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Integration theorem of formal meromorphic functions

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The aim of this short note is to give a ground for using "residue theorem" in an algebraic situation such as formal power series. Once the algebraic version of residue theorem is verified, we do not have to check analytic convergence of power series with which we are dealing.

The aim of this note is to alleviate our sense of guilt we topologists sometimes feel in doing calculations using formal power series. Power series that appear in topology are not necessarily analytic objects. It is not rare that the coefficients of the power series belong to some cohomology groups or other algebraic systems. Even if the coefficients are subfields of the complex number field, it is desirable to do without analytic theories. For examle, when we apply Hirzebruch's index theorem to the $2n$-dimensional complex projective space $\mathbb{C}\mathrm{P}^{2n}$, we must calculate the coefficient of x^{2n} in the power series given by $(x/\tanh x)^{2n+1}$. This calculation is usually performed by using the integration theorem in complex function theory ([2, Lemma 1.5.1]). In this note, we shall replace analytic integration theorem by an algebraic one which does not need any assumption of convergence of the power series we are handling.

We present below the steps to perform our program. Most of the statements upto inverse function theorem are written in the first few pages of Cartan's book([1]).

1. We shall consider the field K with characteristic 0 (e.g. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and the ring of formal power series $K[[X]]$. Although most of the statements in this note are applicable to the characteristic p case under a slight modification, to simplify our statements, we shall solely treat the case with characteristic zero. So in this case, there are many well-known power series such as $\exp(X) = e^X$, $\sin X$, $\cos X$, $\tan X$, $\sinh X$, $\cosh X$, $\tanh X$ and so on.

2.[**Order**] The order $\mathrm{ord}(f)$ of a formal power series $f(x) = \sum_{n\geq 0} a_n X^n$ is defined as the least integer n such that $a_n \neq 0$ if $f(X)$ is not zero, and we define $\mathrm{ord}(f) = \infty$ when $f(X) = 0$. The order satisfies the formula: $\mathrm{ord}(f\cdot g) = \mathrm{ord}(f) + \mathrm{ord}(g)$ Here we adopt the convention $\infty + n = n + \infty = \infty + \infty = \infty$.

3. [**Summability principle**] Let $\{f_\lambda(X)\}_{\lambda\in\Lambda}$ be a family of power series. We can take the sum

$$\sum_{\lambda\in\Lambda} f_\lambda(X)$$

if for each integer n there exist only a finite number of $\lambda \in \Lambda$ such that $\mathrm{ord}(f_\lambda) \leq n$. From this standpoint, we can view the notation $f(X) = \sum_{n\geq 0} a_n X^n$ as the "sum" of the family of power series $\{a_n X^n\}_{n=0,1,2,\ldots}$.

4.[**Substitution**] Let

$$f(X) = \sum_{n\geq 0} a_n X^n \in K[[X]]$$

and $g(Y) \in K[[Y]]$ be two power series in X and Y respectively. The substitution of X by $g(Y)$ in $f(X)$ is possible if $\mathrm{ord}(g) \geq 1$. The resulting power series

$$f(g(Y)) = \sum_{n\geq 0} a_n (g(Y))^n$$

is also denoted by $(f \circ g)(Y)$. In particular, we can always substitute X by 0 and we obtain $f(0) = a_0$. We set $I(X) = X$ the "identity" power series. We have the following formulas on substitution:

1. $f \circ I = f$
2. $I \circ g = g$
3. $(c_1 f_1 + c_2 f_2) \circ g = c_1 f_1 \circ g + c_2 f_2 \circ g$
4. $(\sum_\lambda f_\lambda) \circ g = \sum_\lambda (f_\lambda \circ g)$
5. $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g)$
6. $(f \circ g) \circ h = f \circ (g \circ h)$
7. $\mathrm{ord}(f \circ g) = \mathrm{ord}(f)\,\mathrm{ord}(g) \quad (0 \times \infty = 0, \quad n \times \infty = \infty\ (n > 0), \quad \infty \times n = \infty)$.

