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SOME REMARKS ON THE JOHNSON HOMOMORPHISM OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Dedicated to Professor Yasuhiko Kitada on the occasion of his sixtieth birthday

ABSTRACT. In this paper we construct new obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group. We also determine the structure of the cokernel of the Johnson homomorphism for degrees 2 and 3.

1. Introduction

Let F_n be a free group of rank $n \geq 2$ and $F_n = \Gamma_n(1), \Gamma_n(2), \ldots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$. This filtration was introduced in 1963 with a remarkable pioneer work by S. Andreadakis [1] who showed that $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ is a descending central series of $\mathcal{A}_n(1)$ and each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) = \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. He [1] also computed that $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^k(\mathcal{A}_2)$ for all $k \geq 1$ and $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^2(\mathcal{A}_3)$, and asserted $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_3) = 44$. In Section 5, however, we show that $\mathrm{gr}^3(\mathcal{A}_3) = 43$. Moreover, by a recent remarkable work by A. Pettet [15] we have $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^2(\mathcal{A}_n) = \frac{1}{3}n^2(n^2-4) + \frac{1}{2}n(n-1)$ for all $n \geq 3$. However, it is difficult to compute the rank of $\mathrm{gr}^k(\mathcal{A}_n)$.

Let H be the abelianization of F_n and $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. Let $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ be the free graded Lie algebra generated by H. Then for each $k \geq 1$, a $GL(n, \mathbf{Z})$-equivariant injective

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homomorphim

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

is defined. (For definition, see Section 2.) This is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. The theory of the Johnson homomorphism of a mapping class group of a compact Riemann surface began in 1980 by D. Johnson [6] and has been developed by many authors. There is a broad range of remarkable results for the Johnson homomorphism of a mapping class group. (For example, see [5] and [13].) However, the properties of the Johnson homomorphism of $\mathrm{Aut}\, F_n$ are far from being well understood.

The main interest of this paper is to determine the structure of the cokernel of the Johnson homomorphism τ_k as a $GL(n, \mathbf{Z})$-module. For $k = 1$, it is a well known fact that the first Johnson homomorphism τ_1 is an isomorphism. (See [8].) For $k \geq 2$, the Johnson homomorphism τ_k is not surjective. In fact, a recent remarkable work by Shigeyuki Morita indicates that there is a symmetric product $S^k H_{\mathbf{Q}}$ of $H_{\mathbf{Q}} = H \otimes_{\mathbf{Z}} \mathbf{Q}$ in the cokernel of $\tau_{k,\mathbf{Q}} = \tau_k \otimes id_{\mathbf{Q}}$ for each $k \geq 2$. To show this, he introduced a homomorphism

$$\mathrm{Tr}_k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H,$$

called the trace map, and showed that Tr_k vanishes on the image of τ_k and is surjective after tensoring with $\mathbf{Q}$ for all $k \geq 2$.

The trace maps were introduced in the 1993 by Morita [12] for a Johnson homomorphism of a mapping class group of a surface. He called these maps traces because they were defined using the trace of some matrix representation. Morita's traces are very important to study the Lie algebra structure of the target $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n = \mathrm{Der}(\mathcal{L}_n)$ of the Johnson homomorphisms. Here $\mathrm{Der}(\mathcal{L}_n)$ denotes the graded Lie algebra of derivations of $\mathcal{L}_n$. Morita conjectured that for any $n \geq 3$, the abelianization of the Lie algebra $\mathrm{Der}(\mathcal{L}_n)$ is given by

$$H_1(\mathrm{Der}(\mathcal{L}_n^{\mathbf{Q}})) \simeq (H_{\mathbf{Q}}^* \otimes_{\mathbf{Z}} \Lambda^2 H_{\mathbf{Q}}) \oplus (\bigoplus_{k \geq 2}^{\infty} S^k H_{\mathbf{Q}})$$

where $\mathcal{L}_n^{\mathbf{Q}} = \mathcal{L}_n \otimes_{\mathbf{Z}} \mathbf{Q}$ and the right hand side is understood to be an abelian Lie algebra. Recently, combining a work of Kassabov [7] with the concept of the traces, he [14] showed that the isomorphism above holds up to degree $n(n-1)$.

The subgroup $\mathcal{A}_n(1)$ is called the IA-automorphism group of F_n and denoted by IA_n. The group IA_n is the kernel of the natural map $\mathrm{Aut}\, F_n \to GL(n,\mathbf{Z})$ which is given by the action of $\mathrm{Aut}\, F_n$ on H. The structures of IA_n plays an important role in the study $\mathrm{Aut}\, F_n$. W. Magnus [10] showed that IA_n is finitely generated for all $n \geq 3$. However, it is not known whether IA_n is finitely presented or not for any $n \geq 4$. For $n = 3$, by a remarkable work by S. Krstić and J. McCool [9], it is known that IA_3 is not finitely presented. On the other hand, the abelianization of IA_n is given by

$$IA_n^{\mathrm{ab}} \simeq H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

as a $GL(n,\mathbf{Z})$-module. (See [8].)

