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Generic構造の安定性についてI (Algebraic Types of Generic Graphs)

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Abstract

Generic グラフにおける代数的タイプの性質を調べた（定理 7）．系として，generic グラフの次元による forking の特徴づけが得られた（系 10）．

1 Preliminaries

Many papers [1,2,3,4,5,7] have laid out the basics of generic structures. So we do not explain all of those details here.

Let $R(*,*)$ be a binary relation of a(n undirected) graph, i.e., $\models \forall x(\neg R(x,x))$ and $\models \forall x\forall y(R(x,y)\rightarrow R(y,x))$. For a finite graph A, let $r(A)$ denote the number of the edges of A. For a finite graph AB with $A\cap B=\emptyset$, let $r(A,B)=r(AB)-r(A)-r(B)$.

Let α be a positive real number. For a finite graph A, we write a *predimension* $\delta_\alpha(A)=|A|-r(A)$. For a finite graph AB, we denote $\delta(A/B)=\delta(AB)-\delta(B)$. Let K_α denote $\{A : A$ is a finite graph, $\delta_\alpha(B)\geq 0$ for any $B\subset A\}$. For a finite subgraph A of a graph M, A is said to be *closed in* M (in symbol, $A\leq M$), if $\delta_\alpha(XA)\geq\delta_\alpha(A)$ for any finite $X\subset M-A$. The *closure* of A in M is defined by $\mathrm{cl}_M(A)=\bigcap\{B : A\subset B\leq M, |B|<\omega\}$. We define a *dimension* of A in M by $\mathrm{d}_M(A)=\delta(\mathrm{cl}_M(A))$. Let $\mathbf{K}=(K,\leq)$ be a subclass of $\mathbf{K}_\alpha=(K_\alpha,\leq)$ that is closed under substructures.

Definition 1 A countable graph G is said to be $\mathbf{K}$-*generic*, if it satisfies the following: (i) for finite $A\subset G$, $A\in\mathbf{K}$; (ii) if $A\leq B\in\mathbf{K}$ and $A\leq G$, then there exists a copy B' of B over A with $B'\leq G$.

$\mathbf{K}$ is said to have *finite closures*, if there are no chains $A_0\subset A_1\subset\cdots$ of elements of $\mathbf{K}$ with $\delta_\alpha(A_{i+1})<\delta_\alpha(A_i)$ for each $i<\omega$. If $\mathbf{K}$ has finite closures,

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then we can see that there exists a unique **K**-generic graph G, and moreover that any finite set of G has finite closures. On the other hand, it can be seen that if a **K**-generic graph M is saturated then **K** has finite closures. We summarize our situation.

Assumption $\mathbf{K} = (K, \leq)$ is derived from a dimension function δ_α for a real number α between 0 and 1 such that **K** is closed under substructures. M is a saturated **K**-generic graph.

2 Smallness of algebraic types

Definition 2 Let AB be finite structure. Then
(i) A pair (B, A) is said to be **K**-*normal*, if $A \leq AB \in \mathbf{K}$ and $A \cap B = \emptyset$.
(ii) A **K**-normal pair (B, A) is said to be *minimal*, if $\delta(C/A) > \delta(B/A)$ for any non-empty proper subset C of B.
(iii) A **K**-normal pair (B, A) is said to be *weakly small*, if whenever $A \subset C, B \subset D$ and (D, C) is **K**-normal, then $\delta(D/C) \geq \delta(B/C)$.
(iv) A **K**-normal pair (B, A) is said to be *pseudo-small*, if whenever $A \subset C$ and (B, C) is **K**-normal, then $\delta(B/C) \geq \delta(B/A)$.
(v) A **K**-normal pair (B, A) is said to be *small*, if whenever $A \subset C, B \subset D$ and (D, C) is **K**-normal, then $\delta(D/C) \geq \delta(B/A)$.

Note 3 A **K**-normal pair (B, A) is small if and only if it is weakly small and pseudo-small.