The proofs of the formulas above are mostly straightforward. Formulas 1 to 3 follows directly from the definition. Formula 4 is a consequence of the summability principle. Noting that substituting $X = g(Y)$ in $(I(X))^n = X^n$ is $(g(Y))^n$, formula 5 holds for the special case where $f_1(X) = X^n$ and $f_2(X) = X^m$. The general case follows again by the summability principle. The special case of formula 6 for $f(X) = X^n$ follows from the induction on n. The general case follows from the summability principle. Formula 7 is immediate if neither f nor g is not equal to zero.

5.[**Invertible elements**] A power series $f(X)$ is called invertible if there exists a power series $g(X)$ such that $f(X)g(X) = 1$. If this holds then $f(0)g(0) = 1$ and $f(0) \neq 0$. Conversely, if $f(0) \neq 0$, then we can show that $f(X)$ is invertible. Let $f(0) = a_0 \neq 0$ and let us write $f(X) = a_0(1 - g(X))$ for some $g(X)$ with $\mathrm{ord}(g) \geq 1$. Substituting $Y = g(X)$ in the equality

$$(1 - Y)(1 + Y + Y^2 + \cdots) = 1,$$

we get

$$(1 - g(X)) \sum_{n\geq 0} (g(X))^n = 1.$$

Thus we have

$$f(X)\frac{1}{a_0}\left(\sum_{n\geq 0}(g(X))^n\right) = 1.$$

and $f(X)$ is invertible.

From this result we conclude that $K[[X]]$ is a discrete valuation ring with maximal ideal generated by the unique maximal ideal composed of non-invertible elements, that is, the ideal generated X. Hence the quotient field $K((X))$ of $K[[X]]$ consists of the negatively finite Laurent series

$$\sum_{n\geq r} a_n X^n \ ,$$

where r is an integer. We shall call an element of $K((X))$ a formal meromorphic function. We may extend the definition of order to $K((X))$ in an obvious manner, i.e., $\mathrm{ord}(f/g) = \mathrm{ord}(f) - \mathrm{ord}(g)$, for $f, g \in K[[X]]$ $(g \neq 0)$. Of course the formula $\mathrm{ord}(\varphi \cdot \psi) = \mathrm{ord}(\varphi) + \mathrm{ord}(\psi)$ holds for $\varphi, \psi \in K((X))$. The order takes values in $\mathbb{Z} \cup \{\infty\}$. Given a formal meromorphic function $f(X)$, it is also possible to make a substitution $X = g(Y)$ where g is a formal power series with $\mathrm{ord}(g) \geq 1$.

6.[**Differentiation**] For a power series

$$f(X) = \sum_{n\geq 0} a_n X^n,$$

we define its derivative by

$$f'(X) = \sum_{n\geq 1} n a_n X^{n-1}.$$

It is also denoted by $\dfrac{df(X)}{dX}$ when we want to specify the variable X. In the following we assume that $g(X)$ has positive order and $f_3(X)$ is invertible.

1. $(\sum_\lambda c_\lambda f_\lambda)' = \sum_\lambda (c_\lambda f'_\lambda)$
2. $(f_1\, f_2)' = f'_1\, f_1 + f_1\, f'_2$
3. $\left(\dfrac{f_1}{f_3}\right)' = \dfrac{f'_1 f_3 - f_1 f'_3}{(f_3)^2}$
4. $(f(g(Y)))' = f'(g(Y))\, g'(Y)$

The last formula (chain rule) can be proved by considering the special case $f(X) = X^n$ and the applying summability principle. Once we have verified the above formulas, we can extend the derivation to $K((X))$ in an obvious way and we get the same set of formulas for f_λ, f_1, f_2, f_3, f in $K((X))$. Here we do not need invertibility condition for $f_3(X)$ any more.

7.[**Inverse function theorem**] Given $f(X) \in K[[X]]$, there exists $g(Y) \in K[[Y]]$ such that $f \circ g = I$ if and only if $f(0) = 0$ and $f'(0) \neq 0$, namely, $\mathrm{ord}(f) = 1$. If this is the case, $g(Y)$ is uniquely determined by $f(X)$ and $g \circ f = I$ also holds. We shall say that $g(Y)$ is the inverse function for $f(X)$ and denoted by $f^{-1}(Y)$.