Now let $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, ... be the lower central series of $IA_n = \mathcal{A}_n(1)$ and $\mathrm{gr}^k(\mathcal{A}'_n)$ its graded quotient of it for each $k \geq 1$. In Section 2, we define a $GL(n,\mathbf{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

which is also called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. In this paper, we construct new obstructions of the surjectivity of the Johnson homomorphism τ'_k. Let us denote the tensor products with $\mathbf{Q}$ of a $\mathbf{Z}$-module by attaching a subscript $\mathbf{Q}$ to the original one. For example, $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathcal{L}_n^{\mathbf{Q}}(k) := \mathcal{L}_n(k) \otimes_{\mathbf{Z}} \mathbf{Q}$. Similarly, for a $\mathbf{Z}$-linear map $f : A \to B$ we denote by $f_{\mathbf{Q}}$ the $\mathbf{Q}$-linear map $A_{\mathbf{Q}} \to B_{\mathbf{Q}}$ induced by f. It is conjectured that $\mathrm{Coker}\, \tau'_{k,\mathbf{Q}} = \mathrm{Coker}\, \tau_{k,\mathbf{Q}}$ for $k \geq 1$. It is true for $1 \leq k \leq 3$. In fact, $\mathcal{A}_n(1) = \mathcal{A}'_n(1)$ by definition. We have $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ from the result stated above. (See [8].) Moreover, Pettet [15] showed that $\mathcal{A}'_n(3)$ has a finite index in $\mathcal{A}_n(3)$. Hence, $\mathrm{Coker}\, \tau'_{k,\mathbf{Q}} = \mathrm{Coker}\, \tau_{k,\mathbf{Q}}$ for $1 \leq k \leq 3$. Our main result is

Theorem 1.

(1) $\Lambda^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}$ *for odd* k *and* $3 \leq k \leq n$.

(2) $H_{\mathbf{Q}}^{[2,1^{k-2}]} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}$ *for even* k *and* $4 \leq k \leq n-1$.

Here $\Lambda^k H_{\mathbf{Q}}$ *denotes the* k*-th exterior product of* $H_{\mathbf{Q}}$*, and* $H_{\mathbf{Q}}^{[2,1^{k-2}]}$ *denotes the Schur-Weyl module of* $H_{\mathbf{Q}}$ *corresponding to the partition* $[2, 1^{k-2}]$.

In order to prove this, in Section 3, we introduce homomorphisms defined by

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H,$$

$$\mathrm{Tr}_{[2,1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}) \circ \Phi_2^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$$

and show that these maps vanish on the image of the Johnson homomorphism τ'_k. Since these maps are constructed in a way similar to that of Morita's trace Tr_k, we also call these maps traces.

In Section 5, we determine the $GL(n, \mathbf{Z})$-module structure of the cokernel of the Johnson homomorphism τ_k for 2 and 3. Our result is

Theorem 2. *We have $GL(n, \mathbf{Z})$-equivariant exact sequences*

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

and

$$0 \to \mathrm{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}^{\mathbf{Q}}_n(4) \to S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0$$

for $n \geq 3$.

Thus we have

Corollary 1. *For $n \geq 3$,*

$$\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8).$$

Contents

1. **Introduction** 1
2. **Preliminaries** 5
2.1. Commutators of higher weight 5
2.2. IA-automorphism group 6
2.3. The associated graded Lie algebra 7
2.4. Johnson homomorphism 7
3. **The contractions** 9
3.1. The image of $\Phi^k_1 \circ \tau'_k$ 10
3.2. The image of $\Phi^k_2 \circ \tau'_k$ 11
4. **The trace maps** 12
4.1. Morita's trace (Trace map for $S^k H$) 13
4.2. Trace map for $\Lambda^k H$ 13
4.3. Trace map for $H^{[2,1^{k-2}]}$ 14
5. **The cokernel of the Johnson homomorphism τ_k for $k = 2$ and 3** 14
5.1. The case $k = 2$ 14
5.2. The case $k = 3$ 15
6. **Acknowledgments** 16
References 16

2. Preliminaries

In this section we review some basic facts. First, we note that the group $\mathrm{Aut}\,F_n$ acts on F_n on the right. For any $\sigma \in \mathrm{Aut}\,F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ.

2.1. Commutators of higher weight.

In this paper, we often use basic facts of commutator calculus. The reader is referred to [11] and [16], for example. Let G be a group. For any elements x and y of G, the element

$$xyx^{-1}y^{-1}$$

is called the commutator of x and y, and denoted by $[x, y]$. In general, a commutator of higher weight is recursively defined as follows. First, a commutator of weight 1 is an element of G. For $k > 1$, a commutator of weight k is an element of the type $C = [C_1, C_2]$ where C_j is a commutator of weight a_j ($j = 1, 2$) such that $a_1 + a_2 = k$. The weight of the commutator C is denoted by $\mathrm{wt}\,(C) = k$. The commutator which has elements $g_1, \dots, g_t \in G$ in the bracket components is called the commutator among the components $g_1, \dots, g_t$. For elements $g_1, \dots, g_t \in G$, a commutator of weight k among the components $g_1, \dots, g_t$ of the type

$$[[\cdots[[g_{i_1}, g_{i_2}], g_{i_3}], \cdots], g_{i_k}], \quad i_j \in \{1, \dots, t\}$$

with all of its brackets to the left of all the elements occuring is called a simple k-fold commutator and is denoted by

$$[g_{i_1}, g_{i_2}, \cdots, g_{i_k}].$$

For each $k \geq 1$, the subgroups $\Gamma_G(k)$ of the lower central series of G are defined recursively by

$$\Gamma_G(1) = G, \quad \Gamma_G(k+1) = [\Gamma_G(k), G].$$

We use the following basic lemma in later sections.

