

A MEAN VALUE THEOREM FOR THE SQUARE OF CLASS NUMBER TIMES REGULATOR OF QUADRATIC EXTENSIONS

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ABSTRACT. Let  $k$  be a number field, and  $\Delta_k, h_k$  and  $R_k$  the absolute discriminant, the class number and the regulator, respectively. In this article we will give a survey of [9] in which we found the asymptotic behavior of the mean values of  $h_F^2 R_F^2$  with respect to  $|\Delta_F|$  for certain families of quadratic extensions  $F$  of a fixed number field  $k$ . The global zeta function of prehomogeneous vector space for the space of pairs of quaternions are used to prove the theorem. Also we give some examples of interpretations of set of rational orbits in some inner form representations.

1. INTRODUCTION

We start with our main result. We fix an algebraic number field  $k$ . Let  $\mathfrak{M}, \mathfrak{M}_\infty, \mathfrak{M}_f, \mathfrak{M}_\mathbb{R}$  and  $\mathfrak{M}_\mathbb{C}$  denote respectively the set of all places of  $k$ , all infinite places, all finite places, all real places and all complex places. For  $v \in \mathfrak{M}$  let  $k_v$  denotes the completion of  $k$  at  $v$  and if  $v \in \mathfrak{M}_f$  then let  $q_v$  denote the order of the residue field of  $k_v$ . We let  $r_1, r_2$ , and  $e_k$  be respectively the number of real places, the number of complex places, and the number of roots of unity contained in  $k$ . We denote by  $\zeta_k(s)$  the Dedekind zeta function of  $k$ .

To state our result, we classify quadratic extensions of  $k$  via the splitting type at places of  $\mathfrak{M}_\infty$ . Note that if  $[F : k] = 2$ , then  $F \otimes k_v$  is either  $\mathbb{R} \times \mathbb{R}$  or  $\mathbb{C}$  for  $v \in \mathfrak{M}_\mathbb{R}$  and is  $\mathbb{C} \times \mathbb{C}$  for  $v \in \mathfrak{M}_\mathbb{C}$ . We fix a  $\mathfrak{M}_\infty$ -tuple  $L_\infty = (L_v)_{v \in \mathfrak{M}_\infty}$  where  $L_v \in \{\mathbb{R} \times \mathbb{R}, \mathbb{C}\}$  for  $v \in \mathfrak{M}_\mathbb{R}$  and  $L_v = \mathbb{C} \times \mathbb{C}$  for  $v \in \mathfrak{M}_\mathbb{C}$ . We define

$$\Omega(L_\infty) = \{F \mid [F : k] = 2, F \otimes k_v \cong L_v \text{ for all } v \in \mathfrak{M}_\infty\}.$$

Let  $r_1(L_\infty)$  and  $r_2(L_\infty)$  be the number of real places and complex places of  $F \in \Omega(L_\infty)$ , respectively. (This does not depend on the choice of  $F$ .) For  $v \in \mathfrak{M}_f$  we put

$$E_v = 1 - 3q_v^{-3} + 2q_v^{-4} + q_v^{-5} - q_v^{-6}, \quad E'_v = 2^{-1}(1 - q_v^{-1})^3(1 + 2q_v^{-1} + 4q_v^{-2} + 2q_v^{-3}).$$

The following theorem is a special case of [9, Theorem 10.12].

**Theorem 1.1.** *Let  $n \geq 2$ . We fix an  $L_\infty$  and  $v_1, v_2, \dots, v_n \in \mathfrak{M}_f$ . Then the limit*

$$\lim_{X \rightarrow \infty} \frac{1}{X^2} \sum_{\substack{F \in \Omega(L_\infty) \\ F \text{ not split at } v_1, \dots, v_n \\ |\Delta_{F/k}| \leq X}} h_F^2 R_F^2$$

exists, and the value is equal to

$$\frac{(\text{Res}_{s=1} \zeta_k(s))^3 \Delta_k^2 e_k^2 \zeta_k(2)^2}{2^{r_1+r_2+1} 2^{2r_1(L_\infty)} (2\pi)^{2r_2(L_\infty)}} \cdot \prod_{1 \leq i \leq n} E'_{v_i} \prod_{\substack{v \in \mathfrak{M}_f \\ v \neq v_1, \dots, v_n}} E_v.$$

Date: April 5, 2005.

