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Generalized Lerch formulas

Yoshinori Mizuno (水野 義紀)

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Osaka University (大阪大学理学研究科数学専攻D 3)

1 Generalized Lerch's formulas

The zeta-regularized product of a countable sequence $\{\lambda_k\} \subset \mathbf{C} \setminus \{0\}$ is defined by

$$\widehat{\prod}_k \lambda_k = \exp\left(-\frac{\partial}{\partial s}\sum_k \lambda_k^{-s}\Big|_{s=0}\right),$$

provided that $\Lambda(s) = \sum_k \lambda_k^{-s}$ is continued holomorphically at $s = 0$. Here the branch is chosen so that $-\pi < \arg(\lambda_k) \leq \pi$.

There are several interesting formulas which can be formulated in terms of zeta-regularized products. Typical examples are Lerch's formula

$$\widehat{\prod}_{n=0}^{\infty}(n+x) = \frac{\sqrt{2\pi}}{\Gamma(x)} \tag{1}$$

and Kronecker's limit formula

$$\widehat{\prod}_{(c,d)=1}\frac{|cz+d|}{\sqrt{y}} = (y^6|\Delta(z)|)^{-\frac{1}{6}}. \tag{2}$$

Here $\Gamma(x)$ is Euler's gamma function and $\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}$ is Ramanujan's delta function.

In this paper, we generalize Lerch's formula.

Theorem 1 *For $z_j \in \mathbf{C} \setminus \{0, -1, -2, ...\}$, we have*

$$\widehat{\prod}_{m=0}^{\infty}\left(\prod_{j=1}^{n}(m+z_j)\right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^{n}\Gamma(z_j)} = \prod_{j=1}^{n}\left(\widehat{\prod}_{m=0}^{\infty}(m+z_j)\right).$$

As a part of Theorem 1, we can obtain the formula of Lerch, Kurokawa and Wakayama.

Corollary 1 (Lerch)

$$\widehat{\prod}_{n=0}^{\infty}((n+x)^2+y^2) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)}.$$

Corollary 2 (Kurokawa and Wakayama [5]**)**

$$\widehat{\prod}_{n=0}^{\infty}((n+x)^m - y^m) = \frac{(\sqrt{2\pi})^m}{\prod_{\zeta^m=1}\Gamma(x-\zeta y)}.$$

We would like to mention that our motivation of generalizing Lerch's formula is how $\widehat{\prod}_n(a_n \cdot b_n)$ is connected with $\widehat{\prod}_n a_n \cdot \widehat{\prod}_n b_n$.

Suppose that a_n and b_n depend on some parameters X. In many examples, we know

$$\widehat{\prod}_n(a_n \cdot b_n) = e^{F(X)}\widehat{\prod}_n a_n \cdot \widehat{\prod}_n b_n \tag{3}$$

with some $F(X)$. An interesting question is to understand $F(X)$.

Theorem 1 is an example of the case where $F(X)$ vanishes in (3). In fact we have

Corollary 3 *For monic polynomials $P_j(x)$ such that $P_j(m) \neq 0$ for any $m \in \{0\} \cup \mathbf{N}$, one has*

$$\widehat{\prod}_{m=0}^{\infty}\left(\prod_{j=1}^{n}P_j(m)\right) = \prod_{j=1}^{n}\left(\widehat{\prod}_{m=0}^{\infty}P_j(m)\right).$$

Corollary 3 is remarkable because it is saying that $F(X) = 0$ in (3), which does not hold in general at all. We can see examples for $F(X) \neq 0$ in Corollary 4 which will be given in Section 2 and Lemma 1 of [8].

2 Two dimensional analogue and q-analogue

There are two dimensional analogue and q-analogue of Euler's gamma function, so called Barnes' double gamma functions and Jackson's q-gamma functions (see [1], [7]). Hence it is natural to seek two dimensional analogue and q-analogue of Theorem 1.

Barnes' double gamma function $\Gamma_2^*(z,(\omega_1,\omega_2))$ is defined by

$$\log\Gamma_2^*(z,(\omega_1,\omega_2)) = \frac{\partial}{\partial s}\sum_{m,l=0}^{\infty}(m\omega_1+l\omega_2+z)^{-s}\Big|_{s=0},$$

$$\Gamma_2^*(z,(\omega_1,\omega_2))^{-1} = \widehat{\prod}_{m,l=0}^{\infty}(m\omega_1+l\omega_2+z).$$

We get a two dimensional analogue of Theorem 1 by using the following result.