Lemma 2.1. *If a group G is generated by $g_1, \dots, g_t$, then each of the graded quotients $\Gamma_G(k)/\Gamma_G(k+1)$ for $k \geq 1$ is generated by the cosets of the simple k-fold commutators*

$$[g_{i_1}, g_{i_2}, \dots, g_{i_k}], \quad i_j \in \{1, \dots, t\}.$$

Now, for each $k \geq 1$, let $\Gamma_n(k)$ be the k-th subgroup $\Gamma_{F_n}(k)$ of the lower central series of a free group F_n of rank n and $\mathrm{gr}^k(\Gamma_n)$ its graded quotient $\Gamma_n(k)/\Gamma_n(k+1)$. We denote by $\mathrm{gr}\,(\Gamma_n) = \bigoplus_{k \geq 1} \mathrm{gr}^k(\Gamma_n)$ the associated graded sum. Then the set $\mathrm{gr}\,(\Gamma_n)$ naturally has a structure of a graded Lie algebra over $\mathbf{Z}$ induced from the commtator bracket on F_n. Let H be the abelianization of F_n and $\mathcal{L}_n = \bigoplus_{k\geq 1}\mathcal{L}_n(k)$ the free graded Lie algebla generated by H. It is well known that the Lie algebra $\mathrm{gr}\,(\Gamma_n)$ is isomorphic to $\mathcal{L}_n$ as a graded Lie algebra over $\mathbf{Z}$. Thus, in this paper, we identify $\mathrm{gr}\,(\Gamma_n)$ with $\mathcal{L}_n$. For any element $x \in \Gamma_n(k)$, we also denote by x the coset class of x in $\mathcal{L}_n(k) = \Gamma_n(k)/\Gamma_n(k+1)$. Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then the algebra $T(H)$ is the universal envelopping algebra of the free Lie algebra $\mathcal{L}_n$ and the natural map $\mathcal{L}_n \to T(H)$ defined by

$$[X,Y] \mapsto X\otimes Y - Y\otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective Lie algebra homomorphism. Hence we also regard $\mathcal{L}_n(k)$ as a submodule of $H^{\otimes k}$ for each $k \geq 1$.

2.2. IA-automorphism group.

The kernel of the natural map $\mathrm{Aut}\,F_n \to GL(n,\mathbf{Z})$ which is given by the action of $\mathrm{Aut}\,F_n$ on H is called the IA-automorphism group of F_n and denoted by IA_n. Let $\{x_1,\ldots,x_n\}$ be a basis of a free group F_n. Magnus [10] showed that IA_n is finitely generated by automorphisms

$$K_{ab}: \begin{cases} x_a & \mapsto x_b^{-1}x_a x_b, \\ x_t & \mapsto x_t, \quad (t\neq a) \end{cases}$$

and

$$K_{abc}: \begin{cases} x_a & \mapsto x_a x_b x_c {x_b}^{-1} {x_c}^{-1}, \\ x_t & \mapsto x_t, \quad (t\neq a) \end{cases}$$

for any distinct a, b and $c \in \{1,2,\ldots,n\}$. It is known that the abelianization IA_n^{ab} of the IA-automorphism group is free abelian group with generators K_{ab} for distinct a and b, and K_{abc} for distinct a,b,c and $b<c$. More precisely, if we denote by $H^* = \mathrm{Hom}_{\mathbf{Z}}(H,\mathbf{Z})$ the dual group of H, we have a $GL(n,\mathbf{Z})$-module isomorphism $IA_n^{\mathrm{ab}} \simeq H^*\otimes_{\mathbf{Z}}\Lambda^2 H$. (For details, see [8].)

2.3. The associated graded Lie algebra.

Here we consider two descending filtrations of IA_n. The first one is $\{\mathcal{A}_n(k)\}_{k\geq 1}$ defined as above. Since the series $\mathcal{A}_n(1), \mathcal{A}_n(2), \ldots$ is central, the associated graded sum $\mathrm{gr}(\mathcal{A}_n) = \bigoplus_{k\geq 1}\mathrm{gr}^k(\mathcal{A}_n)$ naturally has a structure of a graded Lie algebla over $\mathbf{Z}$ induced from the commutator bracket on $\mathcal{A}_n(1)$. For each $k \geq 1$, the group $\mathcal{A}_n(0) = \mathrm{Aut}\,F_n$ naturally acts on

$\mathcal{A}_n(k)$ by conjugation, hence on $\mathrm{gr}^k(\mathcal{A}_n)$. Since the group $\mathcal{A}_n(1) = IA_n$ trivially acts on $\mathrm{gr}^k(\mathcal{A}_n)$, we see that the group $GL(n,\mathbf{Z}) \simeq \mathcal{A}_n(0)/\mathcal{A}_n(1)$ naturally acts on $\mathrm{gr}^k(\mathcal{A}_n)$.