Lemma 4 Let (B, A) be a **K**-minimal pair with $A \leq AB \leq M$. If $\text{tp}(B/A)$ is algebraic, then (B, A) is weakly small.

Proof Suppose by way of contradiction that (B, A) is not weakly small. Then there are $C \supset A$ and $D \supset B$ such that (D, C) is **K**-normal and $\delta(D/C) < \delta(B/C)$.
Claim 1: There is a set $\{B_i\}_{i<\omega}$ of copies of B with the following conditions:
(i) $B_i \cong_{CB_0...B_{i-1}} B$ for each $i < \omega$;
(ii) $CB_0...B_i, CB_0...B_{i-1}D \leq CB_0...B_iD \in \mathbf{K}$ for each $i < \omega$;
(iii) $D, B_0, B_1, B_2, ...$ are pairwise disjoint.
Proof of Claim 1: We construct $\{B_i\}_{i<\omega}$ inductively. Suppose that $\{B_i\}_{i\leq n}$ has been defined. By (ii), $CB_0...B_n \leq CB_0...B_nD \in \mathbf{K}$, and so we have $CB_0...B_n \leq CB_0...B_nB \in \mathbf{K}$. By the amalgamation property, we can take a copy B_{n+1} of B over $CB_0...B_n$ such that
(*) $CB_0...B_nD, CB_0...B_nB_{n+1} \leq CB_0...B_nB_{n+1}D \in \mathbf{K}$.

Hence B_{n+1} satisfies (i) and (ii). On the other hand, $B_0, B_1, ..., B_{n+1}$ are pairwise disjoint, since $B_{n+1} \cong_{CB_0...B_n} B$ and $B \subset D$. So, to see that (iii) holds it is enough to show that $B' = B_{n+1} \cap D = \emptyset$. If $B' = B_{n+1}$ would hold, then we have $B_{n+1} \subset D$, and so $CB_{n+1} \not\leq CD$, since $\delta(D/C) < \delta(B/C) = \delta(B_{n+1}/C)$. This contradicts (*), and hence we have $B' \neq B_{n+1}$. By (*) again, we have $CB_0...B_nD \leq CB_0...B_nB_{n+1}D$, and so $AB' \leq AB_{n+1}$. Since (B, A) is a minimal pair, we have $B' = \emptyset$. (End of Proof of Claim 1)

Claim 2: $AB, AB_j \leq AB_0...B_iB (\in \mathbf{K})$ for $j \leq i < \omega$

Proof: We prove by induction on i. By (ii) of claim 1, $AB_0...B_iB \leq AB_0...B_{i+1}B$. By induction hypothesis, we have $AB, AB_j \leq AB_0...B_iB$ for $j \leq i$. Hence $AB, AB_j \leq AB_0...B_{i+1}B$ for $j \leq i$. So, it is enough to show that $AB_{i+1} \leq AB_0...B_{i+1}B$. By induction hypothesis again, we have $AB \leq AB_0...B_iB$. From (i) of claim 1, it follows that $AB_{i+1} \leq AB_0...B_{i+1}$. By (ii) of claim 1, $AB_0...B_{i+1} \leq AB_0...B_{i+1}B$. Hence we have $AB_{i+1} \leq AB_0...B_{i+1}B$. (End of Proof of Claim 2)

We show that $\mathrm{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $AB, AB_j \leq ABB_0...B_i \leq M$ for each i, j with $j \leq i < \omega$. So we have $\mathrm{tp}(B_j/A) = \mathrm{tp}(B/A)$ for each $j \leq i$. By (iii) of claim 1, B_j's are pairwise disjoint. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Lemma 5 Assume that $\mathbf{K}$ is closed under subgraphs. Let (B, A) be a $\mathbf{K}$-normal pair with $A \leq AB \leq M$. If $\mathrm{tp}(B/A)$ is algebraic, then (B, A) is pseudo-small.

Proof Suppose by way of contradiction that (B, A) is not pseudo-small. Then there is $C \supset A$ such that (B, C) is $\mathbf{K}$-normal and $\delta(B/C) < \delta(B/A)$.

