数理解析研究所講究録 1449 巻 2005 年 27-32

CLASSIFICATION OF QUASITORIC MANIFOLDS OVER A CUBE

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I report some results obtained as a joint work in progress with Taras Panov and some with Dong Youp Suh.

1. Bott tower

For a complex vector bundle $E \to X$, we denote its projectivization by $P(E)$. We consider the following sequence:

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}$$

where $B_k = P(\mathbf{1} \oplus L_k)$, L_k is a holomorphic line bundle over B_{k-1} and $\mathbf{1}$ denotes the product complex line bundle. If every line bundle L_k is trivial, then $B_n = (\mathbb{C}P^1)^n$. Each $\pi_k \colon B_k \to B_{k-1}$ is a $\mathbb{C}P^1$-bundle and it has two natural cross sections which correspond to the zero sections of L_k and $\mathbf{1}$. The above sequence together with these natural cross sections is called a *Bott tower* in [5]. In this article we are only concerned with the top space B_n of a Bott tower and call B_n a *Bott manifold.* Our starting point is

Problem. Classify Bott manifolds B_n's up to diffeomorphism.

It follows from Borel-Hirzebruch formula that

$$H^*(B_k) = H^*(B_{k-1})[y_k]/(y_k^2 - c_1(L_k)y_k)$$

where y_k is the first Chern class of the canonical line bundle over B_k associated with the fibration $\pi_k \colon B_k \to B_{k-1}$. Therefore

$$H^*(B_k) \cong H^*((\mathbb{C}P^1)^k) \quad \text{as groups}$$

but not as rings in general. Since $H^2(B_k)$ is additively generated by $y_1, \dots, y_k$ over $\mathbb{Z}$, L_{k+1} is parameterized by $\mathbb{Z}^k$ so that there is a canonical surjection

$$\mathbb{Z} \oplus \mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^{n-1} = \mathbb{Z}^{n(n-1)/2} \to \{B_n\text{'s}\}. \tag{1.1}$$

Example. When $n = 2$, we have a surjection $\mathbb{Z} \to \{B_2\text{'s}\}$ and $L_2 = \gamma^m$ for some $m \in \mathbb{Z}$ where γ is the canonical line bundle over $B_1 = \mathbb{C}P^1$. It is well-known that

$$P(\gamma^m \oplus \mathbf{1}) \cong P(\gamma^{m'} \oplus \mathbf{1}) \Longleftrightarrow m \equiv m' \pmod 2.$$

The proof goes as follows. We note that $P(E) \cong P(E \otimes \eta)$ for any complex line bundle η. Suppose $m \equiv m' \pmod 2$. Then $m' - m = 2\ell$ for some $\ell \in \mathbb{Z}$ and we have

$$P(\gamma^m \oplus \mathbf{1}) \cong P((\gamma^m \oplus \mathbf{1}) \otimes \gamma^\ell) = P(\gamma^{m+\ell} \oplus \gamma^\ell).$$

Here $\gamma^{m+\ell} \oplus \gamma^\ell$ and $\gamma^{m'} \oplus \mathbf{1}$ are over $\mathbb{C}P^1$ and have the same first Chern class, so they are isomorphic. Hence the last space above is same as $P(\gamma^{m'} \oplus \mathbf{1})$. This proves the implication $\Longleftarrow$ above.

On the other hand, it is not difficult to see that if $H^*(P(\gamma^m \oplus \mathbf{1})) \cong H^*(P(\gamma^{m'} \oplus \mathbf{1}))$ as rings, then $m \equiv m' \pmod 2$. $\square$

The example above shows that cohomology ring detects diffeomorphism types of Bott manifolds B_n's when $n = 2$. One can check that this is also the case when $n = 3$. So we are led to ask

Question. Are Bott manifolds B_n and B_n' diffeomorphic if and only if $H^*(B_n) \cong H^*(B_n')$ as rings?

The following proposition gives a partial affirmative answer to the question above.

