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Another Interpretation of Measurable Norms and Cylindrical Measures.

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1 Introduction

Gross [5] presented a new notion "measurable norm", which was the start of the splendid and fantastic theory of "abstract Wiener spaces". This gives a condition satisfying that the Gauss cylindrical measure extends to a measure. In 1971, Dudley-Feldman-LeCam [3] defined another notion of measurable norms. If the norm is continuous, then this is a necessary and sufficient condition for satisfying that a general cylindrical measure extends to a measure.

Badrikian and Chevet [1] offered the following question:

"Do these notions of two measurable norms coincide with each other for every cylindrical measure?"

In 1984 Maeda [15] solved this problem negatively. These two notions are very close, but subtly different.

Many mathematicians noted conditions involved with these two measurable norms and investigated them. They gave similar conditions to these measurable norms. In Section 3, we give seven conditions approximating to measurable norms. And we show the relation of seven conditions.

We introduce a new definition of a Gauss cylindrical measures which was introduced by Baxendale [2]. There we compare it with the Gauss cylindrical measure defined in the original sense. The Gauss cylindrical measures introduced by Baxendale contain ones without variance, so that they extends the original ones. We study Baxendale's Gauss cylindrical measures in detail.

2 Preliminaries

Let X be a locally convex Hausdorff topological vector space over the real field $\mathbb{R}$, X' its topological dual, $(\cdot,\cdot)$ the natural pairing between X and X' and $\mathcal{B}(X)$ the Borel σ-algebra of X. Let $\{\xi_1,\dots,\xi_n\}$ be a finite system of elements of X'. Then by Ξ we denote the operator from X into $\mathbb{R}^n$ mapping x onto the vector $((x,\xi_1),\dots,(x,\xi_n))$. A set $Z\subset X$ is said to be a cylindrical set if there are $\xi_1,\dots,\xi_n\in X'$ and $B\in\mathcal{B}(\mathbb{R}^n)$

such that $Z = \Xi^{-1}(B)$. Let $\mathcal{C}_X$ denote the collection of all cylindrical sets of X.

A map μ from $\mathcal{C}_X$ into $[0,1]$ is called a cylindrical measure if it satisfies the following conditions:

(1) $\mu(X) = 1$;

(2) Restrict μ to a σ-algebra of cylindrical sets which are generated by a fixed finite system of functionals. Then each such restriction is countably additive.

By putting $\mu_{\xi_1,\ldots,\xi_n}(B) = \mu(\Xi^{-1}(B))$ each cylindrical measure μ defines a family of Borel probability measures.

Next we introduce two kinds of measurable norms defined on a Hilbert space. Let H be a real separable Hilbert space with norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $\mathcal{F}$ be the partially ordered set of all finite dimensional orthogonal projections of H and $FD(H)$ the family of all finite dimensional subspaces of H. $P > Q$ means $PH \supset QH$ for $P, Q \in \mathcal{F}$. Also a subset E of H of the following form is a cylindrical set, $E = \{x \in H; Px \in F\}$, where $P \in \mathcal{F}$ and F is a Borel subset of PH.

First, we define the canonical Gauss cylindrical measure and two measurable norms.

Definition 2.1 *The canonical Gauss cylindrical measure is a cylindrical measure γ from $\mathcal{C}_H$ into $[0,1]$ defined as follows:*

If $E = \{x \in H; Px \in F\}$, then

$$\gamma(E) = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n \int_F e^{-\frac{|x|^2}{2}}\, dx,$$

where $n = \dim PH$ and dx is the Lebesgue measure on PH.

Remark 2.2 *If H is an infinite dimensional space, then γ is finitely additive, but is not σ-additive. In general, we denote by $\gamma^t(Z) = (\frac{1}{\sqrt{2\pi t}})^n \int_F e^{-\frac{|x|^2}{2t}}\, dx$ the Gauss cylindrical measure with parameter t $(0 < t < \infty)$. γ^1 is the canonical Gauss cylindrical measure. In this paper, we denote the canonical Gauss cylindrical measure by γ.*

We define two measurable norms.

Definition 2.3 *A semi-norm $\|\cdot\|$ in H is said to be γ-(G)measurable if for every $\varepsilon > 0$, there exists $P_0 \in \mathcal{F}$ such that $\gamma(\{x \in H; \|Px\| > \varepsilon\}) < \varepsilon$ for $\forall P \perp P_0$ and $P \in \mathcal{F}$.*

This concept was introduced by Gross [7] in 1962. It was the starting point of the successive research concerning the abstract Wiener space. In Definition 2.3, we can replace γ with μ which is any cylindrical measure defined on H. In such a case we say that $\|\cdot\|$ is μ-(G)measurable. We can also redefine the above concept as follows:

We say that $\|\cdot\|$ is μ-(G)measurable if for every $\varepsilon > 0$, there exists $G \in FD(H)$ such that $\mu(N_\varepsilon \cap F + F^\perp) \geq 1 - \varepsilon$ whenever $F \in FD(H)$ and $F \perp G$, where $N_\varepsilon = \{x \in H; \|x\| \leq \varepsilon\}$ and $F^\perp$ is the orthogonal complement of F.

