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Weakly o-minimal algebraic structures

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1 Introduction

Let M be a linearly ordered structure and A a subset M. The set A is said to be *convex* if for all $a, b \in A$ and $c \in M$ with $a < c < b$ we have $c \in A$. A linearly ordered structure M is said to be *o-minimal* if every definable subset of M is a finite union of intervals (possibly with infinite endpoints). A linearly ordered structure M is said to be *weakly o-minimal* if every definable subset of M is a finite union of convex sets. A theory T is said to be *weakly o-minimal* if every model of T is weakly o-minimal. Henceforth, a linearly ordered structure is abbreviated as an ordered structure.

It is well-known the following fact.

Fact 1 *Let M be an ordered structure. Then the following is equivalent:*

1. $\mathrm{Th}(M)$ *is weakly o-minimal;*

2. *for each formula $\varphi(x, \overline{y})$ there exists some $n \in \omega$ such that for each tuple $\overline{a}$ from M the set $\varphi(M, \overline{a})$ can be written as a union of at most n many convex sets.*

Fact 2 *Let M be a weakly o-minimal structure. If M is ω-saturated, then* $\mathrm{Th}(M)$ *is weakly o-minimal.*

Fact 3 [BP] *Let M be an expansion of an o-minimal structure by convex subsets. Then* $\mathrm{Th}(M)$ *is weakly o-minimal.*

2 Monoids and groups

In this section, we study weakly o-minimal monoids and groups. It is well-known the following fact.

Fact 4 [MMS] *Let G be a weakly o-minimal group. Suppose that H is a definable subgroup of G. Then, the following holds:*

1. *G is abelian and divisible;*
2. *H is convex.*

Let G be a weakly o-minimal group. Suppose that H is a definable subgroup of G. Then,by Fact 4, H is divisible.

We call an ordered group $(G, 0, +, <, \ldots)$ *Archimedian* if for all elements a, b with $b > 0$ there exists some $n \in \omega$ such that $a < nb$.

Lemma 5 *Let $\mathcal{G} = (G, 0, +, <, \ldots)$ be a weakly o-minimal Archimedian group. Suppose that H is a definable subgroup of $\mathcal{G}$. Then H is either $\{0\}$ or $\mathcal{G}$.*

Proof. Let $a \in G$. Without loss of generality, we may assume $a > 0$. Let $H \neq \{0\}$. Then, there exists some $b \in H$ such that $b > 0$. Since the group $\mathcal{G}$ is Archimedian, there exists some $n \in \omega$ such that $a < nb$. Hence, by Fact 4, we have $a \in H$. □

From now on, we study monoids.

Proposition 6 *Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly o-minimal monoid. Then $\mathcal{N}$ is commutative.*

Proof. For all $a \in N$, let $C_N(a) := \{x \in N \mid x + a = a + x\}$.
Claim $C_N(a)$ is convex.

Clearly, $0 \in C_N(a)$ and, if $x, y \in C_N(a)$ then $x + y \in C_N(a)$. By weak o-minimality, $C_N(a)$ is the union of finitely many maximal convex subsets. Let X be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in X$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in C_N(a)$. By $x < y + x < 2x$ and $2x \in X$, we have $y + x \in X$. Hence $(y + x) + a = a + (y + x)$. By $x \in C_N(a)$, we have $(y + a) + x = (a + y) + x$. Hence, we have $y + a = a + y$. Thus, $y \in C_N(a)$, as desired.

Let $b, c \in N$ with $b < c$. Then the following is equivalent:

- b and c are commutative;
- b and $b+c$ are commutative;
- $b+c$ and c are commutative.

Now $b, b+c \leq 0$ or $b+c, c \geq 0$. Hence we may assume $0 < b < c$. Then, as $C_N(c)$ is convex, we have $b \in C_N(c)$. Therefore $\mathcal{N}$ is commutative. □

Let $\mathcal{N} = (N, 0, +, <, \dots)$ be an ordered monoid. Suppose that $\mathrm{I}_N := \{x \in N \mid \mathcal{N} \models \exists y(x+y=0)\}$. Clearly, I_N contains 0. We call an ordered monoid $(N, 0, +, <, \dots)$ *Archimedian* if for all elements a, b with $b > 0$ there exists some $n \in \omega$ such that $a < nb$, and for all elements a, b with $b < 0$ there exists some $n \in \omega$ such that $nb < a$.

