数理解析研究所講究録 1452 巻 2005 年 11-21

Conditions for convergence theorems in non-additive measure theory*

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Abstract: This paper discusses convergence theorems describing implications between six convergence concepts with respect to non-additive measure: almost everywhere convergence, pseudo-almost everywhere convergence, almost uniform convergence, pseudo-almost uniform convergence, convergence in measure, and convergence pseudo-in measure. The paper shows several new convergence theorems and organizes them together with existing convergence theorems by using ordinality and duality. In addition, it gives a new necessary condition and a new sufficient condition for the Egoroff theorem to hold.

1 INTRODUCTION

Since Sugeno [15] introduced the concept of non-additive measure, which he called a fuzzy measure, non-additive measure theory has been constructed along the lines of the classical measure theory [1, 13, 21]. Generally, theorems in the classical measure theory no longer hold in non-additive measure theory, so that to find necessary and/or sufficient conditions for such theorems to hold is very important for the construction of non-additive measure theory.

In the classical measure theory, there are several different convergences of a sequence of measurable functions such as almost everywhere convergence, almost uniform convergence, and convergence in measure, and theorems that describe implication relationship between such convergence concepts (e.g, the Egoroff, Lebesgue, and Riesz theorems) are fundamental and important. In non-additive measure theory, these theorems do not hold without additional conditions.

This paper discusses necessary and sufficient conditions for the implications between almost everywhere convergence, pseudo-almost everywhere convergence [20], almost uniform convergence, pseudo-almost uniform convergence [20], convergence in measure, and convergence pseudo-in measure [20] in non-additive measure theory. So far, most of the implications have been established [4, 6, 8, 10, 12, 14, 17, 19, 20, 21]. Section 4 of this paper clarifies the remaining ones, that is, necessary and sufficient conditions for almost uniform convergence to imply pseudo-almost everywhere convergence, pseudo-almost uniform convergence, and convergence pseudo-in measure. Section 5 summarizes necessary and sufficient conditions for implications between the six convergences into Tables 2 and 3. Section 6 introduces a new notion called condition (M), and show that condition (M) is a necessary condition for the Egoroff theorem, which asserts that almost everywhere convergence implies almost uniform convergence, and that the conjunction of condition (M) and null-continuity is a sufficient condition for the theorem.

2 DEFINITIONS

Throughout the paper, $(X, \mathcal{S})$ is assumed to be a measurable space. All subsets of X and functions on X referred to are assumed to be measurable.

Definition 2.1 A *non-additive measure* on $(X, \mathcal{S})$ is a set function $\mu : \mathcal{S} \to [0, \infty]$ satisfying the following two conditions:

- $\mu(\emptyset) = 0$,
- $A, B \in \mathcal{S},\ A \subset B \ \Rightarrow\ \mu(A) \le \mu(B)$.

*Partial financial support from the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for the 21st Century COE Program "Creation of Agent-Based Social Systems Sciences" is gratefully acknowledged.

A non-additive measure μ is said to be *finite* if $\mu(X) < \infty$. If μ is a non-additive measure on $(X, \mathcal{S})$, the triplet $(X, \mathcal{S}, \mu)$ is called a *non-additive measure space*. For each $A \in \mathcal{S}$, the restriction $\mu \upharpoonright (\mathcal{S} \cap A)$ is a non-additive measure on $(A, \mathcal{S} \cap A)$, where $\mathcal{S} \cap A = \{E \cap A \mid E \in \mathcal{S}\}$, and $(A, \mathcal{S} \cap A, \mu \upharpoonright (\mathcal{S} \cap A))$ is called a *subspace* of $(X, \mathcal{S}, \mu)$.

Hereinafter, μ is assumed to be a non-additive measure on $(X, \mathcal{S})$.

In the following definitions, each label in bold face stands for the corresponding term; for example, "$\downarrow A$" means "continuity from above at A" (Definition 2.2 (i)).

Definition 2.2 (i) **$\downarrow A$:** μ is said to be *continuous from above at* A if $A_n \downarrow A$ implies $\mu(A_n) \to \mu(A)$.

$\downarrow$: μ is said to be *continuous from above* if μ is continuous from above at every measurable set.

(ii) **$\downarrow \emptyset$:** [2] μ is said to be *order continuous* if μ is continuous from above at the empty set.

(iii) **$\downarrow 0$:** [3] μ is said to be *strongly order continuous* if $N_n \downarrow N$ and $\mu(N) = 0$ together imply $\mu(N_n) \to 0$.

(iv) **T$\downarrow$0:** [12] μ is said to be *strongly order totally continuous* if, for every decreasing net $\mathcal{B}$ of measurable sets such that $\bigcap \mathcal{B}$ is measurable and $\mu(\bigcap \mathcal{B}) = 0$, it holds that $\inf_{B \in \mathcal{B}} \mu(B) = 0$.

(v) **$\uparrow A$:** μ is said to be *continuous from below at* A if $A_n \uparrow A$ implies $\mu(A_n) \to \mu(A)$.

$\uparrow$: μ is said to be *continuous from below* if μ is continuous from below at every measurable set.