Theorems of this kind are called density theorems. They assert that the arithmetic objects in the question are distributed regularly in some sense. Today many density theorems are known. The asymptotic behavior of the number of  $\mathrm{SL}(2, \mathbb{Z})$ -equivalence classes of primitive integral binary quadratic forms conjectured by Gauss and proved by Lipschitz and Siegel may be one of the most famous examples among them.

One relatively new method to obtain density theorems is the use of the theory of zeta functions associated with prehomogeneous vector spaces. This was first carried out by Shintani [8] to improve the estimate of the Gauss conjecture mentioned above. There are some advantages to using this theory. For example, at the moment this approach is the only possible way that allows the ground field to be a general number field rather than just  $\mathbb{Q}$ , as is done in [2], [1], or [4].

Before we indicate our approach, we recall a more famous topic which is on the average density of class number times regulator of quadratic extensions. The following theorem is proved by Goldfeld-Hoffstein [3] in the case  $k = \mathbb{Q}$ , and extended to a general number field by Datskovsky [1] using the theory of global zeta functions of prehomogeneous vector spaces. (and he also corrected an error in the constant of Goldfeld-Hoffstein's formula.)

**Theorem 1.2** (Datskovsky). *Let  $L_\infty = (L_v)_{v \in \mathfrak{M}_\infty}$  be a  $\mathfrak{M}_\infty$ -tuple. Then we have*

$$\lim_{X \rightarrow \infty} \frac{1}{X^{3/2}} \sum_{\substack{F \in \mathcal{Q}(L_\infty) \\ |\Delta_{F/k}| \leq X}} h_F R_F = \frac{(\mathrm{Res}_{s=1} \zeta_k(s))^2 \Delta_k e_k \zeta_k(2)}{3 \cdot 2^{r_1+r_2-1} 2^{r_1(L_\infty)} (2\pi)^{r_2(L_\infty)}} \prod_{v \in \mathfrak{M}_f} (1 - q_v^{-2} - q_v^{-3} + q_v^{-4}).$$

We could prove this theorem by using the theory of the space of binary quadratic forms. Let us consider  $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$ , and its linear representation on

$$V = \mathrm{Sym}^2 k^2 = \{x = x(u, v) = x_0 u^2 + x_1 uv + x_2 v^2 \mid x_0, x_1, x_2 \in k\}.$$

Explicitly, the  $\mathrm{GL}(2)$ -part acts on  $V$  by the linear change of variables, and the  $\mathrm{GL}(1)$ -part by the usual scalar multiplication. The relation between  $(G, V)$  and Theorem 1.2 is clarified by the following proposition.

**Proposition 1.3.** (1) *Let  $P(x) = x_1^2 - 4x_0x_2$  which is a polynomial in  $V$ , and  $\chi(g) = (\det g)^2$  which is a character of  $G$ . Then we have  $P(gx) = \chi(g)P(x)$  for all  $g \in G, x \in V$ .*

(2) *Let  $V' = \{x \in V \mid P(x) \neq 0\}$ . For  $x \in V'_k$ , we let  $k(x)$  be the splitting field of  $x(u, v)$  if it is irreducible, and  $k(x) = k \times k$  if  $x(u, v)$  is reducible. Then the isomorphism class of  $k(x)$  depends only  $G_k$  orbit of  $x$ , and this gives a bijection between  $G_k \backslash V'_k$  and the set of isomorphism classes of étale quadratic extensions of  $k$ .*

(3) *For  $x \in V'_k$ ,  $G_x^\circ \cong k(x)^\times$  as an algebraic group over  $k$ .*

The statement (2) explains why  $(G, V)$  concerns to quadratic extensions of  $k$ . On the other hand, if we put  $T = \ker(G \rightarrow \mathrm{GL}(V))$ , we immediately see  $T \cong \mathrm{GL}(1)$ . Hence, from (3) we could see that the unnormalized Tamagawa number of  $G_x^\circ/T$  is more or less equal to  $h_{k(x)} R_{k(x)}$ . We call propositions of this form the *rational orbit decomposition* for  $(G, V)$ .

For the comparison of Theorem 1.1 and Theorem 1.2, if there exist a linear representation of an algebraic group satisfying the corresponding proposition, then with an appropriate theory, we could expect theorems of the form Theorem 1.1. In fact, the representation is already known by the work of Wright and Yukie [11], namely the space of pairs of  $2 \times 2$  matrices.