Example 7 *Let* $\mathcal{M} = (\{0\} \cup \mathbb{Q}^{\geq 1}, 0, +, <, P)$, *where* $\mathbb{Q}^{\geq 1} = \{a \in \mathbb{Q} \mid a \geq 1\}$ *and the unary predicate symbol* P *is interpreted by the convex set* $P^{\mathcal{M}} = (\sqrt{2}, 3) \cap \mathbb{Q}$. *Then,* $\mathcal{M}$ *is a weakly o-minimal Archimedian monoid and not divisible. Moreover* $I_{\mathcal{M}} = \{0\}$.

Hence, in generally a weakly o-minimal Archimedian monoid is not a group. However the following holds.

Proposition 8 *Let* $\mathcal{N} = (N, 0, +, <, \dots)$ *be a weakly o-minimal Archimedian monoid. Suppose that* $\mathrm{I}_N \neq \{0\}$. *Then* $\mathcal{N}$ *is a group.*

Proof. Clearly $0 \in \mathrm{I}_N$. Let $x, y \in \mathrm{I}_N$. Then, there exist x_1, y_1 such that $x + x_1 = 0$ and $y + y_1 = 0$. Then $(x+y) + (y_1 + x_1) = 0$. Thus, $x + y \in \mathrm{I}_N$.
Claim I_N is convex.

By weak o-minimality, I_N is the union of finitely many maximal convex subsets. Let C be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in \mathrm{I}_N$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. Hence, there exists some $z \in N$ such that $(y+x)+z = 0$. So $y + (x+z) = 0$. Thus, $y \in \mathrm{I}_N$, as desired.

Let $g \in N$. By $\mathrm{I}_N \neq \{0\}$, there exists some $a \in \mathrm{I}_N$ such that $a \neq 0$. Without loss of generality, we may assume that $g > 0$ and $a > 0$. As N is Archimedian, there exists some $n \in \omega$ such that $0 < g < na$. Since I_N is convex, we have $g \in I_N$. Therefore $I_N = N$. □

Let N be an ordered monoid and A a subset N. The ordered monoid N is said to be *rich*, if for all $a, b \in N$ if $0 \leq a \leq b$ or $b \leq a \leq 0$, then there exists some $c \in N$ such that $b = a + c$. The set A admits *right elimination*, if for all $a \in A$ and all $b \in N$ if $b + a \in A$, then $b \in A$.

Example 9 *Let $\mathcal{M} = (\mathbb{Q}^{\geq 0}, 0, +, <, P)$, where $\mathbb{Q}^{\geq 0} = \{a \in \mathbb{Q} \mid a \geq 0\}$ and the unary predicate symbol P is interpreted by the convex set $P^{\mathcal{M}} = (\sqrt{2}, 3) \cap \mathbb{Q}$. Then, $\mathcal{M}$ is a weakly o-minimal rich monoid and divisible.*

Proposition 10 *Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly o-minimal monoid. Then the following is equivalent:*

1. *$\mathcal{N}$ is divisible;*
2. *for all $n \in \omega$, nN admits right elimination;*
3. *for all $n \in \omega$, nN is convex.*

Proof. ($1 \Rightarrow 2$) It is clear.
($2 \Rightarrow 3$) Let $n \in \omega$. Let $x, y \in nN$. Then there exist $x_1, y_1 \in N$ such that $x = nx_1$ and $y = ny_1$. By Proposition 6, we have $x + y = nx_1 + ny_1 = n(x_1 + y_1)$ Hence, $x + y \in nN$. Now, by weak o-minimality, nN is the union of finitely many maximal convex subsets. Let C be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We may show that $y \in nN$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. As nN admits right elimination, we have $y \in nN$, as desired.
($3 \Rightarrow 1$) Let n be a nonzero natural nunber. For all positive $a \in N$, we have $0 < a < na$. As nN is convex, we have $a \in nN$. Hence $\mathcal{N}$ is divisible. □

Proposition 11 *Let $\mathcal{N} = (N, 0, +, <, \ldots)$ be a weakly o-minimal monoid. If $\mathcal{N}$ is rich, then $\mathcal{N}$ is divisible.*