The other is the lower central series $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, ... of $A_n(1)$. Let $\mathrm{gr}^k(\mathcal{A}'_n) = \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$ be the graded quotient for each $k \geq 1$. Similarly the associated graded sum $\mathrm{gr}(\mathcal{A}'_n) = \bigoplus_{k\geq 1}\mathrm{gr}^k(\mathcal{A}'_n)$ has a structure of a graded Lie algebra structure on $\mathbf{Z}$. Moreover, each graded quotient $\mathrm{gr}^k(\mathcal{A}'_n)$ is a $GL(n,\mathbf{Z})$-module. It is clear that $\mathcal{A}'_n(k) \subset \mathcal{A}_n(k)$ for every $k \geq 1$. In particular, we have $\mathcal{A}'_n(k) = \mathcal{A}_n(k)$ for $1 \leq k \leq 2$ and Pettet [15] showed that $\mathcal{A}'_n(3)$ has finite index in $\mathcal{A}_n(3)$ as mentioned in section 1. From Lemma 2.1, for each $k \geq 1$, the graded quotient $\mathrm{gr}^k(\mathcal{A}'_n)$ is generated by (the cosets of) the simple k-fold commutators among the components K_{ab} and K_{abc}.

2.4. **Johnson homomorphism.**

Here we define the Johnson homomorphisms of $\mathrm{Aut}\,F_n$. For each $k \geq 1$, let $\tau_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ be the map defined by

$$\sigma \mapsto (x \mapsto x^{-1}x^{\sigma}) \tag{1}$$

for $\sigma \in \mathcal{A}_n(k)$ and $x \in H$. Then the map τ_k is a homomorphism and the kernel of τ_k is just $\mathcal{A}_n(k+1)$. Hence, identifying $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, we obtain an injective $GL(n,\mathbf{Z})$-equivariant homomorphism, also denoted by τ_k,

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

This homomorphism is called the k-th Johnson homomorphism of $\mathrm{Aut}\,F_n$. Similarly, for each $k \geq 1$, we can define a homomorphism $\tau'_k : \mathcal{A}'_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ as (1). Since $\mathcal{A}'_n(k+1)$ is contained in the kernel of τ'_k, we obtain a $GL(n,\mathbf{Z})$-equivariant homomorphism, also denoted by τ'_k,

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

We also call the map τ'_k the Johnson homomorphism of $\mathrm{Aut}\,F_n$.

Let $\{x_1, \ldots, x_n\}$ be a basis of F_n. It defines a basis of H as a free abelian group, also denoted by $\{x_1, \ldots, x_n\}$. Let $\{x_1^*, \ldots, x_n^*\}$ be the dual basis of H^*. For any $\sigma \in \mathrm{A}'_n(k)$, if we set $s_i(\sigma) := x_i^{-1}x_i^{\sigma} \in \mathcal{L}_n(k+1)$ $(1 \leq i \leq n)$ then we have

$$\tau_k(\sigma) = \tau'_k(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma) \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

Let Der $(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of Der $(\mathcal{L}_n)$ is expressed as Der $(\mathcal{L}_n)(k) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k)$. Thus we sometimes identify Der $(\mathcal{L}_n)$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$. Then the Johnson homomorphism $\tau = \bigoplus_{k \geq 1} \tau_k$ is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\sigma$ the element of Der $(\mathcal{L}_n)$ corresponding to an element $\sigma \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$ and write the action of $\partial\sigma$ on $X \in \mathcal{L}_n$ as $X^{\partial\sigma}$ then we have

$$\tau'_{k+l}([\sigma, \tau]) = \sum_{i=1}^{n} x_i^* \otimes (s_i(\sigma)^{\partial\tau} - s_i(\tau)^{\partial\sigma}). \tag{2}$$

for any $\sigma \in \mathrm{A}'_n(k)$ and $\tau \in \mathrm{A}'_n(l)$.

In general, each $s_i(\sigma) \in \mathcal{L}_n(k+1)$ cannot be uniquely written as a sum of commutators among the components $x_1, \ldots, x_n$. In this paper, each $s_i(\sigma)$ is recursively computed in the following way. First, for $\sigma = K_{abc}$, we can set

$$s_a(K_{abc}) = [x_b, x_c], \quad s_t(K_{abc}) = 0 \quad \text{if} \quad t \neq a.$$

For $\sigma = K_{ab}$, we see that

$$x_t^{-1} x_t^{\sigma} = \begin{cases} [x_a^{-1}, x_b^{-1}] & \text{if} \quad t = a, \\ 1 & \text{if} \quad t \neq a \end{cases}$$

in F_n. Since $[x_a^{-1}, x_b^{-1}] = [x_a, x_b]$ in $\mathcal{L}_n(2)$, so we can set

$$s_a(K_{ab}) = [x_a, x_b], \quad s_t(K_{ab}) = 0 \quad \text{if} \quad t \neq a.$$