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2. THE SPACE OF PAIRS OF  $2 \times 2$  MATRICES (ORIGINAL APPROACH)

For a while, let  $k$  be an arbitrary field. Let

$$(2.1) \quad G = \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2), \quad V = k^2 \otimes k^2 \otimes k^2.$$

There is an identification  $V \cong \mathrm{M}(2, 2) \oplus \mathrm{M}(2, 2)$  and hence we call this space as the space of pairs of  $2 \times 2$  matrices. We put  $T = \ker(G \rightarrow \mathrm{GL}(V))$ . We immediately see  $T \cong \mathrm{GL}(1) \times \mathrm{GL}(1)$ . The following proposition is proved in [6] and [11].

**Proposition 2.2.** (1) *There exists a non-zero polynomial  $P$  of  $V$  and a rational character  $\chi$  on  $G$  such that  $P(gx) = \chi(g)P(x)$ .*

(2) *Let  $V' = \{x \in V \mid P(x) \neq 0\}$ . Then there exists the canonical bijection between  $G_k \backslash V'_k$  and the set of isomorphism classes of étale quadratic extensions of  $k$ . For  $x \in V'_k$ , we denote by  $k(x)$  the corresponding algebra.*

(3) *For  $x \in V'_k$ ,  $G_x^\circ \cong k(x)^\times \times k(x)^\times$  as an algebraic group over  $k$ .*

From the similar observation as in the case of binary quadratic forms, we can expect that an appropriate theory for this space leads the density of  $h_F^2 R_F^2$  of quadratic extensions  $F$  of  $k$ . This observation, due to [11], is the starting point of our work.

Next we recall the definition of the zeta function for this prehomogeneous vector space. Let  $k$  be a number field and  $\mathbb{A}$  the adèle ring of  $k$ . We let

$$L = \{x \in V'_k \mid k(x) \not\cong k \times k\},$$

which is a  $G_k$ -invariant subset of  $V'_k$ . Note that  $G_k \backslash L$  corresponds bijectively to the set of quadratic extensions of  $k$ .

**Definition 2.3.** For a Schwartz-Bruhat function  $\Phi$  on  $V_{\mathbb{A}}$  and a complex variable  $s$ , we define the global zeta function as

$$Z(\Phi, s) = \int_{G_{\mathbb{A}}/T_{\mathbb{A}}G_k} |\chi(g)|_{\mathbb{A}}^s \sum_{x \in L} \Phi(gx) dg,$$

where  $dg$  is an invariant measure on  $G_{\mathbb{A}}/T_{\mathbb{A}}$ .

The integral converges absolutely and locally uniformly if  $\Re(s)$  is sufficiently large.

Roughly speaking, from the Proposition 2.2 we see that the global zeta function has the following expansion

$$(2.4) \quad \sum_{L_\infty} \left( \Gamma_{L_\infty}(\Phi_\infty, s) \times \sum_{F \in Q(L_\infty)} \frac{h_F^2 R_F^2}{|\Delta_{F/k}|^s} \right)$$

where  $L_\infty$  runs through all the splitting type at  $\mathfrak{M}_\infty$ , and  $\Gamma_{L_\infty}(\Phi_\infty, s)$  are the gamma factors. Hence from the analytic properties of  $Z(\Phi, s)$ , by Tauberian theorem, we could get the mean value of  $h_F^2 R_F^2$ . Actually our zeta function is slightly different from the above form. We will discuss on this difference in Section 4.

Let us consider the principal parts of the global zeta function. The standard tool to study the global zeta function is the Fourier analysis. We choose a suitable inner product  $[\ , \ ]: V \times V \rightarrow k$ . Let  $g^t$  denote the contragredient representation and  $\widehat{\Phi}$  the Fourier transform with respect to  $[\ , \ ]$ . Then by the Poisson summation formula, we have

$$Z(\Phi, s) = Z_+(\Phi, s) + Z_+(\widehat{\Phi}, 2-s) + I(\Phi, s)$$

where  $Z_+(\Phi, s), Z_+(\widehat{\Phi}, 2 - s)$  are the entire functions and  $I(\Phi, s)$  is given by

$$(2.5) \quad I(\Phi, s) = \int_{\substack{G_{\mathbb{A}}/T_{\mathbb{A}}G_k \\ |\chi(g)|_{\mathbb{A}} \leq 1}} \left( |\chi(g)|_{\mathbb{A}}^{s-2} \sum_{x \in V'_k \setminus L} \widehat{\Phi}(g^t x) - |\chi(g)|_{\mathbb{A}}^s \sum_{x \in V'_k \setminus L} \Phi(gx) \right) dg.$$

