数理解析研究所講究録 1449 巻 2005 年 33-44

ISOVARIANT BORSUK-ULAM TYPE RESULTS AND THEIR CONVERSE

Ikumitsu Nagasaki (大阪大学大学院理学研究科・長崎 生光)
Department of Mathematics, Graduate School of Science
Osaka University

0. The Borsuk-Ulam theorem

In this note, we first make a brief survey of Borsuk-Ulam type theorems, and next introduce some results on the isovariant Borsuk-Ulam theorem and its converse from [22, 23].

K. Borsuk (1905–82) showed the following three results in 1933.

Theorem 0.1 ([21]).

(B1) *If $f : S^n \to S^n$ is antipodal, i.e., $f(-x) = -f(x)$ for all $x \in S^n$, then f is essential, i.e., f is not null-homotopic.*

(B2) *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.*

(B3) *Suppose $S^n = \bigcup_{i=0}^n F_i$, F_i: nonempty closed sets. Then some F_i contains an antipodal pair; $\{x_0, -x_0\} \subset F_i$. (Lusternik-Schnirelmann 1930)*

The second result was conjectured by S. Ulam; so it is usually called the Borsuk-Ulam theorem. It is known that the Borsuk-Ulam theorem has various equivalent statements; indeed, the above statements (B1)–(B3) are equivalent, and in addition, the following statements are also equivalent to the Borsuk-Ulam theorem.

(B4) If $f : S^n \to \mathbb{R}^n$ is antipodal, then $f^{-1}(0) \neq \emptyset$.

(B5) If $f : S^n \to S^m$ is antipodal, then $n \leq m$.

0.1. **Generalization.** Each of (B1) – (B5) has various generalizations and related topics. Indeed (B1) says that the degree of f is nonzero; in fact, it is well known that $\deg f$ is odd. Thus (B1) is related to the degree of (equivariant) maps or degree theory. Recently Hara [11] and Inoue [13] obtained a natural extension of (B1) for equivariant maps between Stiefel manifolds with standard $O(n)$- or $\mathbb{Z}_p^k$-action.

This research is partially supported by Grant-in-Aid for Scientific Research ((C) No.17540075), Japan Society for the Promotion of Science.

Statements (B2) and (B4) are related to coincidence theory or fixed point theory, and there are various researches in this field; see, for example, Gonçalves-Jaworowski-Pergher [8], Gonçalves et al. [9], Gonçalves-Wong [10].

Statement (B3) is related to the Lusternik-Schnirelmann category or Lusternik-Schnirelmann theory, which provides lower estimate for the number of critical points of a smooth function. For example, (B3) implies $\operatorname{cat}\mathbb{R}P^n \geq n$ and so we obtain $\operatorname{cat}\mathbb{R}P^n = n$, where $\operatorname{cat} X$ denotes the Lusternik-Schnirelmann category of X, i.e., $\operatorname{cat} X := \min\{n | X = \bigcup_{i=0}^n F_i$, each F_i is closed and contractible in $X\}$.

0.2. **Equivariant generalization.** From the viewpoint of transformation groups, (B5) can be rephrased as follows: If there is a $\mathbb{Z}_2$-map $f : S^n \to S^m$, then $n \leq m$ holds, where $\mathbb{Z}_2$ acts antipodally on the spheres. This formulation has a lot of equivariant generalizations; see, for example, Jaworowski [14], Dold [6], Fadell-Husseini [7], Marzantowicz [18], Bartsch [1], Komiya [16], Hara-Minami [12], etc. We recall some well-known equivariant generalizations. A direct generalization of (B5) is the following.

Theorem 0.2. *Suppose that $G \neq 1$ acts freely on S^n, S^m. If there is a G-map $f : S^n \to S^m$, then $n \leq m$ holds. (Dold [6], Kobayashi [15], Laitinen [17] etc.)*

The proof of Theorem 0.2 is reduced to the case $G = \mathbb{Z}_p$. An important fact is that the degree of a self G-map $f : S^n \to S^n$ is nonzero; in fact $\deg f \equiv 1 \mod p$.

Remark. This result still holds for free finite G-CW complexes homotopy equivalent to spheres.

In nonfree case, the following is known.

Theorem 0.3. *If there is a $\mathbb{Z}_p^k$-map (or T^k-map) $f : S^n \to S^m$, where $\mathbb{Z}_p^k$ or T^k acts fixed-point-freely on spheres, then $n \leq m$ holds. (Fadell-Husseini [7], Marzantowicz [18], etc.) Moreover this result still holds for $\mathbb{Z}_p$ (or $\mathbb{Q}$)-homology spheres. (Clapp-Puppe [4].)*

A euclidean space V with linear G-action is called a G-representation. We may suppose that the action is orthogonal. Let SV denote the unit sphere of a G-representation V. In this case, we say that G acts linearly on SV or that SV is a linear G-sphere.