(vi) **$\uparrow \mu(A)$:** [8] μ is said to be *strongly continuous from below at* A if $B_n \uparrow B \subset A$ and $\mu(B) = \mu(A)$ together imply $\mu(B_n) \to \mu(B)$.

(vii) **$\uparrow 0$:** [18] μ is said to be *null-continuous* if $N_n \uparrow N$ and $\mu(N_n) = 0$ for every n together implies $\mu(N) = 0$.

The value of $\mu(A)$ is not substituted for $\mu(A)$ in "$\uparrow \mu(A)$"; if $\mu(A) = 0.5$ for example, we write not "$\uparrow 0.5$" but "$\uparrow \mu(A)$".

Definition 2.3 (i) **0-sub. A:** μ is said to be *null-subtractive at* A if $\mu(N) = 0$ implies $\mu(A \setminus N) = \mu(A)$.

0-add.: [19] μ is said to be *null-additive* if $\mu(N) = 0$ implies $\mu(A \cup N) = \mu(A)$ for every $A \in \mathcal{S}$.

(ii) **c.0-add. A:** μ is said to be *converse-null-additive at* A if $\mu(A) = \mu(A \setminus N)$ implies $\mu(A \cap N) = 0$.

c.0-add.: μ is said to be *converse-null-additive* if μ is converse-null-additive at every measurable set.

Null-subtractivity at every measurable set is equivalent to null-additivity [21]. Converse-null-additivity defined above is stronger than the original in [19]: $\mu(A) = \mu(A \setminus N) < \infty$ implies $\mu(A \cap N) = 0$.

Definition 2.4 (i) **auto.$\uparrow$ A:** μ is said to be *autocontinuous from below at* A if $\mu(N_n) \to 0$ implies $\mu(A \setminus N_n) \to \mu(A)$.

auto.$\uparrow$: [19] μ is said to be *autocontinuous from below* if μ is autocontinuous from below at every measurable set.

(ii) **c.auto.$\uparrow$ A:** μ is said to be *converse-autocontinuous from below at* A if $\mu(A \setminus N_n) \to \mu(A)$ implies $\mu(A \cap N_n) \to 0$.

c.auto.$\uparrow$: μ is said to be *converse-autocontinuous from below* if μ is converse-autocontinuous from below at every measurable set.

(iii) **m.auto.$\uparrow$ A:** μ is said to be *monotone autocontinuous from below at* A if $N_n \downarrow$ and $\mu(N_n) \to 0$ together imply $\mu(A \setminus N_n) \to \mu(A)$.

m.auto.↑: μ is said to be *monotone autocontinuous from below* if μ is monotone autocontinuous from below at every measurable set.

(iv) **c.m.auto.↑ A:** μ is said to be *converse-monotone autocontinuous from below at A* if $N_n \downarrow$ and $\mu(A \setminus N_n) \to \mu(A)$ together imply $\mu(A \cap N_n) \to 0$.

c.m.auto.↑: μ is said to be *converse-monotone autocontinuous from below* if μ is converse-monotone autocontinuous from below at every measurable set.

(v) **s.m.auto.↑ A:** μ is said to be *strongly monotone autocontinuous from below at A* if $N_n \downarrow N$ and $\mu(N) = 0$ together imply $\mu(A \setminus N_n) \to \mu(A)$.

s.m.auto.↑: μ is said to be *strongly monotone autocontinuous from below* if μ is strongly monotone autocontinuous from below at every measurable set.

(vi) **s.c.m.auto.↑ A:** μ is said to be *strongly converse-monotone autocontinuous from below at A* if $N_n \downarrow N$ and $\mu(A \setminus N) = \mu(A)$ together imply $\mu(A \cap N_n) \to 0$.

s.c.m.auto.↑: μ is said to be *strongly converse-monotone autocontinuous from below* if μ is strongly converse-monotone autocontinuous from below at every measurable set.

Converse-autocontinuity from below defined above is stronger than the original in [20]: $\mu(A \setminus N_n) \to \mu(A) < \infty$ implies $\mu(A \cap N_n) \to 0$. In [14] strong converse-monotone autocontinuity from below is called pseudo-order continuity.

Definition 2.5 (i) **(S):** [17] μ said to have *property (S)* if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i}\right) = 0$.

(ii) **(PS) A:** μ said to have *property (PS) at A* if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i}\right) = \mu(A)$.

(PS): [16] μ said to have *property (PS)* if μ has property (PS) at every measurable set.

(iii) **(TS) A:** μ said to have *property (TS) at A* if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i}\right) = \mu(A)$.

(TS): μ said to have *property (TS)* if μ has property (TS) at every measurable set.

(iv) **(TPS) A:** μ said to have *property (TPS) at A* if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} N_{n_i} \cap A\right) = 0$.

(TPS): μ said to have *property (TPS)* if μ has property (TPS) at every measurable set.

Definition 2.6 (i) **(S1):** [10] μ said to have *property (S1)* if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} N_{n_i}\right) \to 0$ as $k \to \infty$.

(ii) **(S2) A:** μ said to have *property (S2) at A* if $\mu(N_n) \to 0$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} N_{n_i}\right) \to \mu(A)$ as $k \to \infty$.

