数理解析研究所講究録 1451 巻 2005 年 112-119

A geometric nonabelian class field theory over the field of complex numbers and its application

Ken-ichi SUGIYAMA *†

February 7, 2005

1 The geometric abelian class field theory

1.1 A review of the geometric alelian class field theory

Let X be a smooth projective connected curve defined over $\mathbb{C}$ of genus g. We will choose and fix a base point x_0 of X. The Jacobian variety of X will be denoted by $\mathrm{Jac}(X)$.

The abelian class field theory over $\mathbb{C}$ shows that there is one to one correspondence between the isomorphism classes of characters of $\pi_1(X, x_0)$ and one of flat line bundles over $\mathrm{Jac}(X)$, which we will recall now.

Suppose we are given a character

$$\pi_1(X, x_0) \xrightarrow{\chi} \mathbb{C}^{\times}.$$

Since the first homology group of X is isomorphic to the abelization of the fundamental group:

$$\pi_1(X, x_0)^{ab} \simeq H_1(X, \mathbb{Z}), \tag{1}$$

χ factors through the homomorphism:

$$H_1(X, \mathbb{Z}) \xrightarrow{\chi^{ab}} \mathbb{C}^{\times}.$$

*Address : Ken-ichi SUGIYAMA, Department of Mathematics and Informatics, Faculty of Science, Chiba University, 1-33 Yayoi-cho Inage-ku, Chiba 263-8522, Japan

†e-mail address : sugiyama@math.s.chiba-u.ac.jp

Since the fundamental group of the Jacobian is isomorphic to $H_1(X, \mathbb{Z})$:

$$H_1(X, \mathbb{Z}) \simeq \pi_1(\mathrm{Jac}(X), 0), \tag{2}$$

χ^{ab} will define a flat line bundle $\mathcal{L}_\chi$ on $\mathrm{Jac}(X)$.

Coversely let $\mathcal{L}$ be a flat line bundle on $\mathrm{Jac}(X)$. The monodromy representation yields a group homomorphism:

$$\pi_1(\mathrm{Jac}(X), 0) \xrightarrow{\chi_{\mathcal{L}}} \mathbb{C}^\times,$$

and by (1) and (2) we have a character:

$$\pi_1(X, x_0) \xrightarrow{\chi_{\mathcal{L}}} \mathbb{C}^\times.$$

1.2 The geometric abelian class field theory in terms of a hamiltonian system

We want to generalize the correspondence to a non-abelian case. In order to do so, it is necessary to formulate the geometric abelian class field theory in terms of a hamiltonian system.

Since the cotangent bundle $T^*\mathrm{Jac}(X)$ of $\mathrm{Jac}(X)$ is trivial, we have a natural projection:

$$T^*\mathrm{Jac}(X) \simeq \mathrm{Jac}(X) \times H^0(X, \Omega_X) \xrightarrow{p} H^0(X, \Omega_X).$$

Here we have identified the cotangent space of the Jacobian variety at the origin with $H^0(X, \Omega_X)$ by the deformation theory. The compactness of $\mathrm{Jac}(X)$ implies that p induces an isomorphism:

$$\Gamma(H^0(X, \Omega_X), \mathcal{O}) \overset{p^*}{\simeq} \Gamma(T^*\mathrm{Jac}(X), \mathcal{O}). \tag{3}$$

On the other hand let $D(\mathrm{Jac}(X))$ be the ring of global differential operators on $\mathrm{Jac}(X)$. Taking symbols we have an isomorphism

$$D(\mathrm{Jac}(X)) \overset{\sigma}{\simeq} \Gamma(T^*\mathrm{Jac}(X), \mathcal{O}), \tag{4}$$

and the composition this with (3) implies

$$D(\mathrm{Jac}(X)) \simeq \Gamma(H^0(X, \Omega_X), \mathcal{O}). \tag{5}$$

Now let χ be a character of $\pi_1(X, x_0)$ and $\mathcal{L}_\chi$ be the corresponding flat line bundle on X with a flat connection

$$\nabla = d + A_\chi, \quad A_\chi \in H^0(X, \Omega_X).$$

The connection form A_χ defines a homomorphism

$$D(\mathrm{Jac}(X)) \simeq \Gamma(H^0(X, \Omega_X), \mathcal{O}) \xrightarrow{f_{A_\chi}} \mathbf{C},$$

which defines a D-module $\mathcal{M}_\chi$ on $\mathrm{Jac}(X)$:

$$\mathcal{M}_\chi = \mathcal{D}_{\mathrm{Jac}(X)}/(\mathrm{Ker}\, f_{A_\chi}).$$