Proof. Let n be a nonzero natural nunber. Now, by weak o-minimality, nN is the union of finitely many maximal convex subsets. Let C be the greatest of these convex components with respect to the ordering induced by $<$. Let $x \in C$ with $x > 0$. Suppose that $y \in N$ with $0 < y < x$. We show that $y \in nN$. By $x < y + x < 2x$ and $2x \in C$, we have $y + x \in C$. So there exist $z_1, z_2 \in N$ with $0 < z_1 < z_2$ such that $x = nz_1$ and $y + x = nz_2$.

As $\mathcal{N}$ is rich, there exists some $a \in N$ such that $a + z_1 = z_2$. Hence, we have $y + nz_1 = na + nz_1$. Therefore we have $y = na \in nN$. It follows that $nN = N$. □

Proposition 12 [T] *Let N be an ordered monoid. Suppose that* $\mathrm{Th}(N)$ *is weakly o-minimal. Then there exists an extending ordered group G of N such that* $\mathrm{Th}(G)$ *is weakly o-minimal.*

Proof. Let N_1 be an ω-saturated elementary extension of N. Define the following relation on $N_1 \times N_1$:

$$(a, b) \sim (a', b') \iff a + b' = a' + b.$$

Then $\sim$ is an equivalence relation on $N_1 \times N_1$. For each $(a, b) \in N_1 \times N_1$, let $[(a, b)]$ denote the $\sim$-class of (a, b). Let $G := N_1 \times N_1 / \sim$. Then G can be naturally expanded to an ω-saturated ordered group. We may treat N_1 as a substructure of G by identifying $a \in N_1$ and $[(a, 0)] \in G$. We may show that G is weakly o-minimal. By way of a contradiction, assume that G is not weakly o-minimal. Then there exists a definable subset $A \subseteq G$ and a monotone sequence $\{a_i \in G \mid i \in \omega\}$ such that for all $i \in \omega$, $a_i \in A$ if and only if i is even. As G is an eq-object of N_1, there exists a formula $\varphi(x, y)$ (parameters from N_1) such that $[(b, c)] \in A$ if and only if $N_1 \models \varphi(b, c)$. For all $i \in \omega$, let $a_i := [(b_i, c_i)]$. Then we have

$$N_1 \models \varphi(b_i, c_i) \iff i \text{ is even}.$$

For all $n \in \omega$, let $d_i := \Sigma_{j=0, j \neq i}^{2n} c_i$ and $e := \Sigma_{j=0}^{2n} c_i$. Then we have

$$N_1 \models \varphi(b_i + d_i, e) \iff i \text{ is even}.$$

Hence, the set $\varphi(N_1, e)$ can not be written as the union of n convex sets, contradicting that $\mathrm{Th}(N)$ is weakly o-minimal. □

3 Rings and fields

In this section, we study weakly o-minimal rings and fields.

A commutative ordered domain R is said to be *real closed* if R has intermediate value property, that is, for any polynomial $p(x)$ with coefficients in

R and any $a, b \in R$ such that $a < b$ and $p(a) \cdot p(b) < 0$, there exists some $c \in R$ so that $a < c < b$ and $p(c) = 0$.

It is well-known the following fact.

Fact 13 [MMS]

1. *If a commutative ordered ring R is weakly o-minimal, then R is a real closed ring;*

2. *If an ordered field F is weakly o-minimal, then F is a real closed field.*

In [PS1], it is shown that an o-minimal ring is a real closed field. However, in generally a weakly o-minimal ordered ring is not a field. We shall show that if a weakly o-minimal ordered ring R which may not be associative is Archimedian, then R is a real closed field.

Lemma 14 *If $\mathcal{R} = (R, 0, 1, +, \cdot, <, \dots)$ is a weakly o-minimal ring, then $\mathcal{R}$ is commutative.*

Proof. For all $a \in R$, let $C_R(a) := \{x \in R \mid xa = ax\}$. Then, $C_R(a)$ is a definable additive subgroup. Hence, by Fact 4, $C_R(a)$ is convex. Let $g, h \in R$. Without loss of generality, we may assume that $0 < g < h$. As $C_R(h)$ is convex, we have $g \in C_R(h)$. It follows that $\mathcal{R}$ is commutative. □

We call an ordered ring $(R, 0, 1, +, \cdot, <, \dots)$ *standard* if for all nonzero $a \in R$ there exists $b \in R$ such that $1 < ab$. Clearly, an Archimedian ordered ring is standard.