Definition 2.4 *A semi-norm $\|\cdot\|$ is said to be μ-(D)measurable if for every $\varepsilon > 0$ there exists $G \in FD(H)$ such that $\mu(P_F(N_\varepsilon) + F^\perp) \geq 1 - \varepsilon$ whenever $F \in FD(H)$ and $F \perp G$, where P_F is the orthogonal projection of H onto F.*

This was introduced by Dudley-Feldman-LeCam [3] in 1971.
They showed that these two measurable norms are equivalent for the canonical Gauss cylindrical measure [3].

Theorem 2.5 ([3]) *Let $\|\cdot\|$ be a continuous semi-norm in a Hilbert space H, and γ the canonical Gauss cylindrical measure on H. Then the following statements are equivalent.*

(i) $\|\cdot\|$ *is γ-(D)measurable.*

(ii) $\|\cdot\|$ *is γ-(G)measurable.*

Let E be the completion of H with respect to the norm $\|\cdot\|$ and i the inclusion map of H into E. If $\|\cdot\|$ is γ-measurable, then the triple (i, H, E) is called an *abstract Wiener space.*

Theorem 2.6 ([2]) *Let $\|\cdot\|$ be a continuous norm defined on a Hilbert space H, μ a cylindrical measure on H, and B the completion of H with respect to $\|\cdot\|$. Then the following are equivalent.*

(i) $\|\cdot\|$ *is μ-(D)measurable.*

(ii) $i(\mu)$, where $i(\mu)$ is the image of μ under the map i, is countably additive and is extensible to a measure.

It is easy to see that (G)measurability implies (D)measurability. But the converse is false generally ([15], this is the 1984-Example).

We define the characteristic functions and the continuity of cylindrical measures.

Definition 2.7 *The characteristic function ϕ of a cylindrical measure μ on H is defined by*

$$\phi(\xi) = \int_H e^{i\langle \xi, x\rangle}\, \mu(dx)$$

where $\xi \in H$.

We define the continuity of cylindrical measures.

Definition 2.8 *Let μ be a cylindrical measure on H. The μ is said to be continuous if the characteristic function of μ is continuous.*

We introduce the following proposition. It is useful to check the continuity of cylindrical measures.

Proposition 2.9 ([12]) *Let μ be a cylindrical measure on H. Then the following are equivalent.*
(i) μ is continuous.
(ii) If the generalized sequence $\{\xi_\alpha\} \subset H$ tends to zero, then we have

$$\lim_{\alpha} \mu_{\xi_\alpha}([-\sigma, \sigma]) = 1$$

for some (each) $\sigma > 0$.
(iii) Suppose $\lim_\alpha \xi_\alpha = 0$. Then it follows that μ_{ξ_α} weakly converges to δ_0.

3 Measurable norms and seven conditions

In 1980's, Maeda compared two measurable norms (Gross' and D.F.L.'s sense respectively). She considered only two measurabilities but not other conditions. However there exist several conditions approximating to these measurabilities. In this section, we treat those conditions and research the relation between them.

In the following theorem, Baxendale showed that (i), (ii), (iii) and (vi) are equivalent for the canonical Gauss cylindrical measure. We have that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ for general cylindrical measures by a similar method. Condition (iv) was given by Yan and Gong. Yan proved that $(iv) \Rightarrow (vii)$ for any reflexive Banach space B. After that, Gong showed it for an arbitrary Banach space B. The equivalence of (v) and (vi) was given in Theorem 2.7. The following theorem is a gathering of the above.

Theorem 3.1 *Let H be a real separable Hilbert space with norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$, μ a cylindrical measure on H, $\|\cdot\|$ a continuous norm defined on H, B the completion of H with respect to $\|\cdot\|$ and i the inclusion map from H into B. Moreover, let Y be the bidual B'' of B with weak*-topology $\sigma(B'', B')$ and j be the inclusion map from H into Y. Then the following seven conditions satisfy the relations: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, $(i) \Rightarrow (ii) \Rightarrow (v) \Leftrightarrow (vi) \Rightarrow (vii)$.*
If μ is continuous (this means that the characteristic function of μ is continuous on H), then the following conditions satisfy the relations: $(iii) \Rightarrow (vi)$ and $(iv) \Rightarrow (vii)$.