A fundamental question is: For which finite groups does a Borsuk-Ulam type result hold? T. Bartsch [1] answered this question as follows.

Theorem 0.4 ([1]). *Suppose that G is a finite group. The "weak" Borsuk-Ulam theorem for linear G-spheres holds if and only if G is a p-group. Namely G has the following property (W) if and only if G is a p-group.*

(W) : *There exists a monotonely increasing function φ_G diverging to infinity such that for any linear G-spheres SV, SW ($V^G = W^G = 0$) with a G-map $f : SV \to SW$, the inequality $\varphi_G(\dim SV) \le \dim SW$ holds.*

By Theorem 0.3, one can take the identity map as φ_G for $G = \mathbb{Z}_p^k$, which is the best possible function satisfying (W); such a function φ_G is called the Borsuk-Ulam function. In general, it is difficult to determine the Borsuk-Ulam function, but a few results are known; see [1] for relevant results.

For other topics on the Borsuk-Ulam theorem, see also Steinlein [25, 26], Matoušek [19].

1. The isovariant Borsuk-Ulam theorem

Let G be a compact Lie group. Let X, Y be G-spaces, and V, W G-representations.

Definition 1. A continuous map $f : X \to Y$ is called *G-isovariant* (or isovariant) if f is G-equivariant and preserves the isotropy groups, i.e., $G_{f(x)} = G_x$ for any $x \in X$.

A. G. Wasserman [27] first studied an isovariant version of the Borsuk-Ulam theorem. Using the Borsuk-Ulam theorem for free $\mathbb{Z}_p$-actions, one can obtain the following result.

Theorem 1.1 (Isovariant Borsuk-Ulam theorem). *Let G be a solvable compact Lie group. If there is an isovariant map $f : SV \to SW$, then*

$$\dim SV - \dim SV^G \le \dim SW - \dim SW^G.$$

We note that this result still holds for semilinear actions on spheres.

Definition 2. The smooth G-action on a (homotopy) sphere M is called *semilinear* if for any $H \le G$, M^H is a (homotopy) sphere or $\emptyset$. We call such a G-manifold M a *semilinear G-sphere.*

Theorem 1.2 ([21]). *Let G be a solvable compact Lie group and let M, N be semilinear G-spheres. If there is an isovariant map $f : M \to N$, then*

$$\dim M - \dim M^G \le \dim N - \dim N^G.$$

It is still open whether Theorem 1.1 holds for an arbitrary compact Lie group, but Theorem 1.2 does not hold if G is nonsolvable.

Theorem 1.3 ([21]). *Let G be a nonsolvable compact Lie group. There are fixed-point-free semilinear G-spheres M_n, $n \ge 1$, with $\lim_{n\to\infty} \dim M_n = \infty$ and a representation sphere SW such that there is an isovariant maps $f_n : M_n \to SW$ for every n.*

Consequently, we obtain a Bartsch type result for semilinear actions; namely, the isovariant Borsuk-Ulam theorem for semilinear G-spheres holds if and only if G is solvable.

Remark. Bartsch's result, Theorem 0.4, still holds for semilinear G-spheres.

2. The converse of the isovariant Borsuk-Ulam theorem

Let G be a solvable compact Lie group. A subgroup means a closed subgroup. As mentioned in the previous section, the isovariant Borsuk-Ulam theorem holds for G. We would like to consider the converse.

If there is an isovariant map $f : SV \to SW$, then $f^H : SV^H \to SW^H$, $H \lhd K \leq G$, is K/H-isovariant. Since K/H is also solvable, we can apply the isovariant Borsuk-Ulam theorem to f^H. Hence we have

Proposition 2.1. *Let G be a solvable compact Lie group. If there is an isovariant map $f : SV \to SW$, then*

$(C_{V,W})$: $\dim SV^H - \dim SV^K \leq \dim SW^H - \dim SW^K$ *for any pair of closed subgroups $H \lhd K$.*

We formulate the converse problem of the isovariant Borsuk-Ulam theorem as follows.

Question. Let G be a solvable compact Lie group. Suppose that a pair (V, W) of G-representations satisfies

(a) $\mathrm{Iso}\, SV \subset \mathrm{Iso}\, SW$,
(b) $(C_{V,W})$.

Is there a G-isovariant map $f : SV \to SW$ (or $f : V \to W$)?