Theorem 2 *Assume that* $q_j,\tau_j,z_j\in\mathbf{C}$ *satisfy that* $\Re(q_j)>0,\Re(\tau_j)>0,\Re(z_j)>0$, *and* $q_j\neq q_k,\tau_j\neq\tau_k,q_j\tau_k\neq q_k\tau_j$ *for* $j\neq k$. *The function of* s *defined by*

$$H_2(s) = \sum_{m,l=0}^{\infty}\prod_{j=1}^{n}(mq_j+l\tau_j+z_j)^{-s}$$

is continued meromorphically to all s*-plane.* $H_2(s)$ *is holomorphic at* $s=0$ *and we have the following formula for* $\frac{\partial}{\partial s}H_2(s)\Big|_{s=0}$,

$$\begin{aligned}\frac{\partial}{\partial s}H_2(s)\Big|_{s=0} &= \sum_{j=1}^{n}\log\Gamma_2^*(z_j,(q_j,\tau_j))\\ &+ \frac{1}{2n}\sum_{1\le j<k\le n}\left\{\frac{q_j\tau_k-\tau_jq_k}{q_jq_k}(\log q_k-\log q_j)B_2\left(\frac{q_jz_k-q_kz_j}{q_j\tau_k-\tau_jq_k}\right)\right.\\ &+ \left.\frac{q_k\tau_j-\tau_kq_j}{\tau_j\tau_k}(\log\tau_k-\log\tau_j)B_2\left(\frac{\tau_jz_k-\tau_kz_j}{\tau_jq_k-q_j\tau_k}\right)\right\}.\end{aligned}$$

Here $B_2(x)=x^2-x+1/6$ *is the second Bernoulli polynomial. We choose the principal branch for* $\log q_i,\log\tau_i$.

This is a generalization of Shintani's result (see [12]). He treated the case $n=2$ to give a new proof of Kronecker's limit formula (2).

We remark that in order to conclude

$$\exp\left(-\frac{\partial}{\partial s}H_2(s)\Big|_{s=0}\right) = \widehat{\prod}_{m,l=0}^{\infty}\left(\prod_{j=1}^{n}(mq_j + l\tau_j + z_j)\right),$$

the equation

$$\left\{\prod_{j=1}^{n}(mq_j + l\tau_j + z_j)\right\}^s = \prod_{j=1}^{n}(mq_j + l\tau_j + z_j)^s \qquad (4)$$

must hold for any $m, l \in \mathbf{N} \cup \{0\}$. We take this remark into account to give a two dimensional analogue of Theorem 1. As an example of q_j, τ_j, z_j which satisfy the equation (4) for any $m, l \in \mathbf{N}\cup\{0\}$, we can take $n = 2h, q_j, \tau_j, z_j \in \mathbf{C}, q_{h+j} = \overline{q_j}, \tau_{h+j} = \overline{\tau_j}, z_{h+j} = \overline{z_j}, j = 1, ..., h$.

Corollary 4 *Fix* $q_j, \tau_j, z_j \in \mathbf{C}$ *such that* $\Re(q_j) > 0, \Re(\tau_j) > 0, \Re(z_j) > 0$, *and* $q_j \neq q_k, \tau_j \neq \tau_k, q_j\tau_k \neq q_k\tau_j$ *for* $j \neq k$. *Suppose that* (4) *is satisfied for any* $m, l \in \mathbf{N} \cup \{0\}$. *Then we have*

$$\begin{aligned}\widehat{\prod}_{m,l=0}^{\infty}\left(\prod_{j=1}^{n}(mq_j + l\tau_j + z_j)\right) &= e^F \prod_{j=1}^{n}\Gamma_2^*(z_j, (q_j, \tau_j))^{-1} \\ &= e^F \prod_{j=1}^{n}\left(\widehat{\prod}_{m,l=0}^{\infty}(mq_j + l\tau_j + z_j)\right),\end{aligned}$$

where

$$\begin{aligned}F &= -\frac{1}{2n}\sum_{1\leq j<k\leq n}\left\{\frac{q_j\tau_k - \tau_j q_k}{q_j q_k}(\log q_k - \log q_j)B_2\left(\frac{q_j z_k - q_k z_j}{q_j\tau_k - \tau_j q_k}\right)\right. \\ &+ \left.\frac{q_k\tau_j - \tau_k q_j}{\tau_j\tau_k}(\log \tau_k - \log \tau_j)B_2\left(\frac{\tau_j z_k - \tau_k z_j}{\tau_j q_k - q_j \tau_k}\right)\right\}.\end{aligned}$$

Next we present q-analogue of Theorem 1. Usually the zeta-regularized product is defined for a sequence $\{\lambda_k\} \subset \mathbf{C} \setminus \{0\}$ such that $\Lambda(s) = \sum_k \lambda_k^{-s}$ can be continued holomorphically at $s = 0$. In case $\Lambda(s)$ is meromorphic at

$s=0$, Kurokawa and Wakayama [6] define the generalized zeta reguralization by

$$\widehat{\prod}_k \lambda_k = \exp\left(-\operatorname*{Res}_{s=0}\frac{\Lambda(s)}{s^2}\right).$$

They obtained several examples of such product, one of which is the following q-analogue of Lerch's formula.