The proof of the inverse function theorem is given in Cartan's book ([1]) and will be omitted. This is also a consequence of the implicit function theorem which will be given later.

Example. Let m be a positive integer greater than 1 and let $g(Y)$ be the inverse function for $f(X) = (1+X)^m - 1$. Then $1 + g(Y)$ gives the formula for $(1+Y)^{1/m}$:

$$1 + \sum_{n\geq 1} \frac{\frac{1}{m}(\frac{1}{m}-1)\cdots(\frac{1}{m}-n+1)}{n!} Y^n$$

There is no inverse function for e^X since its order is zero, whereas the inverse function for $e^X - 1$ is given by

$$\sum_{n\geq 1} (-1)^{n+1} \frac{1}{n} Y^n$$

which is usually written as $\log(1+Y)$.

The facts mentioned so far are mostly written in H. Cartan's text book on complex function theory.

Next we consider formal power series in two variables

$$F(X,Y) = \sum_{m,n\geq 0} a_{m,n} X^m Y^n$$

The order is defined to be the "total" order $\operatorname{ord}(F) = \min\{\, m+n \mid a_{m,n} \neq 0 \,\}$. We also define partial differentiation, $F_X(X,Y)$, $F_Y(X,Y)$ in an obvious manner. As in the case of single variables, the substitution of variables by power series with positive order is also possible and if $F(0,0) = a_{0,0}$ is nonzero then $F(X,Y)$ is invertible.

8.[**Implicit function theorem**]

Let $F(X,Y) \in K[[X,Y]]$ satisfy $F(0,0) = 0$ and $F_Y(0,0) \neq 0$. Then there exists a unique power series $f(X)$ such that $F(X, f(X)) = 0$ and $\operatorname{ord}(f) \geq 1$.

Proof. Let us write $F(X,Y) = \sum a_{m,n} X^m Y^n$ and we assume that $F(0,0) = a_{0,0} = 0$ and $F_Y(0,0) = a_{0,1} \neq 0$. We shall construct a power series $f(X) = \sum_k b_k X^k$ of positive order that satisfies $F(X, f(X)) = 0$. Let us write the n-th power of $f(X)$ as

$$(f(X))^n = \sum_k b_k^{(n)} X^k \ ,$$

then $b_k^{(n)}$ is zero for $k < n$ and for $k \geq n$ it can be expressed by $b_1, b_2, \ldots, b_{k-n+1}$. From the relation

$$\begin{aligned} F(X, f(X)) &= \sum_{m,n} a_{m,n} X^m \sum_k b_k^{(n)} X^k \\ &= \sum_{N\geq 1} \sum_{m+k=N,n} a_{m,n} b_k^{(n)} X^{m+k} \\ &= \sum_{N\geq 1} \left(a_{N,0} + \sum_{1\leq k\leq n\leq N} a_{N-k,n} b_k^{(n)} \right) X^N = 0, \end{aligned}$$

we obtain the coefficient relation

$$a_{N,0} + \sum_{k=1}^{N} a_{N-k,1} b_k + \sum_{2 \le n \le k \le N} a_{N-k}, n b_k^{(n)} = 0 \quad (N \ge 1). \tag{1}$$

Considering the case $N = 1$, we have

$$a_{1,0} + a_{0,1} b_1 = 0$$

and b_1 is uniquely determined. Let us proceed inductively and suppose we have determined b_1, b_2,..., b_{N-1}. Then all $b_k^{(n)}$ for $2 \le n \le k \le N$ are determined since $k - n + 1 \le N - 1$. Therefore the third term in the left hand side of the coefficient relation (1) is determined. And from the same relation, we can uniquely determine b_N. This completes the inductive step and our assertion is proved.

Example. Let m, n be positive integers and consider

$$F(X, Y) = (1 + X)^n - (1 + Y)^m.$$

Let $f(X)$ satisfy $F(X, f(X)) = 0$, then $1 + f(X)$ represents $(1 + X)^{n/m}$.

You can extend implicit funtion theorem to the one in n-variables in an obvious manner which we will not produce here. Other extension of the theory such as various types of chain rules also hold.