Remark. (1): The condition (a) is obviously necessary. However if G is abelian, then one can see that the condition (b) implies (a); so the condition (a) can be omitted.

(2) Note that there exists an isovariant map $f : SV \to SW$ if and only if there exists an isovariant map $f : V \to W$.

Definition 3. If this question is affirmative for G, we say that G has the *complete Borsuk-Ulam property* (or G is a *complete Borsuk-Ulam group*).

Unfortunately the complete answer is not known yet, but there are some partial results. In this note, we would like to give the outline of proof of the following theorem; the full detail will appear in [23].

Theorem 2.2. *The following groups have the complete Borsuk-Ulam property.*

(1) *finite abelian p-group,*

(2) $\mathbb{Z}_{p^n q^m}$,
(3) $\mathbb{Z}_{pqr}$,

where p, q, r are prime numbers.

Let T_k, $k \in \mathbb{Z}$, be the irreducible S^1-representation given by $t \cdot z := t^k z$, $t \in S^1$ ($\subset \mathbb{C}$), $z \in T_k$ ($= \mathbb{C}$). Restricting T_k to $\mathbb{Z}_n \subset S^1$, we have a $\mathbb{Z}_n$-representation, denoted by the same symbol T_k. For simplicity we here treat only complex representations.

2.1. **Proof of Theorem 2.2 (1) (outline).** Let us consider the case $G = \mathbb{Z}_p$. Then T_k, $0 \le k \le p-1$, are all irreducible $\mathbb{Z}_p$-representations. We may suppose $V^G = W^G = 0$. In fact, one can see that there exists an isovariant map $f : V \to W$ if and only if there exists an isovariant map $f : V_G \to W_G$, where V_G denotes the orthogonal complement of V^G in V. Therefore we may set $V = T_{k_1} \oplus \cdots \oplus T_{k_n}$, $W = T_{l_1} \oplus \cdots \oplus T_{l_m}$, where k_i, l_i are prime to $|G|$.

An isovariant map $f : T_k \to T_l$ is defined by $f_{k,l}(z) = \xi^{k'l} z$, where $k'k \equiv 1 \mod |G|$. Since condition $(C_{V,W})$ implies $n \le m$, one can construct an isovariant map $f : V \to W$ using $f_{k,l}$.

For a general abelian p-group, a similar argument shows Theorem 2.2 (1).

2.2. **Proof of Theorem 2.2 (2) (outline).**

Definition 4. A pair of representations (V, W) is called *primitive* if V and W cannot be decomposed into $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such that $(V_i, W_i) \ne (0, 0)$ satisfies (C_{V_i, W_i}), $i = 1, 2$.

If there are isovariant maps $f_i : V_i \to W_i$, then $f_1 \oplus f_2 : V_1 \oplus W_1 \to V_2 \oplus W_2$ is also isovariant; therefore it suffices to construct an isovariant map between each primitive pair.

Let us consider $G = \mathbb{Z}_{pq}$ for example. Clearly $(0, T_s)$ and (T_k, T_l), $(k, |G|) = (l, |G|)$, are primitive, and one can easily construct isovariant maps between these representations as in the proof of (1). In addition, a new primitive pair $(T_1, T_p \oplus T_q)$ appears for $G = \mathbb{Z}_{pq}$. In this case an isovariant map exists; for example, the map defined by $f : z \mapsto (z^p, z^q)$ is isovariant. These pairs mentioned above are essentially all primitive pairs for $\mathbb{Z}_{pq}$. Therefore $\mathbb{Z}_{pq}$ has the complete isovariant Borsuk-Ulam property.

For $\mathbb{Z}_{p^n q^m}$, other primitive pairs appear, but one can directly define isovariant maps in a similar way. For example, $(T_p \oplus T_q,\ T_{p^2} \oplus T_{pq} \oplus T_{q^2})$ is primitive for $\mathbb{Z}_{p^n q^m}$, $n, m \ge 2$. In this case there is an isovariant map; for example $f : (z_1, z_2) \mapsto (z_1^p, z_1^q + z_2^p, z_2^q)$ is isovariant. Thus one can see that $\mathbb{Z}_{p^n q^m}$ has the complete isovariant Borsuk-Ulam property.

2.3. **Proof of Theorem 2.2 (3) (outline).** Next consider the case of $\mathbb{Z}_{pqr}$. The proof is more complicated.

For all primitive pairs except one type, one can directly define isovariant maps as before. The exception is the following type of primitive pair:

$$(T_p \oplus T_q \oplus T_r,\ T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}).$$

If there is an isovariant map for this pair, it turns out that $\mathbb{Z}_{pqr}$ has the complete isovariant Borsuk-Ulam property. It seems, however, difficult to directly define an isovariant map; so we would like to use equivariant obstruction theory.