Theorem 3 (Kurokawa and Wakayama [6]) *For $q>1, x>0$,*

$$\widehat{\prod}_{n=0}^{\infty}[n+x]_q = \frac{C_q}{\Gamma_q(x)}.$$

Here $[x]_q = \frac{q^x-1}{q-1}$ is the q-analogue of number x,

$$\Gamma_q(x) = \frac{\prod_{n=1}^{\infty}(1-q^{-n})}{\prod_{n=0}^{\infty}(1-q^{-(x+n)})}(q-1)^{1-x}q^{\frac{x(x-1)}{2}}$$

is Jackson's q-gamma function,

$$C_q = \widehat{\prod}_{n=1}^{\infty}[n]_q = q^{-\frac{1}{12}}(q-1)^{\frac{1}{2}-\frac{\log(q-1)}{2\log q}}\prod_{n=1}^{\infty}(1-q^{-n}).$$

We obtain the next result which is the q-analogue of Theorem 1 including the above Theorem 3.

Theorem 4 *For $q>1, z_j>1$, we have*

$$\begin{aligned}\widehat{\prod}_{m=0}^{\infty}\left(\prod_{j=1}^{n}[m+z_j]_q\right) &= \frac{C_q^n}{\prod_{j=1}^{n}\Gamma_q(z_j)}q^{-\frac{1}{2n}(\sum_{j=1}^{n}z_j)^2+\frac{1}{2}\sum_{j=1}^{n}z_j^2}\\ &= q^{-\frac{1}{2n}(\sum_{j=1}^{n}z_j)^2+\frac{1}{2}\sum_{j=1}^{n}z_j^2}\prod_{j=1}^{n}\left(\widehat{\prod}_{m=0}^{\infty}[n+z_j]_q\right).\end{aligned}$$

3 Double Hurwitz zeta

For $\beta > \alpha > 0$, let $H_{\alpha,\beta}(s_1, s_2)$ be Dirichlet series defined by

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty}(n+\alpha)^{-s_1}(n+\beta)^{-s_2}.$$

This series converges absolutely for $\Re(s_1 + s_2) > 1$.

$H_{\alpha,\beta}(s_1, s_2)$ is an important object in the theory of the zeta-regularized product. For example, as we presented in Section 1, we know generalized Lerch's formula

$$\exp\left(-\frac{\partial}{\partial s}H_{\alpha,\beta}(s,s)\Big|_{s=0}\right) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}.$$

We know also that the spectral zeta function $Z_n(s)$ of the unit n-sphere S^{n-1} can be written in terms of $H_{\alpha,\beta}(s_1, s_2)$ as

$$Z_n(s) = \sum_{d=0}^{n-1} T_{n,d} H_{1,n}(s-d, s), \tag{5}$$

where

$$T_{n,d} = \frac{1}{n!}\sum_{r=d+1}^{n} s(n,r)\binom{r}{d}(n^{r-d} - (n-2)^{r-d}),$$

$s(r,d)$ denoting the Stirling numbers of the first kind. See Lemma 2 of [4] p.202. We get the formula for the functional determinant of the Laplacian by evaluating $\frac{\partial}{\partial s}Z_n(s)\Big|_{s=0}$. See Theorem 1 of [4] p. 200.

In the results mentioned above, the main target is not $H_{\alpha,\beta}(s_1, s_2)$ itself but evaluating derivative of $H_{\alpha,\beta}(s_1, s_2)$. In this section, we analyze $H_{\alpha,\beta}(s_1, s_2)$ itself. First by applying the method described in [2], we can get the following expression for $H_{\alpha,\beta}(s_1, s_2)$.

$$H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(s_1+s_2)}{\Gamma(s_1)\Gamma(s_2)}\int_0^1 u^{s_2-1}(1-u)^{s_1-1}\zeta(s_1+s_2, \alpha-(\alpha-\beta)u)du, \tag{6}$$

where $\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$ is Hurwitz zeta function. It is very interesting to note that S. Ramanujan already treated the integral of the right hand side on (6) apart from Dirichlet series $H_{\alpha,\beta}(s_1, s_2)$. See (14) of [9] p.166.

Starting from the integral expression (6), we show the following results.