Here $\mathcal{D}_{\mathrm{Jac}(X)}$ is the sheaf of differential operators on $\mathrm{Jac}(X)$. It is easy to see that $\mathcal{M}_\chi$ is nothing but the D-module associated the flat line bundle $\mathcal{L}_\chi$ in the previous section.

Remark 1.1. *The fibration*

$$T^*\mathrm{Jac}(X) \xrightarrow{p} H^0(X, \Omega_X)$$

is a Lagrangian fibration with respect to the canonical symplectic form on $T^*\mathrm{Jac}(X)$.

2 A geometric nonabelian class field theory

2.1 An unramified case

Using the idea of the hamiltonian formulation of the last section, Beilinson and Drinfeld formulated a geometric nonabelian class field theory ([1]). For simplicity we will treat only the SL_2-case. We put $G = SL_2(\mathbf{C})$ and let $\mathfrak{g}$ be its Lie algebra. Let $\mathrm{Bun}_{G,X}$ be the modular stack of principal G bundles on X, which is a smooth stack of dimension $3(g-1)$. It will play a role of $\mathrm{Jac}(X)$ in the geometric abelian class field theory, which classifies $\mathbb{G}_m$-bundles of degree 0 on X. Each of the tangent space and the cotangent space at $P \in \mathrm{Bun}_{G,X}$ becomes

$$T_P(\mathrm{Bun}_{G,X}) \simeq H^1(X, \mathrm{ad}_P(\mathfrak{g})), \quad T_P^*(\mathrm{Bun}_{G,X}) \simeq H^0(X, \mathrm{ad}_P(\mathfrak{g}) \otimes \Omega_X),$$

respectively by the deformation theory. Associating

$$h(A) = \det(A) \in H^0(X, \Omega^{\otimes 2})$$

to $A \in T_P^*(\mathrm{Bun}_{G,X}) \simeq H^0(X, \mathrm{ad}_P(\mathfrak{g}) \otimes \Omega_X)$, we have *the Hitchin map*

$$T^*(\mathrm{Bun}_{G,X}) \xrightarrow{h} H^0(X, \Omega^{\otimes 2}).$$

One can show that the Hitchin map is a Lagrangian fibration with respect to the canonical symplectic form on $T^*(\mathrm{Bun}_{G,X})$. Moreover the following facts are known.

Fact 2.1. *([4])*

1. *h is flat and surjective.*
2. *For generic $q \in H^0(X, \Omega^{\otimes 2})$, $h^{-1}(q)$ is an abelian variety of dimension* $3(g-1)$.

In particular this will imply

$$\Gamma(H^0(X, \Omega^{\otimes 2}), \mathcal{O}) \overset{h^*}{\simeq} \Gamma(T^*(\mathrm{Bun}_{G,X}), \mathcal{O}). \tag{6}$$

Let $\mathcal{D}'$ be the sheaf of differential operators twisted by the half canonical $\omega^{\frac{1}{2}}$ of $\mathrm{Bun}_{G,X}$. Based on the isomorphism (6), Beilinson and Drinfeld have shown the following *quantization* of the Hitchin's theorem.

Fact 2.2. *([1]) There is an isomorphism of* $\mathbf{C}$*-algebras:*

$$\Gamma(\mathrm{Bun}_{G,X}, \mathcal{D}') \simeq \Gamma(H^0(X, \Omega^{\otimes 2}), \mathcal{O}). \tag{7}$$

Note that (6) and (7) are nonabelization of the isomorphism (3) and (5), respectively. Now we can explain a geometric non-abelian class field theory (of SL_2-case).