The question is the following:

Question. Is there a $\mathbb{Z}_{pqr}$-isovariant map

$$f : T_p \oplus T_q \oplus T_r \to T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}?$$

The answer is yes. Actually we shall show the existence of an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}).$$

Therefore we see that $\mathbb{Z}_{pqr}$ has the complete Borsuk-Ulam property.

3. The existence of an isovariant map

We shall discus the above question in a more general setting. Let $G = S^1$ and let M be a rational homology sphere with *pseudofree* S^1-action.

Definition 5 (Montgomery-Yang). An S^1-action on M is *pseudofree* if

(1) the action is effective, and
(2) the singular set $M^{>1} := \bigcup_{1 \neq H \leq S^1} M^H$ consists of finitely many exceptional orbits.

Here an orbit $G(x)$ is called exceptional if $G(x) \cong S^1/C$, $(1 \neq C < S^1)$.

Example 3.1. Let $V = T_p \oplus T_q \oplus T_r$. Then the S^1-action on SV is pseudofree. Indeed it is clearly effective, and

$$\begin{aligned} SV^{>1} &= ST_p \coprod ST_q \coprod ST_r \\ &\cong S^1/\mathbb{Z}_p \coprod S^1/\mathbb{Z}_q \coprod S^1/\mathbb{Z}_r \end{aligned}$$

Remark. There are many "exotic" pseudofree S^1-actions on high-dimensional homotopy spheres. (Montgomery-Yang [20], Petrie [24].)

Let SW be any S^1-representation sphere. We consider an S^1-isovariant map $f : M \to SW$.

The result is the following:

Theorem 3.2. *With the above notation, there is an S^1-isovariant map $f : M \to SW$ if and only if*

(I): $\operatorname{Iso} M \subset \operatorname{Iso} SW$,
(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *when* $1 \neq H \leq C$ *for some* $C \in \operatorname{Iso} M$,
(PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *when* $1 \neq H \not\leq C$ *for every* $C \in \operatorname{Iso} M$.

3.1. **Examples.** We give some examples. Let p, q, r be pairwise coprime integers greater than 1.

Example 3.3. There is an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. (PF1) and (PF2) are fulfilled. One can see $\operatorname{Iso} M = \{1, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r\}$ and $\operatorname{Iso} SW = \{1, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r, \mathbb{Z}_{pq}, \mathbb{Z}_{qr}, \mathbb{Z}_{rp}\}$; hence $\operatorname{Iso} M \subset \operatorname{Iso} SW$. □

Example 3.4. There is not an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. (PF1) is not fulfilled. □

Remark. There is an S^1-equivariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

By Example 3.3, we see that $\mathbb{Z}_{pqr}$ has the complete Borsuk-Ulam property.

3.2. **Proof of Theorem 3.2 (outline).** We shall give the outline of Theorem 3.2. The full detail will appear in [22]. Set $Y := SW \setminus SW^{>1}$. Note that S^1 acts freely on Y. Let N_i be an S^1-tubular neighborhood of each exceptional orbit in M. By the slice theorem, N_i is identified with $S^1 \times_{C_i} DU_i$ $(1 \leq i \leq r)$, where C_i is the isotropy group of the exceptional orbit and U_i is the slice C_i-representation. Set $X := M \setminus (\coprod_i \operatorname{int} N_i)$. Note that S^1 acts freely on X.

The "only if" part is proved by the (isovariant) Borsuk-Ulam theorem. Indeed we can show (PF1) as follows. Take a point $x \in M$ with $G_x = C$ and a C-invariant closed neighborhood B of x C-diffeomorphic to some unit disk DV. Hence we obtain an H-isovariant map $f : SV \to SW$. Applying the isovariant Borsuk-Ulam theorem to f, we have (PF1).

We next show (PF2). Since f is isovariant, f maps M into $SW \setminus SW^H$, and since $SW \setminus SW^H$ is S^1-homotopy equivalent to SW_H, we obtain an S^1-map $g : M \to SW_H$. By Theorem 0.3, we obtain (PF2).

To show the converse, we begin with the following lemma.