Claim: There is a set $\{B_i\}_{i<\omega}$ of copies of B over A with the following conditions:

(i) $C \leq CB_j \leq CB_0B_1 \cdots AB_i \in \mathbf{K}$ for each $j \leq i < \omega$

(ii) $B_0, B_1, B_2, ...$ are pairwise distinct.

(iii) $B_i \cap C = \emptyset$ for each $i < \omega$;

(iv) $\delta(B_i/C) = \delta(B_i/A)$ for each $i < \omega$.

Proof: We construct $\{B_i\}_{i<\omega}$ inductively. Suppose that $\{B_i\}_{i\leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathbf{K}$, and by (i) we have $C \leq CB_0B_1...B_n \in \mathbf{K}$. So we can take a copy B^* of B over C such that $CB_0...B_n, CB^* \leq CB_0...B_nB^* \in \mathbf{K}$. By (iv), $\delta(B_i/C) = \delta(B_i/A)$ for each $i \leq n$. On the other hand, we have $\delta(B^*/C) < \delta(B^*/A)$. So we have $B_i \neq B^*$ for each $i \leq n$. Take a graph B_{n+1} such that $CB_0...B_nB^*$ is an expansion of $CB_0...B_{n+1}$ and $r(B_{n+1}, C) = r(B_{n+1}, A)$. Then we can see that (i)–(iv) hold. (End of Proof of Claim)

By claim, we have $AB_j \leq AB_0...B_i (\in \mathbf{K})$ for $j \leq i < \omega$. So we can assume that $AB_j \leq AB_0...B_i \leq M$ for each i, j with $j \leq i < \omega$. Thus we have $\mathrm{tp}(B_j/A) = \mathrm{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, B_j's are pairwise distinct. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Lemma 6 If (B, A) and (C, BA) are **K**-small, then so is (BC, A).

Proof Take any **K**-normal pair (E, D) such that $BC \subset E$ and $A \subset D$. Then note that $(E - B, BD)$ is **K**-normal. (Proof: Take any $X \subset E - B$. Note that (XB, D) is **K**-normal since (E, D) is so. Since (B, A) is **K**-small, we have $\delta(X/BD) = \delta(XB/D) - \delta(B/D) \geq \delta(XB/D) - \delta(B/A) \geq 0$. Hence $(E - B, BD)$ is **K**-normal.) Since (C, BA) is **K**-small, we have $\delta(E/BD) \geq \delta(C/BA)$. On the other hand, since (B, A) is **K**-small and (B, D) is **K**-normal, we have $\delta(B/D) = \delta(B/A)$. It follows that $\delta(BC/A) = \delta(C/AB) + \delta(B/A) \leq\leq \delta(E/BD) + \delta(B/D) = \delta(E/D)$. Hence (BC, A) is **K**-small.

Thoerem 7 Let **K** be a subclass of $\mathbf{K}_\alpha$ that is closed under subgraphs and M a saturated **K**-generic graph. Let (B, A) be **K**-normal with $AB \leq M$. If $\mathrm{tp}(B/A)$ is algebraic, then (B, A) is **K**-small.

Proof Let $\mathrm{tp}(B/A)$ be algebraic. Take a sequence $A = B_0 \leq B_0B_1 \leq \ldots \leq B_0B_1\ldots B_n = AB$ with (B_{i+1}, B_i) **K**-minimal for each i. Since each $\mathrm{tp}(B_{i+1}/B_0\ldots B_i)$ is algebraic, it is **K**-small by lemma 4 and 5. So, by lemma 6, $(B, A) = (B_0B_1\ldots B_n, B_0)$ is **K**-small.

3 Forking and dimension

Lemma 8 Assume that **K** is closed under subgraphs. Let $A \leq B \leq M$. Then $B \cup \mathrm{acl}(A) \leq M$.