To compute  $I(\Phi)$ , it seems natural to divide the index set  $V_k \setminus L$  of the summation into its  $G_k$ -orbits and perform integration separately. However, we cannot put this into practice because the corresponding integrals diverge. This is the main difficulty when one calculates the global zeta functions of prehomogeneous vector spaces. To surmount this problem Shintani [7] introduced a smoothed Eisenstein series of  $GL(2)$ . He used this series to determine the principal parts of the global zeta functions for the space of binary cubic forms. Later A. Yukie [12] generalized the theory of Eisenstein series to the groups of products of  $GL(n)$ 's, and determined the principal parts of the global zeta functions for the space of quadratic forms  $(GL(1) \times GL(n), \text{Sym}^2 k^n)$  and the space of pairs of ternary quadratic forms  $(GL(3) \times GL(2), \text{Sym}^2 k^3 \otimes k^2)$ . The latter space is known as a significantly interesting case such that the rational orbit space parameterize étale quartic extensions (see [11]) and the determination of the principal parts is worth remarkable.

The author's original approach was to apply their method to our case (2.5) but could not succeed in computing. After a while the author modified the approach as follows.

### 3. THE SPACE OF A PAIR OF QUATERNION ALGEBRAS (MODIFIED APPROACH)

Let  $\mathcal{B}$  be a quaternion algebra over  $k$ . Let us consider the representation

$$(3.1) \quad G = \mathcal{B}^\times \times (\mathcal{B}^{\text{op}})^\times \times GL(2), \quad V = \mathcal{B} \otimes k^2 = \mathcal{B} \oplus \mathcal{B}.$$

We regard (3.1) as a representation of the algebraic group  $G$  over  $k$ . This is an inner form representation of (2.1), and if  $\mathcal{B} \cong M(2, 2)$  over  $k$  then they are equivalent.

For this representation, instead of Proposition 2.2 the following holds.

- Proposition 3.2.** (1) *There exists a non-zero polynomial  $P$  of  $V$  and a rational character  $\chi$  on  $G$  such that  $P(gx) = \chi(g)P(x)$ .*  
 (2) *Let  $V' = \{x \in V \mid P(x) \neq 0\}$ . Then there exists the canonical bijection between  $G_k \backslash V'_k$  and the set of isomorphism classes of étale quadratic extensions of  $k$  those are embeddable into  $\mathcal{B}$ . For  $x \in V'_k$ , we denote by  $k(x)$  the corresponding algebra.*  
 (3) *For  $x \in V'_k$ ,  $G_x^\circ \cong k(x)^\times \times k(x)^\times$  as an algebraic group over  $k$ .*

If  $k$  is a number field, then whether a quadratic extension  $F$  of  $k$  is embeddable into  $\mathcal{B}$  or not can be determined by finitely many local conditions of  $F$ . This reflects to the condition " $n \geq 2$ " in Theorem 1.1.

We define the global zeta function  $Z(\Phi, s)$  and the "principal parts"  $I(\Phi, s)$  similarly. One advantage of non-split cases is that the global theory becomes much easier. In general, the analysis of the global zeta function becomes much more complicated as the  $k$ -rank the group growth. If  $\mathcal{B}$  is non-split, then the  $k$ -rank of  $G$  in (3.1) is 1, and in this case we could succeed in computing the principal parts. The following theorem is proved in [10].

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**Theorem 3.3.** *Let  $\mathcal{B}$  be a non-split quaternion algebra. Then*

$$I(\Phi, s) = \tau(G/T) \left( \frac{\widehat{\Phi}(0)}{s-2} - \frac{\Phi(0)}{s} \right) + \frac{Z_{\mathcal{B}}(R\widehat{\Phi}, 1/2)}{s-3/2} - \frac{Z_{\mathcal{B}}(R\Phi, 1/2)}{s-1/2},$$

where  $\tau(G/T)$  is the Tamagawa number of  $G/T$ ,  $R\Phi$  the suitable restriction of  $\Phi$  to  $\mathcal{B}_A$ , and  $Z_{\mathcal{B}}$  the zeta function of simple algebra associated to  $\mathcal{B}$ .