Lemma 3.5. *There is an S^1-isovariant map $\tilde{f}_i : N_i \to SW$.*

Proof. Let $N_i = N = S^1 \times_C DV$, where C is the isotropy group of the exceptional orbit and V is the slice representation. Similarly take a closed S^1-tubular neighborhood $N'$of an exceptional orbit with isotropy group C, and set $N' = S^1 \times_C DV'$. By (PF1), we see that $\dim SV + 1 \le \dim SV' - \dim SV'^{>1}$. Since C acts freely on SV, by obstruction theory, there is an C-map $g : SV \to SV' \setminus SV^{>1} \subset SW$, and so we obtain a C-isovariant map $g : SV \to SW$. Taking a cone, we have a C-isovariant map $\tilde{g} : DV \to DV'$; hence there is an S^1-isovariant map $\bar{f} = S^1 \times_C \tilde{g} : N \to N' \subset SW$. □

Set $f_i := \tilde{f}_i|_{\partial N_i} : \partial N_i \to Y$, and $f := \coprod_i f_i : \partial X \to Y$. If f is extended to an S^1-map $F : X \to Y$, by gluing the maps, we obtain an S^1-isovariant map

$$F \cup (\coprod_i \tilde{f}_i) : M \to SW.$$

Thus it suffices to investigate the following question:

(Q) Is there an extension $F : X \to Y$ of $f : \partial X \to Y$?

Since S^1 acts freely on X and Y, the obstruction to an extension lies in

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(Y)).$$

Set $k = \dim SW - \dim SW^{>1}$. A standard computation shows

Lemma 3.6. (1) *Y is $(k-2)$-connected and $(k-1)$-simple.*

(2) *$\pi_{k-1}(Y) \cong H_{k-1}(Y) \cong \oplus_{H \in \mathcal{A}} \mathbb{Z}$, where $\mathcal{A} := \{H \in \operatorname{Iso} SW \mid \dim SW^H = \dim SW^{>1}\}$, and generators are represented by SW_H, $H \in \mathcal{A}$.*

Note that $\dim M - 1 \le k$ by (PF1) and (PF2). We divide into two cases.

Case I: $\dim M - 1 < k$ (i.e., $\dim X/S^1 < k$). In this case, we see that

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(Y)) = 0$$

by dimensional reason. Hence the obstruction vanishes and there exists an extension $F : X \to Y$.

Case II: $\dim M - 1 = k$ (i.e., $\dim X/S^1 = k$). The obstruction $\gamma_{S^1}(f)$ to an extension lies in

$$H^k(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) \cong \oplus_{H \in \mathcal{A}} \mathbb{Z}.$$
$$(H^l(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) = 0, l \ne k)$$

To detect the obstruction, we introduce the multidegree.

3.3. **Multidegree.** Let $N = S^1 \times_C DU \subset M$, $1 \ne C \in \operatorname{Iso}(M)$, $\dim M - 1 = \dim U = k$, and $f : \partial N \to Y$: S^1-map, $\bar{f} = f|_{SU} : SU \to Y$: C-map.

Definition 6. $\operatorname{Deg} f := \bar{f}_*([SU]) \in \oplus_{H \in \mathcal{A}} \mathbb{Z}$, $\bar{f}_* : H_{k-1}(SU) \to H_{k-1}(Y)$, under identifying $H_{k-1}(Y)$ with $\oplus_{H \in \mathcal{A}} \mathbb{Z}$.

Then the obstruction $\gamma_{S^1}(f)$ is described by the multidegrees.

Proposition 3.7. *Let $F_0 : X \to Y$ be a fixed S^1-map (not necessary extending f). Set $f_{0,i} = F_0|_{\partial N_i}$. Then*

$$\gamma_{S^1}(f) = \sum_{i=1}^{r} (\operatorname{Deg} f_i - \operatorname{Deg} f_{0,i})/|C_i|.$$

Remark. (1) There always exists F_0 .

(2) $\operatorname{Deg} f_i - \operatorname{Deg} f_{0,i} \in \oplus_{H\in\mathcal{A}}|C_i|\mathbb{Z}$ by the equivariant Hopf type result. (See the next section.)

Using this proposition and equivariant Hopf type results in the next section, we can choose S^1-isovariant maps $\tilde{f}_i : N_i \to SW$ so that $\gamma_{S^1}(f) = 0$.

4. Equivariant Hopf type results

Let $N = S^1 \times_C DU$ ($\subset M$), $\dim M - 1 = k$ as before. Then the following Hopf type theorem holds.

Theorem 4.1 ([22]). (1) *$\operatorname{Deg} : [\partial N, Y]_{S^1} \to \oplus_{H\in\mathcal{A}}\mathbb{Z}$ is injective.*

(2) *The image of* $\operatorname{Deg} - \operatorname{Deg} f_0$ *coincides with $\oplus_{H\in\mathcal{A}}|C|\mathbb{Z}$, where f_0 is any fixed S^1-map.*

The next result shows the extendability of $f : \partial N = S^1 \times_C SU \to Y$. Set $\operatorname{Deg} f = (d_H(f))_{H\in\mathcal{A}} \in \oplus_{H\in\mathcal{A}}\mathbb{Z}$.