(S2): [10] μ said to have *property (S2)* if μ has property (S2) at every measurable set.

(iii) **(PS1) A:** μ said to have *property (PS1) at A* if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} N_{n_i} \cap A\right) \to 0$ as $k \to \infty$.

(PS1): [10] μ said to have *property (PS1)* if μ has property (PS1) at every measurable set.

(iv) **(PS2) A:** μ said to have *property (PS2) at A* if $\mu(A \setminus N_n) \to \mu(A)$ implies that there exists a subsequence $\{N_{n_i}\}$ of $\{N_n\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} N_{n_i}\right) \to \mu(A)$ as $k \to \infty$.

(PS2): [10] μ said to have *property (PS2)* if μ has property (PS2) at every measurable set.

Definition 2.7 (i) **(Ec):** [12] μ is said to satisfy *the Egoroff condition* if, for every doubly-indexed sequence $N_{m,n}$ such that $N_{m,n} \supset N_{m',n'}$ for $m \geq m'$ and $n \leq n'$ and $\mu(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} N_{m,n}) = 0$, and for every positive number ε, there exists a sequence $\{n_m\}$ such that $\mu(\bigcup_{m=1}^{\infty} N_{m,n_m}) < \varepsilon$.

(ii) **(E):** [4] μ is said to satisfy *condition (E)* if $N_n^m \downarrow N^m$ as $n \to \infty$ for every m and $\mu(\bigcup_{m=1}^{\infty} N^m) = 0$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} N_{n_i}^{m_i}\right) \to 0$ as $k \to \infty$.

(iii) **(TE)** $\boldsymbol{A}$**:** μ is said to satisfy *condition (TE) at A* if $N_n^m \downarrow N^m$ as $n \to \infty$ for every m and $\mu\left(\bigcup_{m=1}^{\infty} N^m\right) = 0$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} N_{n_i}^{m_i}\right) \to \mu(A)$ as $k \to \infty$.

(TE): μ is said to satisfy *condition (TE)* if μ satisfies condition (TE) at every measurable set.

(iv) **(PE)** $\boldsymbol{A}$**:** μ is said to satisfy *condition (PE) at A* if $N_n^m \downarrow N^m$ as $n \to \infty$ for every m and $\mu\left(A \setminus \bigcup_{m=1}^{\infty} N^m\right) = \mu(A)$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(A \setminus \bigcup_{i=k}^{\infty} N_{n_i}^{m_i}\right) \to \mu(A)$ as $k \to \infty$.

(PE): μ is said to satisfy *condition (PE)* if μ satisfies condition (PE) at every measurable set.

(v) **(TPE)** $\boldsymbol{A}$**:** μ is said to satisfy *condition (TPE) at A* if $N_n^m \downarrow N^m$ as $n \to \infty$ for every m and $\mu\left(A \setminus \bigcup_{m=1}^{\infty} N^m\right) = \mu(A)$ together imply that there exist strictly increasing sequences $\{n_i\}$ and $\{m_i\}$ such that $\mu\left(\bigcup_{i=k}^{\infty} N_{n_i}^{m_i}\right) \to 0$ as $k \to \infty$.

(TPE): μ is said to satisfy *condition (TPE)* if μ satisfies condition (TPE) at every measurable set.

The Egoroff condition is equivalent to condition (E), and each is a necessary and sufficient condition for the Egoroff theorem to hold in non-additive measure theory [4, 12]. Condition (TE) at X is called pseudo-condition (E) in [8].

Condition (M) below is defined by this research, and it is discussed in Section 6.

Definition 2.8 (M): μ is said to satisfy *condition (M)* if $\mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} N_i) = 0$ implies that for every positive number ε there exists a strictly increasing sequence $\{m_n\}$ such that $\mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{m_n} N_i) < \varepsilon$.

Definition 2.9 **a.e.:** $\{f_n\}$ is said to converge to f *almost everywhere*, written $f_n \xrightarrow{\text{a.e.}} f$, if there exists N such that $\mu(N) = 0$ and $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X \setminus N$.

p.a.e.: [20]$\{f_n\}$ is said to converge to f *pseudo-almost everywhere*, written $f_n \xrightarrow{\text{p.a.e.}} f$, if there exists N such that $\mu(X \setminus N) = \mu(X)$ and $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X \setminus N$.

a.u.: $\{f_n\}$ is said to converge to f *almost uniformly*, written $f_n \xrightarrow{\text{a.u.}} f$, if for every $\varepsilon > 0$ there exists N_ε such that $\mu(N_\varepsilon) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on $X \setminus N_\varepsilon$.

p.a.u.: [20] $\{f_n\}$ is said to converge to f *pseudo-almost uniformly*, written $f_n \xrightarrow{\text{p.a.u.}} f$, if for every $\xi < \mu(X)$ there exists N_ξ such that $\xi < \mu(X \setminus N_\xi)$ and $\{f_n\}$ converges to f uniformly on $X \setminus N_\xi$.

in meas.: $\{f_n\}$ is said to converge to f *in measure*, written $f_n \xrightarrow{\mu} f$, if for every $\varepsilon > 0$ it holds that $\mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$.

p. in meas.: [20] $\{f_n\}$ is said to converge to f *pseudo-in measure*, written $f_n \xrightarrow{\text{p.}\mu} f$, if for every $\varepsilon > 0$ it holds that $\mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\}) \to \mu(X)$.