## 4. FILTERING PROCESS

If  $Z(\Phi, s)$  had the expansion of the form (2.4), then the Tauberian theorem would allow us to extract the mean value of the coefficients from Theorem 3.3. However, our global zeta function contains an additional factor in each term. More precisely speaking, the expansion of  $Z(\Phi, s)$  is of the form

$$(4.1) \quad Z(\Phi, s) = \sum_{L_{\infty}} \left( \Gamma_{L_{\infty}}(\Phi_{\infty}, s) \times \sum_{F \in Q(L_{\infty})} \frac{h_F^2 R_F^2}{|\Delta_{F/k}|^s} L_F(\Phi, s) \right)$$

where  $L_F(\Phi, s)$  is a Dirichlet series. To surmount this difficulty, Datskovsky-Wright [2] and Datskovsky [1] formulated the method so called the *filtering process*. Roughly speaking, we approximate (2.4) by (4.1) by choosing a sequence of Schwartz-Bruhat functions  $\{\Phi_n\}_{n \geq 1}$  such that  $L_F(\Phi_n, s)$  goes uniformly to 1 as  $n \rightarrow \infty$ . We use the Tauberian theorem at each step and take the limit of the formulae to prove the desired density theorem. This is the reason of the Euler product in the formula of Theorem 1.1. In [9] we followed Datskovsky's approach [1] to obtain the density theorem.

## 5. THE CORRELATION COEFFICIENTS

As an interesting application of Theorem 1.1, combined with the result of Kable-Yukie [4], we also obtain the asymptotic behavior of the correlation coefficients for class number times regulator of certain families of quadratic extensions. For simplicity we state our result in the case  $k = \mathbb{Q}$ . Note that  $R_F = 1$  for imaginary quadratic fields. The following is a special case of [9, Theorem 11.2].

**Theorem 5.1.** *We fix a prime number  $l$  satisfying  $l \equiv 1(4)$ . For any quadratic field  $F = \mathbb{Q}(\sqrt{m})$  other than  $\mathbb{Q}(\sqrt{l})$ , we put  $F^* = \mathbb{Q}(\sqrt{ml})$ . For a positive number  $X$ , we denote by  $\mathcal{A}_l(X)$  the set of quadratic fields  $F$  such that  $-X < D_F < 0$  and  $F \otimes \mathbb{Q}_l$  is the quadratic unramified extension of  $\mathbb{Q}_l$ . Then we have*

$$\lim_{X \rightarrow \infty} \frac{\sum_{F \in \mathcal{A}_l(X)} h_F h_{F^*}}{\left( \sum_{F \in \mathcal{A}_l(X)} h_F^2 \right)^{1/2} \left( \sum_{F \in \mathcal{A}_l(X)} h_{F^*}^2 \right)^{1/2}} = \prod_{\left(\frac{\cdot}{l}\right) = -1} \left( 1 - \frac{2p^{-2}}{1 + p^{-1} + p^{-2} - 2p^{-3} + p^{-5}} \right),$$

where  $\left(\frac{\cdot}{l}\right)$  is the Legendre symbol and  $p$  runs through all the primes satisfying  $\left(\frac{\cdot}{l}\right) = -1$ .

It is an interesting phenomenon that the index set of the product of the density consists of primes  $p$  such that  $\left(\frac{\cdot}{l}\right) = -1$ . For example, we can observe that if we choose  $l$  such that  $\left(\frac{\cdot}{l}\right) = 1$  for all small primes  $p$  then  $h_F$  and  $h_{F^*}$  have strong relation, and if we choose  $l$  such that  $\left(\frac{\cdot}{l}\right) = -1$  for all small primes  $p$  then the relations between  $h_F$  and  $h_{F^*}$  become weak.

## 6. FURTHER PROBLEMS

In the monumental work [11], Wright and Yukie considered the problem of rational orbit decomposition for 8 cases including our case (2.1), and discussed the expected density theorems for those cases. On the other hand, in the process [10] and [9] to prove Theorem 1.1, the technical heart is to consider the inner form (3.1) of (2.1). The  $k$ -forms of irreducible reduced regular prehomogeneous vector spaces over local and global fields are classified by H. Saito [5], and we could see that some other cases of [11] have inner forms. In this section, we will discuss the rational orbit decomposition for some inner form representations. The proof may appear in the forthcoming paper. In this section let  $k$  be an arbitrary field. Let  $\mathcal{E}_i$  be the set of isomorphism classes of étale extensions of  $k$  of degree  $i$ .