Theorem 4.2 ([22]). (1) *$f : \partial N \to Y$ is extendable to an S^1-isovariant map $\tilde{f} : N \to SW$ if and only if $d_H(f) = 0$ for any $H \in \mathcal{A}$ with $H \not\leq C$.*

(2) *For any extendable f and for any $(a_H) \in \oplus_{H\in\mathcal{A}}|C|\mathbb{Z}$ satisfying $a_H = 0$ for $H \in \mathcal{A}$ with $H \not\leq C$, there exists an S^1-map $f' : \partial N \to Y$ such that f' is extendable to an S^1-isovariant map $\tilde{f}' : N \to SW$ and* $\operatorname{Deg} f' = \operatorname{Deg} f + (a_H)$.

4.1. **Example of multidegrees.** Finally we give some examples. Take S^1-representations $V = T_q \oplus T_q \oplus T_r$ and $W = T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}$, where p, q, r are distinct primes. Let us consider linear spheres SV, SW. Let N_i be a closed S^1-tubular neighborhood of the exceptional orbit $ST_i \cong S^1/\mathbb{Z}_i$ in SV, where $i = p, q, r$, and then N_i is identified with $ST_i \times D(T_j \oplus T_k) \cong S^1 \times_{\mathbb{Z}_i} D(T_j \oplus T_k)$. Thus we may set

$$N_p = \{(z_1, z_2, z_3) \in V \mid |z_1| = 1, \|(z_2, z_3)\| \leq 1\},$$
$$N_q = \{(z_1, z_2, z_3) \in V \mid |z_2| = 1, \|(z_1, z_3)\| \leq 1\},$$
$$N_r = \{(z_1, z_2, z_3) \in V \mid |z_3| = 1, \|(z_1, z_2)\| \leq 1\}.$$

We have $\mathcal{A} = \{\mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r\}$; hence we can set

$$\operatorname{Deg} f = (d_{\mathbb{Z}_p}(f), d_{\mathbb{Z}_q}(f), d_{\mathbb{Z}_r}(f)) \in \mathbb{Z}^3.$$

Take positive integers α, β, γ, δ, ξ, η such that $\alpha p - \beta q = 1$, $\gamma q - \delta r = 1$, $\xi r - \eta p = 1$.

Example 4.3. We define $g_i : V \to W$ as follows:

$$g_p(z_1, z_2, z_3) = (z_3^{\xi}\bar{z}_1^{\eta}, z_1^q, z_2^r, z_1^r),$$
$$g_q(z_1, z_2, z_3) = (z_1^{\alpha}\bar{z}_2^{\beta}, z_2^p, z_2^r, z_3^p),$$
$$g_r(z_1, z_2, z_3) = (z_2^{\gamma}\bar{z}_3^{\delta}, z_1^q, z_3^q, z_3^p).$$

Restricting g_i to N_i, we obtain an S^1-map $h_i := g_{i|} : N_i \to W$. Since $h_i^{-1}(0) = \emptyset$, we have an S^1-map $\tilde{f}_i := h_i/\|h_i\| : N_i \to SW$. Moreover $\tilde{f}_i$ is an S^1-isovariant. Set $f_i = \tilde{f}_{i|\partial N_i}$. Then $d_{\mathbb{Z}_p}(f_p)$ is equal to the degree of the map $f_p' : S(T_q \oplus T_r) \to S(T_1 \oplus T_{qr})$;

$$(z_2, z_3) \mapsto (z_3^{\xi}\bar{z}_1^{\eta}, z_2^r)/\|(z_3^{\xi}\bar{z}_1^{\eta}, z_2^r)\|,$$

where z_1 is any fixed nonzero number. Hence we have $d_{\mathbb{Z}_p}(f_p) = \xi r = 1 + \eta p$. Similarly one can see that $d_{\mathbb{Z}_q}(f_p) = d_{\mathbb{Z}_r}(f_p) = 0$. Thus we obtain

$$\operatorname{Deg} f_p = (1 + \eta p, 0, 0).$$

In a similar way, we have

$$\operatorname{Deg} f_q = (0, 1 + \beta q, 0),$$
$$\operatorname{Deg} f_r = (0, 0, 1 + \delta r).$$