Next, if $\sigma = [\tau, K_{ab}]$ for k-fold simple commutator τ, following from (2), we can set

$$s_i(\sigma) = s_i(\tau)^{\partial K_{ab}} - s_i(K_{ab})^{\partial\tau}$$

for each i. Furthermore, since a commutator bracket of weight l is considered as a l-fold multilinear map from the cartesian product of l copies of $\mathcal{L}_n(1)$ to $\mathcal{L}_n(l)$, we can also set

$$s_i(\sigma) = \sum_{p=1}^{\alpha(i)} (-1)^{e_{i,p}} C_{i,p}$$

where $e_{i,p} = 0$ or 1, and $C_{i,p}$ is a commutator of degree $k+1$ among the components $x_1, \ldots, x_n$. We compute $s_i([\tau, K_{abc}])$ for $\sigma = [\tau, K_{abc}]$ similarly. These computations are perhaps easiest explained with examples, so we give two here. For distinct a, b, c and d, we have

$$\tau_2'([K_{ab}, K_{bac}]) = x_a^* \otimes ([x_a, x_b])^{\partial K_{bac}} - x_b^* \otimes ([x_a, x_c])^{\partial K_{ab}},$$
$$= x_a^* \otimes [x_a, [x_a, x_c]] - x_b^* \otimes [[x_a, x_b], x_c]$$

and

$$\begin{aligned}\tau_3'([K_{ab}, K_{bac}, K_{ad}]) &= x_a^* \otimes ([x_a, [x_a, x_c]])^{\partial K_{ad}} - x_b^* \otimes ([[x_a, x_b], x_c])^{\partial K_{ad}} \\ &\quad - x_a^* \otimes ([x_a, x_d])^{\partial [K_{ab}, K_{bac}]}, \\ &= x_a^* \otimes [[x_a, x_d], [x_a, x_c]] + x_a^* \otimes [x_a, [[x_a, x_d], x_c]] \\ &\quad - x_b^* \otimes [[[x_a, x_d], x_b], x_c] \\ &\quad - x_a^* \otimes [[x_a, [x_a, x_c]], x_d].\end{aligned}$$

3. The contractions

For $k \geq 1$ and $1 \leq l \leq k+1$, let $\varphi_l^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{l-1}} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota_n^{k+1} : \mathcal{L}_n(k+1) \to H^{\otimes(k+1)}$, we obtain a $GL(n, \mathbf{Z})$-equivariant homomorphism

$$\Phi_l^k = \varphi_l^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call the map Φ_l^k contraction.

Here we introduce one of methods of the computation of $\Phi_l^k(x_i^* \otimes C)$ for a commutator $C \in \mathcal{L}_n(k+1)$ among the components $x_1, \ldots, x_n$. In this paper, whenever we compute $\Phi_l^k(x_i^* \otimes C)$, we use the following method. First, if x_i does not appear among the components of C, then $\Phi_l^k(x_i^* \otimes C) = 0$. On the other hand, if x_i appears among the components of C m times, then we distinguish them and write such x_i's as $x_{i_1}, \ldots, x_{i_m}$ in C. Then $\Phi_l^k(x_i^* \otimes C)$ is given by rewriting $x_{i_1}, \ldots, x_{i_m}$ as x_i in

$$\sum_{j=1}^{m} \Phi_l^k(x_{i_j}^* \otimes C).$$

Thus it suffices to compute $\Phi_l^k(x_i^* \otimes C)$ for a commutator C which has only one x_i in its components. Now, C is written as $[X, Y]$ for some commutators X and Y. Rewriting the commutator C as $-[Y, X]$ if x_i appears in Y, we may always consider $C = \pm[X, Y]$ such that x_i appears among the components of X. By a recursive argument, we have $C =$

$\pm[x_i, C_1, \dots, C_t]$ where each C_j $(1 \leq j \leq t)$ is a commutator of weight d_j and $d_1 + \cdots + d_t = k$.

Lemma 3.1. *For a commutator* $[x_i, C_1, \dots, C_t] \in \mathcal{L}_n(k+1)$ *as above,*

$$\Phi_1^k(x_i^* \otimes [x_i, C_1, \dots, C_t]) = C_1 \otimes \cdots \otimes C_t.$$

Lemma 3.2. *For a commutator* $[x_i, C_1, \dots, C_t] \in \mathcal{L}_n(k+1)$ *as above,*

$$\Phi_2^k(x_i^* \otimes [x_i, C_1, \dots, C_t]) = - \sum_{\mathrm{wt}\,(C_j)=1} C_j \otimes C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_t.$$

Let $T(H) = \bigoplus_{k\geq 1} H^{\otimes k}$ and $S(H) = \bigoplus_{k\geq 1} S^k H$ be the tensor algebra and the symmetric algebra on H respectively. Then the kernel of a natural map $T(H) \to S(H)$ is a graded ideal of $T(H)$, and denoted by $I(H) = \bigoplus_{k\geq 1} I^k(H)$. For each $k \geq 2$, let $\mathcal{U}_n(k)$ be the $GL(n, \mathbf{Z})$-submodule of $H^{\otimes k}$ generated by elements type of

$$[A, B] := A \otimes B - B \otimes A$$

for $A \in H^{\otimes a}$, $B \in H^{\otimes b}$ and $a + b = k$. If we put $\mathcal{U}_n = \bigoplus_{k\geq 1} \mathcal{U}_n(k)$, then $\mathcal{U}_n$ is the kernel of the abelianizaton $T(H) \to T(H)^{\mathrm{ab}}$ as a Lie algebra. We have

$$\mathcal{L}_n(k) \subset \mathcal{U}_n(k) \subset I^k(H) \subset H^{\otimes k}.$$

3.1. The image of $\Phi_1^k \circ \tau_k'$.

Here we prove

Proposition 3.1. *For* $n \geq 3$ *and* $k \geq 2$, $\mathrm{Im}\,(\Phi_1^k \circ \tau_k') \subset \mathcal{U}_n(k)$.