Let $A \in \mathcal{S}$. For each convergence defined above, if $\{f_n \restriction A\}$ convergences to $f \restriction A$ on the subspace $(A, \mathcal{S} \cap A, \mu \restriction (\mathcal{S} \cap A))$, we say $\{f_n\}$ convergences to f on A and write $f_n \xrightarrow[A]{*} f$, where $f \restriction A$ denotes the restriction of f to A and $*$ stands for a.e., p.a.e., a.u., p.a.u., μ, or p.μ.

Table 1: Dual pairs

(a)	0-sub.X	$\leftrightarrow$	c.0-add.X
(b)	auto.$\uparrow X$	$\leftrightarrow$	c.auto.$\uparrow X$
(c)	m.auto.$\uparrow X$	$\leftrightarrow$	c.m.auto.$\uparrow X$
(d)	s.m.auto.$\uparrow X$	$\leftrightarrow$	s.c.m.auto.$\uparrow X$
(e)	(S)	$\leftrightarrow$	(PS) X
(f)	(TS) X	$\leftrightarrow$	(TPS) X
(g)	(S1)	$\leftrightarrow$	(PS2) X
(h)	(S2) X	$\leftrightarrow$	(PS1) X
(i)	(E)	$\leftrightarrow$	(PE) X
(j)	(TE) X	$\leftrightarrow$	(TPE) X
(k)	$\downarrow \emptyset$	$\leftrightarrow$	$\uparrow X$
(l)	$\downarrow 0$	$\leftrightarrow$	$\uparrow \mu(X)$
(m)	$\downarrow$	$\leftrightarrow$	$\uparrow$
(n)	a.e.	$\leftrightarrow$	p.a.e.
(o)	a.u.	$\leftrightarrow$	p.a.u.
(p)	in meas.	$\leftrightarrow$	p. in meas.

3 DUALITY AND ORDINALITY

Definition 3.1 The *conjugate* $\overline{\mu}$ of a finite non-additive measure μ on $(X, \mathcal{S})$ is defined by

$$\overline{\mu}(A) = \mu(X) - \mu(X \setminus A) \qquad (A \in \mathcal{S}).$$

For every finite non-additive measure μ on $(X, \mathcal{S})$, its conjugate $\overline{\mu}$ is a finite non-additive measure on $(X, \mathcal{S})$ and $\overline{\overline{\mu}} = \mu$.

We denote the class of all non-additive measure spaces by **FMS**, and the class of all finite non-additive measure spaces by **fFMS**. Note that **FMS** and **fFMS** are proper classes, i.e., they are not sets.

Definition 3.2 [11] Let P and Q be conditions concerning a non-additive measure space. P is said to be *dual* to Q when, for every $(X, \mathcal{S}, \mu) \in$ **fFMS**, $(X, \mathcal{S}, \mu)$ satisfies P iff $(X, \mathcal{S}, \overline{\mu})$ satisfies Q.

For each $(X, \mathcal{S}, \mu) \in$ **FMS**, we denote by $\Phi_{(X,\mathcal{S},\mu)}$ the family of continuous, strictly increasing functions $\varphi : [0, \mu(X)] \to [0, \infty]$ satisfying $\varphi(0) = 0$. If $(X, \mathcal{S}, \mu) \in$ **FMS** and $\varphi \in \Phi_{(X,\mathcal{S},\mu)}$, then the composite function $\varphi \circ \mu$ is a non-additive measure on $(X, \mathcal{S})$.

Definition 3.3 [11] A condition P concerning a non-additive measure space is said to be *ordinal* if $(X, \mathcal{S}, \varphi \circ \mu)$ satisfies P for every $\varphi \in \Phi_{(X,\mathcal{S},\mu)}$ whenever $(X, \mathcal{S}, \mu)$ satisfies P.

Ordinal Duality Principle [11]:
An ordinal proposition concerning a (not necessarily finite) non-additive measure space holds, then its dual also holds.

Now we examine the duality and ordinality of concepts defined in the previous section.

Proposition 3.1 *Each pair in Table 1 is dual.*

For example, (a) in Table 1 means that null-subtractivity at the whole set is dual to converse-null-additivity at the whole set. (k) and (l) are pointed out in [8], (m) is in [11], and (n)–(p) are in [9].

Proposition 3.2 *Every concept in Table 1 is ordinal.*

By the above two propositions, Ordinal Duality Principle can apply to the concepts in Table 1.

4 RELATIONS FROM (PSEUDO-)ALMOST UNIFORM CONVERGENCE

The following propositions and corollaries give implication relations of (pseudo-)almost uniform convergence to other convergences. Proposition 4.1 is obviously derived from [21]. In each of the propositions (i) and (ii) are dual to each other; one is derived from the other by Ordinal Duality Principle. On the other hand, (i) and (ii) in each of the corollaries are *not* dual in the sense of Definition 3.2.