Example 4.4. Next we consider the following S^1-maps $g_i' : V \to W$:

$$g_p'(z_1, z_2, z_3) = (z_2^{\gamma}\bar{z}_3^{\delta}, z_1^q, z_2^r + z_3^q, z_1^r),$$
$$g_q'(z_1, z_2, z_3) = (z_3^{\xi}\bar{z}_1^{\eta}, z_2^p, z_2^r, z_3^p + z_1^r),$$
$$g_r'(z_1, z_2, z_3) = (z_1^{\alpha}\bar{z}_2^{\beta}, z_1^q + z_2^p, z_3^q, z_3^p).$$

Then by restriction and normalization, we obtain S^1-isovariant maps $\tilde{f}_i' : N_i \to SW$ and $f_i' : \partial N_i \to SW$, respectively. In this case, one can see that

$$\operatorname{Deg} f_p' = (1, 0, 0),$$
$$\operatorname{Deg} f_q' = (0, 1, 0),$$
$$\operatorname{Deg} f_r' = (0, 0, 1).$$

In fact, for example, $d_{\mathbb{Z}_p}(f_p') = 1$ is showed as follows. Consider the map $\psi : T_q \oplus T_r \setminus 0 \to T_1 \oplus T_{qr} \setminus 0$; $(z_2, z_3) \mapsto (z_2^{\gamma}\bar{z}_3^{\delta}, z_1^q, z_2^r + z_3^q)$. One can see that $\psi^{-1}(1, 0) = \{((-1)^{\delta}, (-1)^{\gamma})\}$ and the Jacobian is $\gamma q + r\delta > 0$; hence $(1, 0) \in T_1 \oplus T_{qr} \setminus 0$ is a regular value, and so $\deg \psi = 1$.

$Y = SW \setminus SW^{>1}$. Let $[\partial N_i, Y]_{S^1}^{\text{ext}}$, $i = p, q, r$, denote the set of S^1-homotopy classes of S^1-maps extended to S^1-isovariant maps from N_i to SW. By Theorems 4.1 and 4.2, we see the following.

Proposition 4.5. *The map* $D_i : [\partial N_i, Y]_{S^1}^{\text{ext}} \to \mathbb{Z}, [f] \mapsto (d_{\mathbb{Z}_i}(f) - 1)/i$, *is a bijection for* $i = p, q, r$.

For the above maps, we have $D_p(f_p) = \eta$ and $D_p(f'_p) = 0$.

Example 4.6. We next define another S^1-map $f_{0,i}$ as follows. Define an S^1-map $g_0 : V \to W$ by setting

$$g_0(z_1, z_2, z_3) = (z_1^\alpha \bar{z}_2^\beta + z_2^\gamma \bar{z}_3^\delta + z_3^\xi \bar{z}_1^\eta, z_1^q, z_2^r, z_3^p).$$

Since g_0 maps the free part of V into the free part of W, by restriction and normalization, we have an S^1-map $f_{0,i} : \partial N_i \to Y$. In this case we have

$$\operatorname{Deg} f_{0,p} = (1 + \eta p, -\beta p, 0),$$

$$\operatorname{Deg} f_{0,q} = (0, 1 + \beta q, -\delta q),$$

$$\operatorname{Deg} f_{0,r} = (-\eta r, 0, 1 + \delta r).$$

By Theorem 4.2, each $f_{0,i}$ cannot be isovariantly extended on N_i.

However, restricting g_0 on $X = SV \setminus \operatorname{int}(N_p \cup N_q \cup N_r)$, one can regard g_0 as an S^1-map from X to Y. Consequently it turns out that $\coprod_i f_{0,i}$ can be extended on X. Consider the S^1-maps $f = \coprod_i f_i : \partial N_i \to Y$ and $f' = \coprod_i f'_i : \partial N_i \to Y$ in Examples 2 and 3. By Proposition 3.7, the obstruction $\gamma_{S^1}(f)$ to an extension on X is described as $\gamma_{S^1}(f) = (\eta, \beta, \delta)$ and $\gamma_{S^1}(f') = (0, 0, 0)$; hence f cannot be extended on X, but f' can.

We also note the following.

Proposition 4.7. *An S^1-isovariant map $\tilde{h} = \coprod_i \tilde{h}_p : \coprod_i N_i \to SW$ is isovariantly extended on SV if and only if* $\operatorname{Deg} h_p = (1, 0, 0)$, $\operatorname{Deg} h_q = (0, 1, 0)$ *and* $\operatorname{Deg} h_r = (0, 0, 1)$, *where* $h_i = \tilde{h}_{i|\partial N_i}$.