Proposition 1.1. *Bott manifolds B_n and $(\mathbb{C}P^1)^n$ are diffeomorphic if and only if $H^*(B_n) \cong H^*((\mathbb{C}P^1)^n)$ as rings.*

Proof. We prove the the "if part" by induction on n. When $n = 1$, the statement is trivial and we assume $n \geq 2$. From

$$H^*(B_n) = H^*(B_{n-1})[y_n]/(y_n^2 - c_1(L_n)y_n)$$

one can conclude that $H^*(B_{n-1}) \cong H^*((\mathbb{C}P^1)^{n-1})$, so B_{n-1} is diffeomorphic to $(\mathbb{C}P^1)^{n-1}$ by induction assumption. Let $x_1, \dots, x_{n-1} \in H^2(B_{n-1})$ be generators with square zero and write $c_1(L_n) = \sum_{i=1}^{n-1} a_i x_i$. Then

$$H^*(B_n) = \mathbb{Z}[x_1, \dots, x_{n-1}, y_n]/\big(x_1^2, \dots, x_{n-1}^2, y_n^2 - (\textstyle\sum a_i x_i)y_n\big).$$

Since $H^*(B_n) \cong H^*((\mathbb{C}P^1)^n)$, there must be an element of the form $y_n + \sum b_i x_i$ with square zero:

$$0 = (y_n + \sum b_i x_i)^2 = \sum (a_i + 2b_i) x_i y_n + (\sum b_i x_i)^2.$$

This holds only when at most one a_i is non-zero and even because $x_i x_j$ $(i < j)$ and $x_i y_n$ form an additive basis of $H^4(B_n)$. Therefore L_n is the pullback of γ^{-2b_i} over $\mathbb{C}P^1$ by a projection $B_{n-1} = (\mathbb{C}P^1)^{n-1} \to \mathbb{C}P^1$. Since $P(\gamma^{-2b_i} \oplus \mathbf{1})$ is a product bundle as observed in the example above, so is $P(L_n \oplus \mathbf{1})$, proving the proposition. $\square$

2. Equivariant classification of Bott manifolds

Each B_k admits an effective action of $(\mathbb{C}^*)^k$ constructed inductively as follows. Suppose B_{k-1} admits an action of $(\mathbb{C}^*)^{k-1}$. Then it lifts to an action on L_k. On the other hand, the product bundle $\mathbf{1}$ admits an action of $\mathbb{C}^*$ by scalar multiplication. These define an action of $(\mathbb{C}^*)^k$ on $\mathbf{1} \oplus L_k$ and induce an action of $(\mathbb{C}^*)^k$ on B_k.

It turns out that B_k with the action of $(\mathbb{C}^*)^k$ is a *compact nonsingular toric variety* of complex dimension k. A toric variety of complex dimension k is a normal algebraic variety with an algebraic action of $(\mathbb{C}^*)^k$ having a dense orbit ([4]). The orbit space of B_k by the maximal compact torus T^k of $(\mathbb{C}^*)^k$ is a k-cube. In particular B_n admits an action of $T = T^n$ and its orbits space is an n-cube.

For a T-space X, its equivariant cohomology is by definition

$$H_T^*(X) := H^*(ET \times_T X)$$

where $ET \to BT$ is a universal principal T-bundle and $ET \times_T X$ is the orbit space of $ET \times X$ by the diagonal action of T. $H_T^*(X)$ is not only a ring but also an algebra over $H^*(BT)$ through the projection map $ET \times_T X \to ET/T = BT$.

As is well known, $H_T^*(B_n)$ is isomorphic as a ring to the face ring of (the dual of) the n-cube. So the ring structure of $H_T^*(B_n)$ does not detect the T-equivariant diffeomorphism type of B_n, but the algebra structure does.

Theorem 2.1. *Bott manifolds B_n and B'_n with the above T-actions are equivariantly diffeomorphic if and only if $H_T^*(B_n) \cong H_T^*(B'_n)$ as algebras over $H^*(BT)$.*

3. Quasitoric manifolds over an n-cube

If M is a compact nonsingular toric variety of complex dimension n, then M has an action of $(\mathbb{C}^*)^n$ and the orbit space M/T of M by the restricted action of the maximal compact torus T of $(\mathbb{C}^*)^n$ is a manifold with corners such that every face (even M/T itself) is contractible. In fact, M/T is often a simple convex polytope (e.g. B_n/T is an n-cube) and this is the case when M is projective (see [4]).

Davis-Januszkiewicz [2] introduced a topological counterpart to a compact nonsingular toric variety in algebraic geometry. They used the terminology *toric manifold* for the topological counterpart, but Buchstaber-Panov [1] started calling it a *quasitoric manifold* because the terminology *toric manifold* was already used in algebraic geometry for (compact) nonsingular toric variety. Roughly speaking a quasitoric manifold is a closed smooth manifold M of dimension $2n$ with smooth T-action such that M/T is a simple convex polytope. Not all but many compact nonsingular toric varieties with the restricted action of the maximal compact subtorus of $(\mathbb{C}^*)^n$ provide examples of quasitoric manifolds, and there are quasitoric manifolds which do not arise this way.