Let

$$\nabla = d + A, \quad A \in H^0(X, \Omega^1 \otimes \mathfrak{g})$$

be the flat $\mathfrak{g}$-connection which corresponds to a projective representation of the fundamental group:

$$\pi_1(X, x_0) \xrightarrow{\rho} PSL_2(\mathbb{C}).$$

Note that the trace of A is equal to zero and that its determinant q is a quadratic diffrential:

$$q = \det A \in H^0(X, \Omega^{\otimes 2}).$$

The evaluation at q yields a homomorphism

$$\Gamma(\mathrm{Bun}_{G,X}, \mathcal{D}') \simeq \Gamma(H^0(X, \Omega^{\otimes 2}), \mathcal{O}) \xrightarrow{f_q} \mathbf{C}$$

and we define a D-module $\mathcal{M}_q$ on $\mathrm{Bun}_{G,X}$ to be

$$\mathcal{M}_q = (\mathcal{D}'/\mathcal{D}' \cdot \mathrm{Ker}(f_q)) \otimes \omega^{-\frac{1}{2}}.$$

One may see that its characteristic variety $\mathrm{char}(\mathcal{M}_q)$ coinsides with *the Laumon's global nilpotent variety* $h^{-1}(0)$([5]). Thus the Hitchin's result implies $\mathcal{M}_q$ is holonomic. In fact $\mathcal{M}_q$ should be regular holonomic.

Remark 2.1. *The construction above is one way of the geometric non-abelian class field theory. In order to construct a representaion of the fundamental group from a regular holonomic D-module $\mathcal{M}$ on $\mathrm{Bun}_{G,X}$, $\mathcal{M}$ should be* a Hecke eigenmodule. *The construction of the reverse direction is illustrated in [3].*

2.2 A ramified case

The Beilinson-Drinfeld correspondence may be considered as a geometric non-abelian class field theory which associates a D-module on $\mathrm{Bun}_{G,X}$ to an unramified projective representation of the fundamental group. We will generalize their correspondence to ramified representations. Let $\{z_1, \cdots, z_N\}$ be mutually distinct points of X and we put

$$D = \sum_{i=1}^{N} z_i \in \mathrm{Div}(X).$$

We will choose and fix a local coordinate t_i at z_i.

Definition 2.1. *Let P be a principal G bundle on X. A* D-flag *of P is defined to be an N-tuple $\{B_1, \cdots, B_N\}$, where B_i is a Borel subgroup of $P|_{z_i} \simeq G$.*

Let $\mathrm{Bun}_{G,X}^{D-fl}$ be the modular stack of principal G bundles on X with D-flags, which is a smooth stack of dimension $3(g-1)+N$. In fact it is a $(\mathbb{P}^1)^N$-fibration over $\mathrm{Bun}_{G,X}$:

$$\mathrm{Bun}_{G,X}^{D-fl} \xrightarrow{\pi} \mathrm{Bun}_{G,X}. \qquad (8)$$

As before we may define a Hitchin map

$$T^*\mathrm{Bun}_{G,X}^{D-fl} \xrightarrow{h} H^0(X, \Omega_X^{\otimes 2}(D)).$$

Note that by the Riemann-Roch theorem the dimension of $H^0(X, \Omega_X^{\otimes 2}(D))$ is equal to $3(g-1)+N$.