It suffices to check that the image of any simple k-fold commutator σ among the components K_{ab} and K_{abc} is in $\mathcal{U}_n(k)$. We have

$$\tau_k'(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma), \quad s_i(\sigma) = \sum_{p=1}^{\alpha(i)} (-1)^{e_{i,p}} C_{i,p}.$$

Now, for convenience, for every $t \in \{1, \dots, n\}$, if each $C_{i,p}$ has x_t in its components $\beta(i,p,t)$ times, we distinguish them and write such x_t's as $x_{t_1}, \dots, x_{t_{\beta(i,p,t)}}$ in $C_{i,p}$. We denote by $\bar{C}_{i,p}$ the element $C_{i,p}$ whose componets are distinguished as above. If we denote by $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural$

the element of $H^{\otimes k}$ which is given by rewriting $x_{t_1}, \dots, x_{t_{\beta(i,p,t)}}$ as x_t in $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})$ for all t, then we have

$$\Phi_l^k \circ \tau_k'(\sigma) = \sum_{i=1}^{n} \sum_{p=1}^{\alpha(i)} (-1)^{e_{i,p}} \sum_{q=1}^{\beta(i,p,i)} \Phi_l^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural. \tag{3}$$

Then Proposition 3.1 follows from

Lemma 3.3. *Let k be an integer greater than 1. According to the notation as above, for each i, p and q, one of the following holds:*

(i) $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = 0$,

(ii) $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = X$; *a commutator of weight k in $\mathcal{L}_n(k)$*

or

(iii) *There exist some j, p' and q' such that $(j,p',q') \neq (i,p,q)$,*

$$(-1)^{e_{i,p}} \Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = \pm A \otimes B,$$
$$(-1)^{e_{j,p'}} \Phi_1^k(x_{j_{q'}}^* \otimes \bar{C}_{j,p'})_\natural = \mp B \otimes A$$

where $A \in H^{\otimes \mu}$, $B \in H^{\otimes \nu}$ and $\mu + \nu = k$.

3.2. The image of $\Phi_2^k \circ \tau_k'$.

Here we prove

Proposition 3.2. *For $n \geq 3$ and $k \geq 3$, $\mathrm{Im}\,(\Phi_2^k \circ \tau_k') \subset H \otimes_{\mathbf{Z}} \mathcal{U}_n(k-1)$.*

For each i, p and q in (3), if $\bar{C}_{i,p}$ has x_{i_q}, rewriting $\bar{C}_{i,p}$ as $\pm[x_{i_q}, D_{i,p}^1, \dots, D_{i,p}^{\gamma(i,p,q)}]$ we have,

$$\begin{aligned}&\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p}) \\ &= \sum_{\mathrm{wt}\,(D_{i,p}^t)=1} \mp(D_{i,p}^t \otimes D_{i,p}^1 \otimes \cdots \otimes D_{i,p}^{t-1} \otimes D_{i,p}^{t+1} \otimes \cdots \otimes D_{i,p}^{\gamma(i,p,q)})_\natural.\end{aligned}$$

Set $T(\bar{C}_{i,p}) := \{t \,|\, \mathrm{wt}\,(D_{i,p}^t) = 1\}$. If $\bar{C}_{i,p}$ does not have x_{i_q} or $T(\bar{C}_{i,p}) = 0$ then $\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = 0$. If $T(\bar{C}_{i,p}) = 1$ and $\gamma(i,p,q) = 2$, then

$$\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = \pm x_s \otimes Z \in H \otimes_{\mathbf{Z}} \mathcal{L}_n(k-1)$$

for some commutator Z of weight $k-1$. Then Proposition 3.2 follows from

Lemma 3.4. *Let k be an integer greater than 2. According to the notation above, for each i, p and q, one of the following holds:*

(i) *Either* $\bar{C}_{i,p}$ *does not have* x_{i_q}, *or* $T(\bar{C}_{i,p}) = 0$,
(ii) $T(\bar{C}_{i,p}) = 1$ *and* $\gamma(i,p,q) = 2$,
or
(iii) *For each* $t \in T(\bar{C}_{i,p})$, *there exist some* j, p', q' *and* t', $(j,p',q',t') \neq (i,p,q,t)$, *such that if we set*

$$X := \mp(-1)^{e_{i,p}}(D^t_{i,p} \otimes D^1_{i,p} \otimes \overset{\check{t}}{\cdots} \otimes D^{\gamma(i,p,q)}_{i,p})_\natural,$$

$$Y := \mp(-1)^{e_{j,p'}}(D^{t'}_{j,p'} \otimes D^1_{j,p'} \otimes \overset{\check{t'}}{\cdots} \otimes D^{\gamma(j,p',q')}_{j,p'})_\natural$$

then $X + Y = 0$ *or*

$$X = \pm x_s \otimes A \otimes B, \quad Y = \mp x_s \otimes B \otimes A$$

where $A \in H^{\otimes\mu}$, $B \in H^{\otimes\nu}$ *and* $\mu + \nu = k - 1$.