We think of the left side of (1.1) as a set of upper triangular matrices with 1 as diagonal entries. Obviously all principal minors of such an upper triangular matrix are 1, where the determinant of the matrix itself is considered to be a principal minor. It turns out that any quasitoric manifold over an n-cube is associated with an integer square matrix $C = (c_{ij})$ of size n such that

(3.1) $\qquad c_{ii} = 1$ for any i and all principal minors of C are ± 1.

The correspondence is as follows (cf. [5]). We view S^1 and S^3 as the unit spheres of $\mathbb{C}$ and $\mathbb{C}^2$ respectively. Associated with the matrix $C = (c_{ij})$, we define an action of $(g_1, \dots, g_n) \in (S^1)^n$ on $(S^3)^n$ by

$$(z_1, w_1, \dots, z_n, w_n) \mapsto (g_1 z_1, (\prod_{i=1}^{n} g_i^{c_{i1}}) w_1, \dots, g_n z_n, (\prod_{i=1}^{n} g_i^{c_{in}}) w_n)$$

where $(z_j, w_j) \in S^3 \subset \mathbb{C}^2$ denotes the coordinate of the jth factor of $(S^3)^n$. The condition (3.1) ensures that the action of $(S^1)^n$ on $(S^3)^n$ is free, so that its orbit space is a closed smooth manifold of dimension $2n$, which we denote by $M(C)$. Note that when C is the identity matrix, $M(C) = (\mathbb{C}P^1)^n$. $M(C)$ admits an action of T induced from an action of $(t_1, \dots, t_n) \in T$ on $(S^3)^n$ defined by

$$(z_1, w_1, \dots, z_n, w_n) \mapsto (z_1, t_1 w_1, \dots, z_n, t_n w_n).$$

The orbit space of $M(C)$ by the induced T-action is an n-cube, so that $M(C)$ with this T-action is a quasitoric manifold over an n-cube.

Theorem 3.1. *The following are equivalent.*

(1) *$M(C)$ is equivariantly diffeomorphic to a Bott manifold.*
(2) *All principal minors of C are* 1.
(3) *$M(C)$ admits a T-invariant almost complex structure.*

Example. A simple example of an integer square matrix C which satisfies the condition (3.1) but does not satisfy (2) in the theorem above is $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. In this case $M(C)$ is (equivariantly) diffeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$ (with an appropriate action of T^2).

Theorem 3.2. *Let C' be another integer square matrix of size n satisfying the condition (3.1). Then the following are equivalent.*

(1) *$M(C)$ and $M(C')$ are equivariantly diffeomorphic.*
(2) *C and C' are conjugate by a permutation matrix and a matrix with ± 1 as diagonal entries and* 0 *as off-diagonal entries.*
(3) *$H_T^*(M(C)) \cong H_T^*(M(C'))$ as algebras over $H^*(BT)$.*

We do not know the corresponding results for (non-equivariant) diffeomorphism classification of $M(C)$'s although we can describe explicitly matrices C such that $M(C)$ is diffeomorphic to $(\mathbb{C}P^1)^n$.

An n-cube is a product of n number of 1-simplices. It turns out that most of the results mentioned so far can be extended to quasitoric manifolds over a product of finitely many simplices (with possibly different dimensions). Those quasitoric manifolds are also studied in [3].

4. Torus manifolds

As remarked before, a compact nonsingular toric variety with restricted action of the maximal compact torus is not necessarily a quasitoric manifold and vice versa. A *torus manifold* introduced in [6] is a closed smooth manifold of dimension $2n$ with a smooth T-action having a fixed point. Precisely speaking, orientation data is incorporated in the definition of torus manifold, but we do not care about it. A compact nonsingular toric variety with restricted action of the maximal compact torus and a quasitoric manifold are both a torus manifold, but of a special type. Their odd degree cohomology groups vanish and every fixed point set component of a subtorus is simply connected. It follows from [7] that

Proposition 4.1. *Let M be a torus manifold of dimension $2n$ such that $H^{odd}(M) = 0$ and every fixed point set component of a subtorus of T (even M itself) is simply connected. Then M/T is a manifold with corners such that every face (even M/T itself) is contractible.*

Because of this, a torus manifold satisfying the assumption in the proposition above seems an appropriate topological counterpart to a compact nonsingular toric variety. We conclude this article with the following question.

Question. Let M and M' be torus manifolds satisfying the assumption in the proposition above.

(1) Are they equivariantly diffeomorphic if and only if $H_T^*(M) \cong H_T^*(M')$ as algebras over $H^*(BT)$?
(2) Are they diffeomorphic if and only if $H^*(M) \cong H^*(M')$ as rings?

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