete semialgebraic group structure on S^1 which is not semialgebraically isomorphic to the standard semialgebraic structure on S^1 and not Nash embedable in any $\mathbb{R}^n$.

Example 4.6. [3] Let G be the noncontinuous semialgebraic group $[0,1)$ with the binary operation $a * b = a + b - [a+b]$ as in Remark 2.4. View G as $[0,1) = [0,1/4) \cup [1/4, 2/4) \cup [2/4, 3/4) \cup [3/4, 1)$. Let X be the standard square in $\mathbb{R}^2$, and let $\phi\colon G \to X$ be the obvious semialgebraic map sending each $[i/4, (i+1)/4)$ to each edge of X linearly for $i = 0, 1, 2, 3$ in the obvious way. We can give a semialgebraic group structure on X via ϕ, i.e., $xy = \phi(\phi^{-1}(x)\phi^{-1}(y))$ for $x, y \in X$. Then the group operation in X is continuous. By Proposition 2.5 X is a Nash group, but it can be shown that X can not be Nash embedable in any $\mathbb{R}^n$. Indeed, if on the contrary X can be Nash embedded in some $\mathbb{R}^n$, then one of its coordinate functions of the embedding is a nonconstant Nash function (i.e. smooth and semialgebraic function). But it can been shown that X does not have a nonconstant Nash function. Thus the semialgebraic group X is isomorphic to S^1 as a Lie group but not as a semialgebraic group.

Example 4.4 shows that the identity component G of $S_{k,l}$ with $k, l > 0$ is a semialgebraic group but not an algebraic group. But this does not imply that G is a negative answer to (Q1') because we do not know whether there is some algebraic group which is semialgebraically isomorphic to G. On the other hand the following Proposition shows that the answer to (Q1') is negative.

Proposition 4.7. *Let X be the semialgebaric group in Example 4.6. Then X does not have an algebraic group structure.*

Proof. Suppose X admits an algebraic group structure, i.e., there exists an algebraic group G semialgebraically (Nash) isomorphic to X. Since every compact algebraic group has an algebraic faithful representation, there exists an injective algebraic homomorphism $\phi\colon G \to GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$. This implies that G can be algebraically embedable in the Euclidean space $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Composition of the semialgebraic (Nash) isomorphism from X to G and the injective algebraic homomorphism ϕ, we have a Nash embedding of X into an Euclidean space, and this is a contradiction. Therefore X does not admit an algebraic group structure. □

5. Existence of semialgebraic faithful representation

The existence of faithful representation plays an important role in Lie and algebraic group theory. The existence of semialgebraic faithful representation is also important in semialgebraic theory. For example the authors showed in [4] that any proper semialgebraic G-space can be semialgebraically and equivariantly embedded in a semialgebraic representation space for a linear semialgebraic group G. It can be seen easily that being a linear group is a necessary condition for all G space to be embedded in a representation space.

It is well-known that any compact Lie (resp. algebraic) group admits a smooth (reps. algebraic) faithful representation. So one might ask whether a compact semialgebraic group admit a semialgebraic faithful representation. The answer to the question is negative as the following proposition shows.

Proposition 5.1. *There exists a compact semialgebraic group which does not admit a semialgebraic faithful representation.*

Proof. Let G be the semialgebraic group X in Example 4.6, which is not Nash embedable in any $\mathbb{R}^n$. By Robson [11] G can be (nonequivariantly) semialgebraically embedded in some $\mathbb{R}^n$. The group multiplication of G induces a semialgebraic group multiplication on the image semialgebraic set. By identifying G with the image semialgebraic group, we can view G as a semialgebraic group. We claim that G does not admit a semialgebraic faithful representation. On the contrary suppose that there exists a semialgebraic faithful representation $\phi\colon G \to GL_n(\mathbb{R})$ for some n. Then $\phi(G)$ is a compact semialgebraic subgroup of $GL_n(\mathbb{R})$. By Proposition 3.5 $\phi(G)$ is an algebraic group, hence a Nash group. On the other hand $\phi\colon G \to \phi(G)$ is a semialgebraic isomorphism, hence ϕ is smooth, thus a Nash group isomorphism. But G is not Nash embedable in $\mathbb{R}^m$ for any $m > 0$, while $\phi(G)$ is Nash embedable in $R^{n^2} = M_n(\mathbb{R})$, the set of all $n \times n$ real matrices. This is a contradiction. So G does not have a semialgebraic faithful representation. □

The main reason for existence of compact semialgebraic group without semialgebraic faithful representation in Proposition 5.1 is because the underlying Lie group of the semialgebraic group G has nonunique semialgebraic group structures. We thus have the following proposition.

Proposition 5.2. *Let G be a compact semialgebraic group whose underlying Lie group has the unique semialgebraic group structure. Then G has a semialgebaric faithful representation.*

Proof. Since any compact Lie group has a unique algebraic group structure, and the underlying Lie group of G has the unique semialgebraic group structure, we may view G as an algebraic group. Since any compact algebraic group admits an algebraic faithful representation, G admits an algebraic (hence semialgebraic) faithful representation. □

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Department of Mathematics,Seoul National University,San 56-1, Shillim-dong, Gwanak-gu, Seoul, Korea.

E-mail address: derham@hanmail.net

E-mail address: choimj@math.kaist.ac.kr

Department of Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Guseong-dong, Yuseong-gu, Daejeon 305-701, Korea

E-mail address: dysuh@math.kaist.ac.kr