4. The trace maps

In this section, using the contractions defined in Section 3, we define a homomorphisms called the trace map which vanishes on the image of the Johnson homomorphism. Here we use some basic facts of the representation theory of $GL(n, \mathbf{Z})$. The reader is referred to, for example, Fulton-Harris [4] and Fulton [3].

For any $k \geq 1$ and any partition λ of k, we denote by H^λ the Schur-Weyl module of H corresponding to the partition λ of k. Let $f_\lambda : H^{\otimes k} \to H^\lambda$ be a natural homomorphism. In this paper, we mainly consider the case for $\lambda = [k]$ or $[1^k]$. The modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^k H$ and the exterior product $\Lambda^k H$ respectively. Using the natural map $\iota^k_n : \mathcal{L}_n(k) \to H^{\otimes k}$, we denote $f_{[1^k]} \circ \iota^k_n(C)$ by $\widehat{C}$ for any $C \in \mathcal{L}_n(k)$.

Lemma 4.1. *For any commutator* C *of weight* $k \geq 3$, $\widehat{C} = 0$ *in* $\Lambda^k H$

Lemma 4.2. *For* $1 \leq k \leq n-2$ *and any commutator* C *of weight* $k+1$ *among the components* $x_1, \ldots, x_n$ *except for* x_i, *there exists an element* $\sigma \in \mathcal{A}'_n(k)$ *such that*

$$\tau'_k(\sigma) = x^*_i \otimes C \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

4.1. Morita's trace (Trace map for $S^k H$).

Here we consider the map

$$\mathrm{Tr}_{[k]} = f_{[k]} \circ \Phi^k_1 : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H.$$

By definition, this map coincides with the Morita's trace Tr_k. For $n \geq 3$ and $k \geq 2$, Morita defined the trace map Tr_k using the Magnus representation of $\mathrm{Aut}\, F_n$ and showed that Tr_k vanishes on the image of τ_k. By a recent work, he showed that $\mathrm{Tr}_k^{\mathbf{Q}}$ is surjective. Hence we have

Theorem 4.1. (Morita) *For $n \geq 3$ and $k \geq 2$,*

$$S^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau_{k,\mathbf{Q}}.$$

Corollary 4.1. *For $n \geq 3$ and $k \geq 2$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Coker}\,(\tau_k)) \geq \binom{n+k-1}{k}.$$

4.2. **Trace map for $\Lambda^k H$.**

Here we consider the map

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H.$$

Theorem 4.2.

(1) *For $3 \leq k \leq n$, $\mathrm{Tr}_{[1^k]}$ is surjective,*
(2) $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]} \circ \tau_k') = 0$ *if k is odd and $3 \leq k \leq n$,*
(3) $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]} \circ \tau_k') = 2(\Lambda^k H) \subset \Lambda^k H$ *if k is even and $4 \leq k \leq n-2$.*

Corollary 4.2. *For an odd k and $3 \leq k \leq n$,*

$$\Lambda^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau_{k,\mathbf{Q}}'.$$

Corollary 4.3. *For an odd k and $3 \leq k \leq n$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Coker}\,(\tau_k')) \geq \binom{n}{k}.$$

4.3. **Trace map for $H^{[2,1^{k-2}]}$.**

Here we consider the map

$$\mathrm{Tr}_{[2,1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}^{k-1}) \circ \Phi_2^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H.$$

Let I be the $GL(n, \mathbf{Z})$-submodule of $H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$ defined by

$$I = \langle x \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge y + y \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge x \mid x, y, z_t \in H \rangle.$$

Theorem 4.3. *For an even k and $4 \leq k \leq n-1$,*

(1) $\mathrm{Im}\,(\mathrm{Tr}_{[2,1^{k-1}]}^{\mathbf{Q}}) = I_{\mathbf{Q}}$,
(2) $\mathrm{Im}\,(\mathrm{Tr}_{[2,1^{k-1}]} \circ \tau_k') = 0.$

Now we have $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \simeq H_{\mathbf{Q}}^{[2,1^{k-2}]} \oplus \Lambda^k H_{\mathbf{Q}}$ from the representation theory of $GL(n, \mathbf{Z})$. For even k, since $I_{\mathbf{Q}}$ is contained in the kernel of a natural map $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \to \Lambda^k H_{\mathbf{Q}}$ defined by $x \otimes y_1 \wedge \cdots \wedge y_{k-1} \mapsto x \wedge y_1 \wedge \cdots \wedge y_{k-1}$, we have $I_{\mathbf{Q}} \simeq H_{\mathbf{Q}}^{[2,1^{k-2}]}$.