Theorem 2.1. *([7]) The Hitchin map h is flat and surjective. Moreover for generic $q \in H^0(X, \Omega_X^{\otimes 2}(D))$ $h^{-1}(q)$ is an abelian variety of dimension $3(g-1)+N$. In particular it induces an isomorphism*

$$\Gamma(H^0(X, \Omega_X^{\otimes 2}(D)), \mathcal{O}) \overset{h^*}{\simeq} \Gamma(T^*\mathrm{Bun}_{G,X}^{D-fl}, \mathcal{O}).$$

For

$$\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N,$$

one can construct a line bundle $\mathcal{L}_\lambda^0$ on $\mathrm{Bun}_{G,X}^{D-fl}$ whose restriction to a fibre of (8) is isomorphic to an exterior product of line bundles on $\mathbb{P}^1$:

$$p_1^*\mathcal{O}(\lambda_1) \otimes \cdots \otimes p_N^*\mathcal{O}(\lambda_N),$$

where p_i is the projection to the i-th factor. Let $\mathcal{D}'_{D-fl,\lambda}$ be the sheaf of differential operators on $\mathrm{Bun}_{G,X}^{D-fl}$ twisted by $\mathcal{L}_\lambda^0 \otimes \pi^*\omega^{\frac{1}{2}}$ and the ring of its global sections will be denoted by $D'_{D-fl,\lambda}$.

A quadratic differential $q \in H^0(X, \Omega^{\otimes 2}(2D))$ will be mentioned as λ-*admissible* if it has a Taylor expansion

$$q = \{\frac{\Delta(\lambda_i)}{t_i^2} + \cdots\}dt_i^{\otimes 2}, \quad \Delta(\lambda_i) = \frac{\lambda_i(\lambda_i+2)}{4},$$

at each z_i. The subspace of $H^0(X, \Omega^{\otimes 2}(2D))$ which consists of λ-admissible quadratic differentials will be denoted by $H_{\Delta(\lambda)}$. The following theorem may be considered as *a quantization* of **Theorem 3.1**.

Theorem 2.2. *([7]) There is an isomorphism as* $\mathbb{C}$*-algebra:*

$$\Gamma(H_{\Delta(\lambda)}, \mathcal{O}) \overset{h_\lambda^+}{\simeq} D'_{D-fl,\lambda}. \tag{9}$$

In fact $D'_{D-fl,\lambda}$ has the natural filtration by the degree of differential operators and one can introduce a filtration on $\Gamma(H_{\Delta(\lambda)}, \mathcal{O})$ so that h_λ^+ is a filtered isomorphism. Then the isomorphism of **Theorem 3.1** is nothing but the graded quotient of (9). As before, using **Theorem 3.2**, one can construct a holonomic D-module on $\mathrm{Bun}_{G,X}^{D-fl}$ from a projective representation of the fundamental group of $X \setminus \{z_1, \cdots, z_N\}$ whose the determinant of the corresponding connection is λ-admissible. (We will call such a representation as λ-*admissible.*)

Remark 2.2. *When X is the projective line* $\mathbb{P}^1$*,* **Theorem 3.2** *has been already established by E. Frenkel([2]).*

3 An application

Using the geometric non-abelian class field theory, one may find a mysterious relation between the Knizhnik-Zamolodchikov equation and a λ-admissible representation of the fundamental group of $\mathbb{P}^1 \setminus \{z_1, \cdots, z_N\}$ ([6]).

References

[1] A. Beilinson and V. Drinfeld. Quantization of Hitchin integrable system and Hecke eigensheaf. Preprint.

[2] E. Frenkel. Affine algebras, Langlands duality and Bethe Ansatz. In *Proc. of the Intern. Congr. of Math. and Phys.*, pages 606–642. International Press, 1995.

[3] V. Ginzburg. Perverse sheaves on a loop group and Langlands' duality. alg-geom/9511007, November 2000.

[4] N. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[5] G. Laumon. Un analogue global du cône nilpotent. *Duke Math. Jour.*, 57:647–671, 1988.

[6] K. Sugiyama. The Beilinson-Drinfeld-Frenkel correspondence and the Knizhnik-Zmamolodchikov equation. Preprint, November 2004.

[7] K. Sugiyama. A quantization of the Hitchin hamiltonian system and the Beilinson-Drinfeld correspondence. Preprint, November 2004.

Address : Department of Mathematics and Informatics
Faculty of Science
Chiba University
1-33 Yayoi-cho Inage-ku
Chiba 263-8522, Japan
e-mail address : sugiyama@math.s.chiba-u.ac.jp