Theorem 5 *$H_{\alpha,\beta}(s_1, s_2)$ can be continued meromorphically to all $s_1, s_2 \in \mathbf{C}$.*

Theorem 6 *For $\Re(s_1) < 0, \Re(s_2) < 0, 0 < \alpha < \beta < 1$, we have*

$$H_{\alpha,\beta}(s_1, s_2) = \frac{\Gamma(1 - s_1 - s_2)}{(2\pi)^{1-s_1-s_2}}$$
$$\times \left\{ e^{\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{-2\pi i n\beta} \, {}_1F_1(s_1, s_1 + s_2, 2\pi i n(\beta - \alpha)) \right.$$
$$\left. + \; e^{-\frac{\pi i}{2}(1-s_1-s_2)} \sum_{n=1}^{\infty} n^{s_1+s_2-1} e^{2\pi i n\alpha} \, {}_1F_1(s_2, s_1 + s_2, 2\pi i n(\beta - \alpha)) \right\}. \quad (7)$$

Here ${}_1F_1(a, b, z)$ is the confluent hypergeometric series defined by

$${}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \quad (8)$$

with $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

This is a generalization of well known Hurwitz relation for $\zeta(s, x)$.

Theorem 7 *We have*

$$H_{\alpha,\beta}(s_1, s_2) = \sum_{n=0}^{\infty} \frac{(s_2)_n}{n!} \zeta(s_1 + s_2 + n, \alpha)(\alpha - \beta)^n.$$

This is a special case of Main Theorem of [3]. However we can prove Main Theorem of [3] by quite different manner using the confluent hypergeometric series ${}_1F_1(a, b, z)$.

Next we give the evaluation formula of $H_{\alpha,\beta}(s_1, s_2)$. We can evaluate the values of $H_{\alpha,\beta}(s_1, s_2)$ at any integers s_1, s_2 in terms of the values of Hurwitz zeta function.

Theorem 8 *For $p, q \in \mathbf{N}$, we have*

$$H_{\alpha,\beta}(q, p) = \frac{\Gamma(p + q)}{(p + q - 1)\Gamma(p)\Gamma(q)}$$
$$\times \left\{ \sum_{n=0}^{p+q-3} \left\{ \sum_{m=\max\{n-p+1,0\}}^{q-1} (-1)^m \binom{q-1}{m} \binom{p+m-1}{n} \right\} \right.$$

$$
\begin{aligned}
\times & \quad \frac{n!}{(2-p-q)_n}\zeta(p+q-n-1,\beta)(\alpha-\beta)^{-n-1} \\
- & \quad \sum_{m=0}^{q-2}(-1)^m\binom{q-1}{m}\frac{(p+m-1)!}{(2-p-q)_{p+m-1}}\zeta(q-m,\alpha)(\alpha-\beta)^{-p-m} \\
+ & \quad (-1)^{q-1}\frac{(p+q-2)!}{(2-p-q)_{p+q-2}}(\alpha-\beta)^{-p-q+1}\left(\frac{\Gamma'}{\Gamma}(\beta)-\frac{\Gamma'}{\Gamma}(\alpha)\right)\Bigg\}.
\end{aligned}
$$

Here empty sum is considered as zero.

Theorem 9 *For $p, q \in \mathbf{Z}$ which are not both negative, we have*

$$
\begin{aligned}
H_{\alpha,\beta}(-p,-q) &= \sum_{k=0}^{q}\binom{q}{k}(\beta-\alpha)^k\zeta(-p-q+k,\alpha) \\
&+ \sum_{k=0}^{p}\binom{p}{k}(\alpha-\beta)^k\zeta(-p-q+k,\beta).
\end{aligned}
$$

Here empty sum is considered as zero.

Finally we mention that we can provide another approach to evaluate the determinant $\det\Delta_n$ of the Laplacian on the n-sphere S^{n-1} starting from the integral expression (6). Here $\det\Delta_n$ is defined by

$$
\det\Delta_n = \exp\left(-\sum_{d=0}^{n-1}T_{n,d}\,\frac{\partial}{\partial s}H_{1,n}(s-d,s)\bigg|_{s=0}\right).
$$

See (5) for the definition of $T_{n,d}$.

Theorem 10

$$
\begin{aligned}
\frac{\partial}{\partial s}H_{1,n+1}(s-d,s)\bigg|_{s=0} &= \zeta'(-d)+\sum_{l=0}^{d}(-n)^{d-l}\binom{d}{l}\zeta'(-l,n+1) \\
&- \frac{(-n)^{d+1}}{2(d+1)}\left(\sum_{j=1}^{d}\frac{1}{j}\right).
\end{aligned}
$$

This is simpler than Kumagai's fomula given in Lemma 3 of [4] p.202. Comparing Theorem 10 and Kumagai's result, we get the following identity for harmonic numbers.

Corollary 5 *The following identity holds:*

$$2^{1-d}\sum_{l=1,odd}^{d}\binom{d+1}{l+1}\sum_{j=1,odd}^{l}\frac{1}{j}=\sum_{j=1}^{d}\frac{1}{j}.$$

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