We shall go to the goal of formal integration of formal meromorphic functions.

9. [**Residue(Integration)**] Let $f(X) \in K((X))$ be written as the Laurent series

$$f(X) = \sum_{k \ge r} a_k X^k.$$

We define its residue to be a_{-1} and let us write $\oint f(X)dX = a_{-1}$.

Theorem (a) [Fundamental theorem of calculus] Let $f(X) \in K((X))$, then

$$\oint f'(X)dX = 0$$

(b) [Integration by parts] Let $f(X), g(X) \in K((X))$, then

$$\oint f'(X)g(X)dX = -\oint f(X)g'(X)dX$$

(c) [Change of variables] Let $f(X) \in K((X))$ and let $g(Y) \in K[[Y]]$ have positive order, then

$$\mathrm{ord}(g) \oint f(X)\, dX = \oint f(g(Y))g'(Y)\, dY\ .$$

Proof. It is clear that our residue is a linear functional and general summability principle also applies here. To prove (a), we have only to deal with the special case when $f(X) = X^n$ where n is an arbitrary integer. Then $f'(X) = nX^{n-1}$ does not contain nonzero factors of $1/X$ this shows (a). For (b), applying (1) to $(f(X)g(X))'$

gives the result. To prove (c), we have only to deal with the case $f(X) = X^n$. If $n \geq 0$, then $f(X) = X^n$ and $g(Y)^n g'(Y)$ are both formal power series and both have residue 0. Let $n = -m$ and first suppose $m > 1$, then $f(X)$ has residue 0 and since $f(g(Y))g'(Y) = g'(Y)/g(Y)^m$ is the derivative of $1/(1-m)(g(Y))^{m-1}$, this also has residue 0 by (a). For the final case when $f(X) = 1/X$, $f(X)$ has residue 1. Let $g(Y)$ has order $q > 0$ and put $g(Y) = aY^q(1 + h(Y))$ with $\mathrm{ord}(h) > 0$. Then we have

$$\frac{g'(Y)}{g(Y)} = \frac{qY^{q-1}(1+h(Y)) + Y^q h'(Y)}{aY^q(1+h(Y))} = \frac{q + (qh(Y) + Yh'(Y))}{Y(1+h(Y))},$$

which also has residue 1. This completes our theorem.

Application

The Pontrjagin class of the complex projective space $\mathbb{C}\mathrm{P}^{2n}$ is given by

$$p(\mathbb{C}\mathrm{P}^{2n}) = (1 + x^2)^{2n+1}$$

where x is the generator of $x \in H^2(\mathbb{C}\mathrm{P}^{2n}; \mathbb{Z})$. Then the total L-class of $\mathbb{C}\mathrm{P}^{2n}$ is given by

$$\mathcal{L}(\mathbb{C}\mathrm{P}^{2n}) = \sum_k L_k(p_1, p_2, \cdots, p_k) = \left(\frac{x}{\tanh x}\right)^{2n+1}$$

where we expressed the image of x in $H^2(\mathbb{C}\mathrm{P}^{2n}; \mathbb{Q})$ by the same letter. To verify that the L-genus $\langle L_n(p_1, p_2, \ldots, p_n), [\mathbb{C}\mathrm{P}^{2n}]\rangle = 1$, we must calculate the coefficient of X^{2n} in the power series

$$\left(\frac{X}{\tanh X}\right)^{2n+1}$$

which is given by the residue

$$\oint \left(\frac{1}{\tanh X}\right)^{2n+1} dX.$$

By substitution $\tanh X = Y$ or $X = \tanh^{-1} Y$, and by (c) of our theorem, this residue is calculated as

$$\begin{aligned} &\oint \frac{1}{Y^{2n+1}} \frac{1}{1-Y^2}\, dY \\ &= \oint \frac{1}{Y^{2n+1}} (1 + Y^2 + \cdots + Y^{2n} + \cdots)\, dY \\ &= 1. \end{aligned}$$

References

[1] Cartan, H., Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes, 1963, Hermann, Paris.

[2] Hirzebruch, F., Topological methods in algebraic geometry, 1966, Springer-Verlag.