Proposition 4.1 (i) *Null-subtractivity at the whole set is a necessary and sufficient condition for almost uniform convergence to imply pseudo-almost everywhere convergence; that is,* μ *is null-subtractive at* X *iff* $f_n \xrightarrow{\text{a.u.}} f$ *implies* $f_n \xrightarrow{\text{p.a.e.}} f$.

(ii) *Converse-null-additivity at the whole set is a necessary and sufficient condition for pseudo-almost uniform convergence to imply almost everywhere convergence.*

By Proposition 4.1, we immediately obtain the following corollary.

Corollary 4.1 (i) *Null-additivity is a necessary and sufficient condition that, for every measurable set* A*, almost uniform convergence on* A *implies pseudo-almost everywhere convergence on* A.

(ii) *Converse-null-additivity is a necessary and sufficient condition that, for every measurable set* A*, pseudo-almost uniform convergence on* A *implies almost everywhere convergence on* A.

Proposition 4.2 (i) *The following statements are equivalent.*

(a) *The non-additive measure is monotone autocontinuous from below at the whole set.*

(b) *Almost uniform convergence implies pseudo-almost uniform convergence.*

(c) *Almost uniform convergence implies convergence pseudo-in measure.*

(ii) *The following statements are equivalent.*

(a) *The non-additive measure is converse-monotone autocontinuous from below at the whole set.*

(b) *Pseudo-almost uniform convergence implies almost uniform convergence.*

(c) *Pseudo-almost uniform convergence implies convergence in measure.*

(proof) By Ordinal Duality Principle, it suffices to prove (i).
(a) $\Rightarrow$ (b). If $f_n \xrightarrow{\text{a.u.}} f$, then for every m there exists a strictly increasing sequence $\{n_k^m\}_{k=1}^{\infty}$ such that

$$\mu\left(\bigcup_{k=1}^{\infty}\bigcup_{i=n_k^m}^{\infty}\left\{x \,\middle|\, |f_i(x)-f(x)| \geq \frac{1}{k}\right\}\right) < \frac{1}{m}. \tag{1}$$

Define a doubly-indexed sequence $\{a_k^m\}$ by $a_k^m = \max\,\{n_k^1, n_k^2, \ldots, n_k^m\}$ for each k. We put $N_m = \bigcup_{k=1}^{\infty}\bigcup_{i=a_k^m}^{\infty}\{x \mid |f_i(x)-f(x)| \geq 1/k\}$. If $l \geq m$, then $a_k^l \geq a_k^m$ for all k. Hence $\{N_m\}$ is a decreasing sequence, and from(1) it follows that $\mu(N_m) \to 0$ as $m \to \infty$. By monotone autocontinuity from below at X, we obtain $\mu(X \setminus N_m) \to \mu(X)$ as $m \to \infty$, and obviously f_n converges to f uniformly on $X \setminus N_m$. Therefore we have $f_n \xrightarrow{\text{p.a.u.}} f$.
(b) $\Rightarrow$ (a). Let $N_n \downarrow N$ and $\mu(N_n) \to 0$ as $n \to \infty$, and define a sequence $\{f_n\}$ of measurable functions by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in X \setminus N_n, \\ 1 & \text{if } x \in N_n, \end{cases} \qquad (n \geq 1)$$

and a measurable function f by

$$f(x) = \begin{cases} 0 & \text{if } x \in X \setminus N, \\ 2 & \text{if } x \in N. \end{cases}$$

Then f_n converges to f uniformly on $X \setminus N$, and since $\mu(N) = 0$, we have $f_n \xrightarrow{\text{a.u.}} f$. By hypothesis, it follows that $f_n \xrightarrow{\text{p.a.u.}} f$. Thus, for every $\xi < \mu(X)$, there exists N'_ξ such that $\xi < \mu(X \setminus N'_\xi)$ and f_n converges to f uniformly on $X \setminus N'_\xi$. By the definitions of f_n and f, there exists n such that $X \setminus N'_\xi \subset X \setminus N_n$ and hence $\mu(X \setminus N'_\xi) \leq \mu(X \setminus N_n)$. Therefore $\mu(X \setminus N_n) \to \mu(X)$ as $n \to \infty$.

(b) $\Rightarrow$ (c). From the proof of (a) $\Rightarrow$ (b), if $f_n \xrightarrow{\text{a.u.}} f$, then $f_n \xrightarrow{\text{p.a.u.}} f$. From [21], it is obvious that $f_n \xrightarrow{\text{p.a.u.}} f$ implies $f_n \xrightarrow{\text{p.}\mu} f$.

(c) $\Rightarrow$ (a). It is similar to the proof of (b) $\Rightarrow$ (a). ∎

By Proposition 4.2, we immediately obtain the following corollary.