(I) The case  $(\mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(2), k^3 \otimes k^3 \otimes k^2)$ .

Let  $\mathcal{D}$  be a simple algebra of degree 3 over  $k$ . Then

$$G = \mathcal{D}^\times \times (\mathcal{D}^{\mathrm{op}})^\times \times \mathrm{GL}(2), \quad V = \mathcal{D} \otimes k^2 \cong \mathcal{D} \oplus \mathcal{D}$$

is an inner form. Let  $\mathcal{E}_3(\mathcal{D})$  be the set of isomorphism classes of étale cubic extensions of  $k$  those are embeddable into  $\mathcal{D}$ . Then the following proposition holds.

**Proposition 6.1.** (1) *There exists a non-zero polynomial  $P$  of  $V$  and a rational character  $\chi$  on  $G$  such that  $P(gx) = \chi(g)P(x)$ .*

(2) *Let  $V' = \{x \in V \mid P(x) \neq 0\}$ . Then there exists the canonical bijection between  $G_k \backslash V'_k$  and  $\mathcal{E}_3(\mathcal{D})$ . For  $x \in V'_k$  we denote by  $k(x) \in \mathcal{E}_3(\mathcal{D})$  be the corresponding extension.*

(3) *For  $x \in V'_k$ ,  $G_x^\circ \cong k(x)^\times \times k^\times$  as an algebraic group over  $k$ .*

From this proposition, we may obtain the density of  $h_F R_F$  of cubic extensions  $F$  of  $k$ . In the case  $\mathcal{D}$  is not split, the principal parts of the global zeta function were described in [10]. It has possible simple pole at  $s = 0, 1/6, 4/3, 3/2$ . The local theory and the filtering process to obtain the density theorem are in progress.

(II) The case  $(\mathrm{GL}(4) \times \mathrm{GL}(2), \wedge^2 k^4 \otimes k^2)$ .

Let  $\mathcal{B}$  be the division algebra of  $k$ . We denote by  $H_2(\mathcal{B})$  be the set of binary Hermitian forms over  $\mathcal{B}$ . Then

$$G = \mathrm{GL}(2, \mathcal{B}) \times \mathrm{GL}(2), \quad V = H_2(\mathcal{B}) \otimes k^2$$

is an inner form. For this case the following proposition holds.

**Proposition 6.2.** (1) *There exists a non-zero polynomial  $P$  of  $V$  and a rational character  $\chi$  on  $G$  such that  $P(gx) = \chi(g)P(x)$ .*

(2) *Let  $V' = \{x \in V \mid P(x) \neq 0\}$ . Then there exists the canonical bijection between  $G_k \backslash V'_k$  and  $\mathcal{E}_2$ . For  $x \in V'_k$  we denote by  $k(x) \in \mathcal{E}_2$  be the corresponding extension.*

(3) *For  $x \in V'_k$ ,  $G_x^\circ \cong (\mathcal{B} \otimes k(x))^\times$  as an algebraic group over  $k$ .*

(III) The case  $(\mathrm{GL}(6) \times \mathrm{GL}(2), \wedge^2 k^6 \otimes k^2)$ .

Let  $H_3(\mathcal{B})$  be the set of ternary Hermitian forms over  $\mathcal{B}$ . Then just the same as the above case,

$$G = \mathrm{GL}(3, \mathcal{B}) \times \mathrm{GL}(2), \quad V = H_3(\mathcal{B}) \otimes k^2$$

is an inner form. For this case the following proposition holds.

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- Proposition 6.3.** (1) *There exists a non-zero polynomial  $P$  of  $V$  and a rational character  $\chi$  on  $G$  such that  $P(gx) = \chi(g)P(x)$ .*
- (2) *Let  $V' = \{x \in V \mid P(x) \neq 0\}$ . Then there exists the canonical bijection between  $G_k \backslash V'_k$  and  $\mathcal{E}_3$ . For  $x \in V'_k$  we denote by  $k(x) \in \mathcal{E}_3$  be the corresponding extension.*
- (3) *For  $x \in V'_k$ ,  $G_x^\circ \cong \{g \in (\mathcal{B} \otimes k(x))^\times \mid \mathcal{N}(g) \in k^\times\}$  as an algebraic group over  $k$ .*

The principal parts of the global zeta function for (II) and (III) are not known.

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