Proof It is enough to show that $BA' \leq M$ for any finite $A' \subset \mathrm{acl}(A)$ with $A \leq A' \leq M$. Let $C = A' \cap B (\leq M)$ and $A'' = A' - C$. Since $\mathrm{tp}(A''/C)$ is algebraic, (A'', C) is **K**-small by theorem 7. If $BA' (= BA'') \not\leq M$, then there is a finite $X \subset M - BA''$ with $\delta(X/BA'') < 0$. Then we have $\delta(XA''/B) = \delta(X/BA'') + \delta(A''/B) < \delta(A''/B) \leq \delta(A'/C)$. Note that (XA''/B) is **K**-normal. So this contradict that (A'', C) is **K**-small. Hence $BA' \leq M$.

For $A \subset B, C \subset M$, we say that B and C are *free* over A, if $B \cap C = A$ and $r(B - A, C - A) = 0$.

Fact 9([2,6,7]) Let A, B, C be finite such that $B, C \leq M, A = B \cap C$ and $A = \mathrm{acl}(A)$. Then the following are equivalent.
(i) $\mathrm{d}(B/A) = \mathrm{d}(B/C)$;
(ii) B and C are free over A, and $BC \leq M$;
(iii) $\mathrm{tp}(B/C)$ does not fork over A.

Corollary 10 Let **K** be a subclass of $\mathbf{K}_\alpha$ that is closed under subgraphs and M a saturated **K**-generic graph. Let A, B, C be finite such that $B, C \leq M$ and $A = B \cap C$. Then the following are equivalent.
(i) $\mathrm{d}(B/A) = \mathrm{d}(B/C)$;
(ii) B and C are free over A, and $BC \leq M$;
(iii) $\mathrm{tp}(B/C)$ does not fork over A.

Proof *(iii)* → *(ii)*. Let $A' = \mathrm{acl}(A), B' = B \cup \mathrm{acl}(A)$ and $C' = C \cup \mathrm{acl}(A)$. By lemma 8, $B', C' \leq M$. So, by fact 9, B' and C' are free over A' and $B'C' \leq M$.
Claim 1: B and C are free over A.
Proof: It is clear that B and C are free over A'. Let $B^* = B \cap A' - A$ and $C^* = C \cap A' - A$. Since $\mathrm{tp}(B^*/A)$ is algebraic and $AB^* \leq M$, (B^*, A) is small by lemma. Then we have $\delta(B^*/A) = \delta(B^*/C)$, and this means that B^*A and C are free over A. Similarly we see that B and C^*A are free over A. Hence B and C are free over A. (End of Proof of Claim 1)
Claim 2: $BC \leq M$.
Proof: We have $B'C' = BCA' \leq M$. If not, then $BC \not\leq BCA'$. Let $D = \mathrm{cl}(BC) - BC$. Note that $D \subset A' - BC$ and $\delta(D/BC) < 0$. Then we can see $AD \leq M$. (Proof: If not, then there is non-empty $D' = \mathrm{cl}(DA) - A$. Let $E = D' \cap B$ and $D = D' \cap C$. Note that $D' = DEF \subset BC \cap A'$ and $\delta(D'/A) < \delta(D/A)$. Since $B, C \leq M$, we have $\delta(D/AE) \geq 0$ and $\delta(D/AF) \geq 0$, and so $\delta(DE/A), \delta(DF/A) \geq \delta(D/A)$. Then $\delta(D'/A) = \delta(DE/A) + \delta(DF/A) - \delta(D/A) \geq \delta(D/A)$. A contradiction.) Since $\mathrm{tp}(D/A)$ is algebraic and $AD \leq M$, (D, A) is small by theorem 7. Then we have $\delta(D/BC)\delta(D/B) + \delta(D/C) - \delta(D/A) \geq \delta(D/C) - \delta(D/A) \geq 0$. A contradiction. (End of Proof of Claim 2)
(ii) → *(i)*. Note that $\mathrm{cl}(BC) = BC$ and $\delta(B/C) = \delta(B/A)$ by (ii). So we have $\mathrm{d}(B/C) = \delta(\mathrm{cl}(BC)/C) = \delta(B/C) = \delta(B/A) = \mathrm{d}(B/C)$.
(i) → *(iii)*. By fact 9.

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