Corollary 4.2 (i) *The following statements are equivalent.*

- (a) *The non-additive measure is monotone autocontinuous from below at every measurable.*
- (b) *For every measurble set A, almost uniform convergence on A implies pseudo-almost uniform convergence on A.*
- (c) *For every measurble set A, almost uniform convergence on A implies convergence pseudo-in measure on A.*

(ii) *The following statements are equivalent.*

- (a) *The non-additive measure is converse-monotone autocontinuous from below at every measurable set.*
- (b) *For every measurble set A, pseudo-almost uniform convergence on A implies almost uniform convergence on A.*
- (c) *For every measurble set A, pseudo-almost uniform convergence on A implies convergence in measure on A.*

5 RELATIONS BETWEEN CONVERGENCES

The results in the previous section are summarized together with existing ones [4, 6, 8, 10, 14, 17, 19, 20, 21] into Tables 2 and 3.

Table 2 shows necessary and sufficient conditions for implications between the six convergences on the whole set X. The cell at row r and column c indicates a necessary and sufficient condition for r-type convergence to imply c-type convergence; for example, condition (E) is a necessary and sufficient condition for almost everywhere convergence to imply almost uniform convergence. The symbol $\varnothing$ indicates the implication holds unconditionally. A cell $\boxed{}$ shows a condition for the Riesz-type theorem; for example, property (S) is a necessary and sufficient condition that $f_n \xrightarrow{\mu} f$ implies that there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \xrightarrow{\text{a.e.}} f$ as $i \to \infty$.

Table 3 shows necessary and sufficient conditions for implications between the six convergences on every measurable set; for example, condition (E) is a necessary and sufficient condition that, for every $A \in \mathcal{S}$, $f_n \xrightarrow[A]{\text{a.e.}} f$ implies $f_n \xrightarrow[A]{\text{a.u.}} f$. Table 3 is derived from Table 2.

The results indicating (TS) X and (TPS) X in Table 2 and (TS) and (TPS) in Table 3 are derived from [10, Theorem 5 and its proof] by removing the assumption of continuity of non-additive measures. Each of the other results without reference number follows from the result (and its proof) in the corresponding cell of the other table. For example, "(a.e. $\Rightarrow$ p. in meas.) $\Leftrightarrow$ s.m.auto.$\uparrow$ X" in Table 2 is derived from "$\forall A \in \mathcal{S}$ (a.e. on $A \Rightarrow$ p. in meas. on A) $\Leftrightarrow$ (0-add. & $\uparrow$)" in Table 3 and its proof in [14]; in this case, the equivalence "(0-add. & $\uparrow$) $\Leftrightarrow$ s.m.auto. $\uparrow$" is derived.

6 CONDITIONS FOR THE EGOROFF THEOREM

The Egoroff theorem, which asserts that almost everywhere convergence implies almost uniform convergence, is one of the most important convergence theorems in classical measure theory. In non-additive

Table 2: Necessary and sufficient conditions for implications between convergences on the whole set X

r $\Rightarrow$ c	a.e.	p.a.e.	a.u.	p.a.u.	in meas.	p. in meas.
a.e.	$\varnothing$	0-sub. X	(E) [4]	(TE) X [8]	$\downarrow$0	s.m.auto.$\uparrow$ X
p.a.e.	c.0-add. X	$\varnothing$	(TPE) X [8]	(PE) X [8]	s.c.m.auto.$\uparrow$ X	$\uparrow \mu(X)$
a.u.	$\varnothing$ [19]	0-sub. X [Prop.4.1(i)]	$\varnothing$	m.auto.$\uparrow$ X [Prop.4.2(i)]	$\varnothing$ [19]	m.auto.$\uparrow$ X [Prop.4.2(i)]
p.a.u.	c.0-add. X [Prop.4.1(ii)]	$\varnothing$	c.m.auto.$\uparrow$ X [Prop.4.2()]	$\varnothing$	c.m.auto.$\uparrow$ X [Prop.4.2()]	$\varnothing$
in meas.	(S)	(TS) X	(S1)	(S2) X	$\varnothing$	auto.$\uparrow$ X
p. in meas.	(TPS) X	(PS) X	(PS1) X	(PS2) X	c.auto.$\uparrow$ X	$\varnothing$

Table 3: Necessary and sufficient conditions for implications between convergences on all measurable sets

$\forall A$(r $\Rightarrow$ c)	a.e.	p.a.e.	a.u.	p.a.u.	in meas.	p. in meas.
a.e.	$\varnothing$	0-add. [20, 21]	(E)	(TE)	$\downarrow$0 [14]	0-add. & $\uparrow$ [14]
p.a.e.	c.0-add. [20, 21]	$\varnothing$	(TPE)	(PE)	s.c.m.auto.$\uparrow$ [14]	$\uparrow$ [14]
a.u.	$\varnothing$	0-add. [Cor.4.1(i)]	$\varnothing$	m.auto.$\uparrow$ [Cor.4.2(i)]	$\varnothing$	m.auto.$\uparrow$ [Cor.4.2(i)]
p.a.u.	c.0-add. [Cor.4.1(ii)]	$\varnothing$ [21]	c.m.auto.$\uparrow$ [Cor.4.2(ii)]	$\varnothing$	c.m.auto.$\uparrow$ [Cor.4.2(ii)]	$\varnothing$ [21]
in meas.	(S) [17]	(TS)	(S1) [10]	(S2) [10]	$\varnothing$	auto.$\uparrow$ [20]
p. in meas.	(TPS)	(PS) [6]	(PS1) [10]	(PS2) [10]	c.auto.$\uparrow$ [20]	$\varnothing$