Proposition 15 *Let $\mathcal{R} = (R, 0, 1, +, \cdot, <, \dots)$ be a weakly o-minimal ring. Then, the following is equivalent:*

1. *$\mathcal{R}$ is standard;*

2. *$\mathcal{R}$ is a field.*

Proof. $(2 \Rightarrow 1)$ Let $a \in R$ with $a \neq 0$. Then, as $\mathcal{R}$ is field, there exists a^{-1}. Hence, $1 < a \cdot 2a^{-1} = 2$, as desired.
$(1 \Rightarrow 2)$ Let $a \in R$. Then, as $\mathcal{R}$ is standard, there exists some $b \in R$ such that $1 < ab$. Now aR is a definable additive subgroup. Hence, as aR is convex, we have $1 \in aR$. It follows that $\mathcal{R}$ is a field. □

Corollary 16 *Let $\mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots)$ be a weakly o-minimal Archimedean ring, where $\mathcal{R}$ may not be associative. Then, $\mathcal{R}$ is a real closed field.*

Proof. By Fact 13, Lemma 14 and Proposition 15, we may show that $\mathcal{R}$ is associative. Let $a \in R$ with $a \neq 0$. Suppose that $D_R(a) := \{x \in R \mid (xa)a = x(aa)\}$. Then, as $\mathcal{R}$ is commutative, $D_R(a)$ contains a and is a definable additive subgroup. Hence, by Lemma 5, $D_R(a) = R$. Also, suppose that $E_R(a) := \{x \in R \mid (za)x = z(ax) \text{ for each } z\}$. Then, by $D_R(a) = R$, $E_R(a)$ contains a and is a definable additive subgroup. Thus, by Lemma 5, $E_R(a) = R$. It follows that $\mathcal{R}$ is associative. □

Proposition 17 *Let R be an ordered ring. Suppose that $\mathrm{Th}(R)$ is weakly o-minimal. Then there exists an extending ordered field F of R such that $\mathrm{Th}(F)$ is weakly o-minimal.*

Proof. Let R_1 be an ω-saturated elementary extension of R. Let $R_1^{>0} := \{a \in R_1 \mid a > 0\}$. Define the following relation on $R_1 \times R_1^{>0}$:

$$(a, b) \sim (a', b') \iff ab' = a'b.$$

Then $\sim$ is an equivalence relation on $R_1 \times R_1^{>0}$. For each $(a, b) \in R_1 \times R_1^{>0}$, let $[(a, b)]$ denote the $\sim$-class of (a, b). Let $F := R_1 \times R_1^{>0}/\sim$. Then F can be naturally expanded to an ω-saturated ordered field. We may treat R_1 as a substructure of F by identifying $a \in R_1$ and $[(a, 1)] \in F$. We may show that F is weakly o-minimal. By way of a contradiction, assume that F is not weakly o-minimal. Then there exists a definable subset $A \subseteq F$ and a monotone sequence $\{a_i \in F \mid i \in \omega\}$ such that for all $i \in \omega$, $a_i \in A$ if and only if i is even. As F is an eq-object of R_1, there exists a formula $\varphi(x, y)$ (parameters from R_1) such that $[(b, c)] \in A$ if and only if $R_1 \models \varphi(b, c)$. For all $i \in \omega$, let $a_i := [(b_i, c_i)]$. Then we have

$$R_1 \models \varphi(b_i, c_i) \iff i \text{ is even}.$$

For all $n \in \omega$, let $d_i := \Pi_{j=0, j \neq i}^{2n} c_i$ and $e := \Pi_{j=0}^{2n} c_i$. Then we have

$$R_1 \models \varphi(b_i d_i, e) \iff i \text{ is even}.$$

Hence, the set $\varphi(R_1, e)$ can not be written as the union of n convex sets, contradicting that $\mathrm{Th}(R)$ is weakly o-minimal. □

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