Proof. One can set $\operatorname{Deg} h_p = (1 + np, 0, 0)$, $\operatorname{Deg} h_q = (0, 1 + mq, 0)$ and $\operatorname{Deg} h_r = (0, 0, 1 + lr)$. Then one can see $\gamma_{S^1}(h) = (n, m, l)$, and so $\gamma_{S^1}(h) = 0$ if and only if $(n, m, l) = (0, 0, 0)$. □

References

[1] T. Bartsch, *On the existence of Borsuk-Ulam theorems*, Topology **31** (1992), 533–543.

[2] K. Borsuk, *Drei Sätze über die n-dimensionale Sphäre*, Fund. Math, **20** (1933), 177–190.

[3] G. E. Bredon, Introduction to compact transformation groups, Academic Press 1972.

[4] M. Clapp and D. Puppe, *Critical point theory with symmetries*, J. Reine Angew. Math. **418** (1991), 1–29.

[5] T. tom Dieck, *Transformation groups*, Walte de Gruyter, Berlin, New York 1987.

[6] A. Dold, *Simple proofs of some Borsuk-Ulam results*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982), 65–69, Contemp. Math., 19.

[7] E. Fadell and S. Husseini, *An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bougin-Yang theorems*, Ergodic Theory Dynamical System **8** (1988), 73–85.

[8] D. L. Gonçalves, J. Jaworowski and P. L. Q. Pergher, *G-coincidences for maps of homotopy spheres into CW-complexes*, Proc. Amer. Math. Soc. **130** (2002), 3111–3115.

[9] D. L. Gonçalves et al., *Coincidences for maps of spaces with finite group actions*, Topology Appl. **145** (2004), 61–68.
[10] D. L. Gonçalves and P. Wong, *Homogeneous spaces in coincidence theory. II*, Forum Math. **17** (2005), 297–313.
[11] Y. Hara, *The degree of equivariant maps*, Topology Appl. **148** (2005), 113–121.
[12] Y. Hara and N. Minami, *Borsuk-Ulam type theorems for compact Lie group actions*, Proc. Amer. Math. Soc. **132** (2004), 903–909.
[13] A. Inoue, *Borsuk-Ulam type theorems on Stiefel manifolds*, to appear in Osaka J. Math.
[14] J. Jaworowski, *Maps of Stiefel manifolds and a Borsuk-Ulam theorem*, Proc. Edinb. Math. Soc., II. Ser. **32** (1989), 271-279.
[15] T. Kobayashi, *The Borsuk-Ulam theorem for a Z_q-map from a Z_q-space to S^{2n+1}*, Proc. Amer. Math. Soc. **97** (1986), 714-716.
[16] K. Komiya, *Equivariant K-theoretic Euler classes and maps of representation spheres*, Osaka J. Math. **38** (2001), 239-249.
[17] E. Laitinen, *Unstable homotopy theory of homotopy representations*, Transformation groups, Poznań 1985, 210–248, Lecture Notes in Math., **1217**, Springer, Berlin, 1986,
[18] W. Marzantowicz, *Borsuk-Ulam theorem for any compact Lie group*, J. Lond. Math. Soc., II. Ser. **49** (1994),195-208.
[19] J. Matoušek, *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*, Universitext, Springer, 2003.
[20] D. Montgomery and C. T. Yang, *Differentiable pseudo-free circle actions on homotopy seven spheres*, Proceedings of the Second Conference on Compact Transformation Groups, Part I, 41–101, Lecture Notes in Math., Vol. 298, Springer, Berlin, 1972.
[21] I. Nagasaki, *The weak isovariant Borsuk-Ulam theorem for compact Lie groups*, Arch. Math. **81** (2003), 348–359.
[22] I. Nagasaki, *Isovariant Borsuk-Ulam results for pseudofree circle actions and their converse*, to appear in Trans. Amer. Math. Soc.
[23] I. Nagasaki, *The converse of isovariant Borsuk-Ulam results for some abelian groups*, preprint.
[24] T. Petrie, *Pseudoequivalences of G-manifolds*, Algebraic and geometric topology, 169–210, Proc. Sympos. Pure Math., XXXII, 1978.
[25] H. Steinlein, *Borsuk's antipodal theorem and its generalizations and applications: a survey*, Topological methods in nonlinear analysis, 166–235, Montreal, 1985.
[26] H. Steinlein, *Spheres and symmetry: Borsuk's antipodal theorem*, Topol. Methods Nonlinear Anal. 1 (1993), 15–33.
[27] A. G. Wasserman, *Isovariant maps and the Borsuk-Ulam theorem*, Topology Appl. **38** (1991), 155–161.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka 560-0043, Osaka, Japan
E-mail address: nagasaki@math.sci.osaka-u.ac.jp