measure theory, this theorem does not hold without additional conditions. So far, it has been shown that the Egoroff condition [12] and condition (E) [4] each is a necesary and sufficient condition for the Egoroff theorem to hold in non-additive measure theory. Both of the conditions are described by a doubly-indexed sequence of measurable sets, and no necessary and sufficient condition described by a single-indexed sequence has been given yet. On the other hand, the Egoroff theorem has a necessary condition described by a single-indexed sequence (strong order continuity [7, 12]) and sufficient conditions described by a single-indexed sequence (continuity from above and below [5], the conjunction of strong order continuity and property (S) [7, 12]). In this section we give new conditions described by a single-indexed sequence; condition (M) is a necessary condition stronger than strong order continuity, and the conjunction of condition (M) and null-continuity is a sufficient condition weaker than the above-mentioned two sufficient conditions.

In non-additive measure theory, there are four types of the Egoroff theorem: the original ("a.e. $\Rightarrow$ a.u.") and its three variations ("a.e. $\Rightarrow$ p.a.u.", "p.a.e. $\Rightarrow$ a.u.", and "p.a.e. $\Rightarrow$ p.a.u."). This section discusses the original only; the corresponding results on the three variations can be obtained similarly.

Lemma 6.1 *Condition (M) implies strongly order continuity.*

(proof) Let $N_n \downarrow N$ and $\mu(N) = 0$. Then $\mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} N_i) = \mu(N) = 0$. It follows from condition (M) that for every $\varepsilon > 0$ there exists a strictly increasing sequence $\{m_n\}$ such that $\mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{m_n} N_i) < \varepsilon$. Since $\bigcap_{i=n}^{m_n} N_i = N_{m_n}$ and $\bigcup_{n=1}^{\infty} N_{m_n} = N_{m_1}$, it follows that $\varepsilon > \mu(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{m_n} N_i) = \mu(N_{m_1})$. Therefore μ is strongly order continuous. ■

Proposition 6.1 *The conjunction of condition (M) and null-continuity is a sufficient condition for the Egoroff theorem.*

(proof) If $f_n \xrightarrow{\text{a.e.}} f$, then $\mu(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x \mid |f_i(x) - f(x)| \geq 1/k\}) = 0$. Thus, for every k $\mu(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x \mid |f_i(x) - f(x)| \geq 1/k\}) = 0$. Since μ is strongly order continuous by Lemma 6.1, it follows that $\mu(\bigcup_{i=n}^{\infty}\{x \mid |f_i(x)-f(x)| \geq 1/k\}) \to 0$ as $n \to \infty$. Therefore, there exists an increasing sequence $\{n_k\}$ such that

$$\mu\left(\bigcup_{i=n_k}^{\infty}\left\{x \,\middle|\, |f_i(x)-f(x)| \geq \frac{1}{k}\right\}\right) < \frac{1}{k} \qquad \text{for every } k. \tag{2}$$

If we put $N_k = \bigcup_{i=n_k}^{\infty}\{x \mid |f_i(x)-f(x)| \geq 1/k\}$ for each k, then (2) implies that $\mu(N_k) \to 0$. Since $N_k \supset \bigcap_{i=l}^{\infty} N_i$ for $k \geq l$, it follows that $\mu(\bigcap_{i=l}^{\infty} N_i) = 0$ for every l. By null-continuity, we have $\mu(\bigcup_{l=1}^{\infty}\bigcap_{i=l}^{\infty} N_i) = 0$. Condition (M) implies that for every positive number ε there exists $\{m_l\}$ such that $\mu(\bigcup_{l=1}^{\infty}\bigcap_{k=l}^{m_l} N_k) < \varepsilon$. It follows that

$$\bigcup_{l=1}^{\infty}\bigcap_{k=l}^{m_l} N_k = \bigcup_{l=1}^{\infty}\bigcap_{k=l}^{m_l}\bigcup_{i=n_k}^{\infty}\left\{x \,\middle|\, |f_i(x)-f(x)| \geq \frac{1}{k}\right\} \supset \bigcup_{l=1}^{\infty}\bigcup_{j=n_{m_l}}^{\infty}\left\{x \,\middle|\, |f_j(x)-f(x)| \geq \frac{1}{l}\right\}.$$

Since

$$X \setminus \bigcup_{l=1}^{\infty}\bigcup_{j=n_{m_l}}^{\infty}\left\{x \,\middle|\, |f_j(x)-f(x)| \geq \frac{1}{l}\right\} = \bigcap_{l=1}^{\infty}\bigcap_{j=n_{m_l}}^{\infty}\left\{x \,\middle|\, |f_j(x)-f(x)| < \frac{1}{l}\right\} \supset X \setminus \bigcup_{l=1}^{\infty}\bigcap_{k=l}^{m_l} N_k,$$

it follows that f_n converges f uniformly on $X \setminus \bigcup_{l=1}^{\infty}\bigcap_{i=l}^{m_l} N_i$. This shows $f_n \xrightarrow{\text{a.u.}} f$. ■