Corollary 4.4. *For an even k and $4 \leq k \leq n-1$,*

$$H_{\mathbf{Q}}^{[2,1^{k-2}]} \subset \operatorname{Coker} \tau'_{k,\mathbf{Q}}.$$

Corollary 4.5. *For an even k and $4 \leq k \leq n-1$,*

$$\operatorname{rank}_{\mathbf{Z}}(\operatorname{Coker}(\tau'_k)) \geq (k-1)\binom{n+1}{k}.$$

5. The cokernel of the Johnson homomorphism τ_k for $k = 2$ and 3

5.1. The case $k = 2$.

In this subsection we consider the case where $n \geq 3$. From Theorem 4.1 and $\operatorname{rank}_{\mathbf{Z}}(\operatorname{Coker}(\tau_2)) = \binom{n+1}{2}$ by Pettet [15], we have a $GL(n, \mathbf{Z})$-equivariant exact sequence

$$0 \to \mathrm{gr}^2_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{2,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(3) \to S^2 H_{\mathbf{Q}} \to 0.$$

In this subsection we show that the exact sequence above holds before tensoring with $\mathbf{Q}$. Here are some examples of commutators of degree 2 among the components K_{ab} and K_{abc} and their images by the Johnson homomorphism τ_2.

$$\begin{array}{lll}
(\mathrm{C1}): & [K_{ab}, K_{ac}], & x_a^* \otimes [[x_a, x_c], x_b] - x_a^* \otimes [[x_a, x_b], x_c], \\
(\mathrm{C2}): & [K_{ab}, K_{acd}], & x_a^* \otimes [[x_c, x_d], x_b], \\
(\mathrm{C3}): & [K_{ab}, K_{abc}], & x_a^* \otimes [[x_b, x_c], x_b], \\
(\mathrm{C4}): & [K_{ab}, K_{bac}], & x_a^* \otimes [x_a, [x_a, x_c]] - x_b^* \otimes [[x_a, x_b], x_c], \\
(\mathrm{C5}): & [K_{abc}, K_{bad}], & x_a^* \otimes [[x_a, x_d], x_c] - x_b^* \otimes [[x_b, x_c], x_d], \\
(\mathrm{C6}): & [K_{abc}, K_{bac}], & x_a^* \otimes [[x_a, x_c], x_c] - x_b^* \otimes [[x_b, x_c], x_c].
\end{array}$$

Theorem 5.1. *For $n \geq 3$,*

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

is a $GL(n, \mathbf{Z})$-equivariant exact sequence.

5.2. The case $k = 3$.

Next we compute the cokernel of the Johnson homomorphism $\tau_{3,\mathbf{Q}}$ for $n \geq 3$ using the fact that $\mathrm{Coker}\, \tau_{3,\mathbf{Q}} = \mathrm{Coker}\, \tau'_{3,\mathbf{Q}}$. We use commutators of weight 3 among the components K_{ab} and K_{abc}:

(C1-1): $[[K_{ab}, K_{ac}], K_{bd}]$, (C1-2): $[[K_{ab}, K_{ac}], K_{bc}]$,
(C1-3): $[[K_{ab}, K_{ac}], K_{ba}]$,
(C3-1): $[[K_{ab}, K_{abc}], K_{cab}]$, (C3-2): $[[K_{ab}, K_{abc}], K_{ca}]$,
(C3-3): $[[K_{ab}, K_{abc}], K_{bad}]$,
(C4-1): $[[K_{ab}, K_{bac}], K_{ac}]$, (C4-2): $[[K_{ab}, K_{bac}], K_{ba}]$,
(C4-3): $[[K_{ab}, K_{bac}], K_{cd}]$, (C4-4): $[[K_{ab}, K_{bac}], K_{abc}]$,
(C4-5): $[[K_{ab}, K_{bac}], K_{cab}]$, (C4-6): $[[K_{ab}, K_{bac}], K_{ca}]$,
(C4-7): $[[K_{ab}, K_{bac}], K_{ab}]$, (C4-8): $[[K_{ab}, K_{bac}], K_{cb}]$,
(C4-9): $[[K_{ab}, K_{bac}], K_{ad}]$.

Here are a few examples of their images by τ_3:

(C1-1)': $x_a^* \otimes [[x_a, x_c], [x_b, x_d]] - x_a^* \otimes [[x_a, [x_b, x_d]], x_c]$,
(C3-1)': $x_a^* \otimes [[x_b, [x_a, x_b]], x_b] - x_c^* \otimes [[[x_b, x_c], x_b], x_b]$,
(C4-1)': $x_a^* \otimes [[x_c, [x_a, x_c]], x_a] + x_a^* \otimes [[x_c, x_a], [x_a, x_c]] + x_b^* \otimes [[x_b, [x_a, x_c]], x_c]$
$-x_a^* \otimes [[[x_c, x_a], x_a], x_c]$.

Theorem 5.2. *For $n \geq 3$,*

$$0 \to \mathrm{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}^{\mathbf{Q}}_n(4) \to S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0$$

is a $GL(n, \mathbf{Z})$-equivariant exact sequence.

Corollary 5.1. *For $n \geq 3$,*

$$\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8). \tag{4}$$

In particular, substituting $n = 3$ into (4), we have $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_3) = 43$.

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