Proposition 6.2 *Condition (M) is a necessary condition for the Egoroff theorem.*

(proof) We prove that the Egoroff condition implies condition (M). Let $\{N_n\}$ satisfy $\mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} N_i) = 0$. Define a sequence $\{N_{k,l}\}$ as

$$N_{k,l} = \begin{cases} \displaystyle\bigcup_{i=l}^{k} N_i & \text{if } k > l, \\ \displaystyle\bigcap_{i=k}^{l} N_i & \text{if } k \leq l. \end{cases}$$

Then the sequence $\{N_{m,n}\}$ is increasing with respect to m and decreasing with respect to n. Since $\mu(\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} N_{m,n}) = \mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{\infty} N_i) = 0$, it follows that for every positive number ε there exists a sequence $\{n_m\}$ such that $\mu(\bigcup_{m=1}^{\infty} N_{m,n_m}) < \varepsilon$. We can let $\{n_m\}$ satisfy $n_m \geq m$. Then $\bigcup_{m=1}^{\infty} N_{m,n_m} = \bigcup_{m=1}^{\infty}\bigcap_{i=m}^{n_m} N_i$ and therefore $\mu(\bigcup_{m=1}^{\infty}\bigcap_{i=m}^{n_m} N_i) < \varepsilon$. ■

In [12], the implications between condition (E), or the Egoroff theorem, and related conditions are summarized as Fig. 1. In this diagram, a directed path from A to B means that condition A implies condition B, and the absence of such a directed path means that A does not imply B. An addition of the results in this section to Fig. 1 yields Fig. 2. This diagram shows that the conjunction of condition (M) and null-continuity is strictly weaker than continuity from above and below and the conjunction of strong order continuity and property (S) each, and is independent of strong order total continuity. Since there exists a non-additive measure space where condition (E) is satisfied without strong order total continuity and null-continuity [12, Example 5], condition (E) is strictly weaker than "T $\downarrow$ 0 or '(M) & $\uparrow$ 0' ". In addition, condition (M) is strictly stronger than strong order continuity. The symbol "?" indicates that the implication from condition (M) to condition (E) has not been clear yet.

7 CONCLUDING REMARKS

In this paper, we have summarized the implication relationship between six convergence concepts: almost everywhere convergence, almost uniform convergence, convergence in measure, and their pseudo

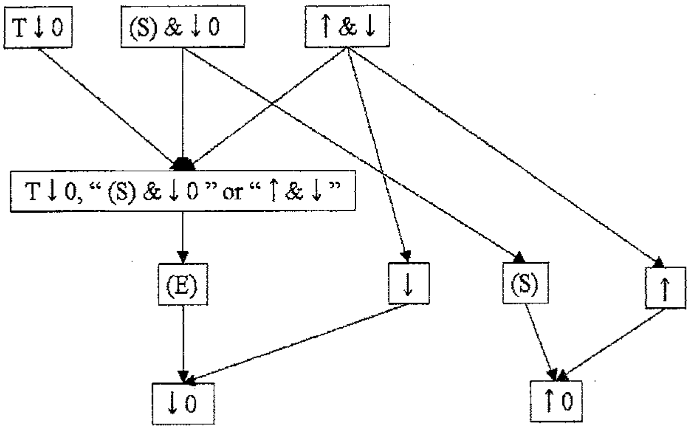

Figure 1: Implication relationship without condition (M) [12]

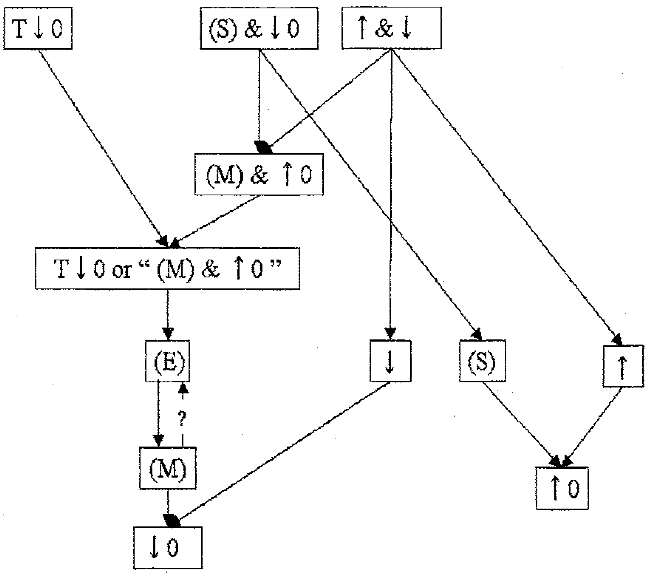

Figure 2: Implication relationship with condition (M)

versions. In addition, for the Egoroff theorem we give a new necessary condition, condition (M), and a new sufficient condition, the conjunction of condition (M) and null-continuity.

The implication from condition (M) to condition (E) has not been clear yet. So the investigation of this implication is an important subject to future research. Another subject is a study of relationship with other convergence concepts such as mean convergence and convergence in distribution.

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