(i) For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, such that $n > m \geq N$ implies

$$\mu(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \varepsilon$$

for every sequence $\{P_n\} \subset \mathcal{F}$ such that $P_n < P_{n+1}$ ($\forall n \in \mathbb{N}$) and P_n converges strongly to the identity map I (we write it $P_n \nearrow I$).
(ii) $\|\cdot\|$ is a μ-(G)measurable norm.
(iii) There exists a sequence $\{P_n\} \subset \mathcal{F}$ with $P_n \nearrow I$ which has the property that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mu(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \varepsilon$$

for every $n > m \geq N$.
(iv) There exists a sequence $\{P_n\} \subset \mathcal{F}$ with $P_n \nearrow I$, which has the property that for any $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $n_\varepsilon \in \mathbb{N}$ such that

$$\mu(\{x \in H;\ \sup_{1 \leq k \leq n} \|P_k x\| > N\}) < \varepsilon$$

for every $N \geq N_\varepsilon$ and every $n \geq n_\varepsilon$.
(v) $\|\cdot\|$ is a μ-(D)measurable norm.
(vi) $i(\mu)$ (i.e. $\mu \circ i^{-1}$) is extensible to a measure.
(vii) $j(\mu)$ is extensible to a measure.

4 The definition of Gauss cylindrical measures introduced by Baxendale

The comparison between Gauss cylindrical measures by Baxendale and the original one was researched in [26]. Here we concretely construct a cylindrical measure on ℓ^2 using the same $\mathbf{a}$ as we introduced in Section 4, and we denote it by $\gamma_{\mathbf{a}}$. We mention the definition of $\gamma_{\mathbf{a}}$ later. It is a Gauss cylindrical measure in the sense of Baxendale but it is not the original Gauss cylindrical measure. In this section, we study seven conditions in Theorem 3.1 comparing γ and $\gamma_{\mathbf{a}}$.

First we define Gauss cylindrical measures by Baxendale.

Definition 4.1 ([2]) *Let E be a separable Banach space.*
(a) A Borel probability measure λ on $\mathbb{R}$ is Gaussian if either
(i) $\lambda = \delta_0$
or
(ii) There exists $t > 0$ such that for $B \in \mathcal{B}(\mathbb{R})$
$$\lambda(B) = \frac{1}{\sqrt{2\pi t}} \int_B e^{-\frac{|x|^2}{2t}}\, dx.$$
(b) A cylindrical measure μ is a Gauss cylindrical measure if every one-dimensional distribution μ_ξ of μ (i.e. the image of μ under $\xi : E \to \mathbb{R}$) is Gaussian on $\mathbb{R}$
(c) A Gauss measure on E is a Borel probability measure on E which restricts to a Gauss cylindrical measure.

Definition 4.2 *Let E be a separable Banach space and μ a Gauss cylindrical measure on E and $\phi(\mu, \xi)$ the characteristic function of μ. Let $L(E, E')$ be the family of all bounded linear operators from E into E'. We say that μ has a variance A if there is a self adjoint $A \in L(E, E')$, such that $\phi(\mu, \xi) = \exp\{-(\xi, A\xi)/2\}$ for all $\xi \in E'$. Here A is called self adjoint if $(\xi, A\eta) = (\eta, A\xi)$ for any ξ, $\eta \in E'$.*

Let $\mathbf{a}$ be an element of ℓ^{2*} such that
$(\mathbf{a}, e_n) = 1$ for $n = 1, 2, \ldots$,
$(\mathbf{a}, e_\alpha) = 0$ for $e_\alpha \in \mathcal{J} \backslash \{e_n\}_{n=1,2,\ldots}$.

Let $\gamma_{\mathbb{R}}$ be the canonical Gauss measure on $\mathbb{R}$, $\gamma_{\mathbf{a}}^*$ the measure induced by a function

$f : \mathbb{R} \to (\ell^2)^*$ $(t \to t\mathbf{a})$. i.e. $\gamma_{\mathbf{a}}^*$ is the image measure of $\gamma_{\mathbb{R}}$ under f. Define a cylindrical measure $\gamma_{\mathbf{a}}$ on ℓ^2 as follows;
For

$$Z = \{x \in (\ell^2)^*; ((x, \xi_1), \ldots, (x, \xi_n)) \in D\}$$

and

$$\tilde{Z} = \{x \in \ell^2; (\langle x, \xi_1\rangle, \ldots, \langle x, \xi_n\rangle) \in D\}$$

Put $\gamma_{\mathbf{a}}(\tilde{Z}) = \gamma_{\mathbf{a}}^*(Z)$, where $\xi_1, \ldots, \xi_n \in \ell^2$ and $D \in \mathcal{B}(\mathbb{R}^n)$.
Then $\gamma_{\mathbf{a}}$ is a Gauss cylindrical measure in the sense of Baxendale, and is not the canonical Gauss cylindrical measure. Further, it does not coincide with γ^t and does not have a variance.

We have the following results for $\|\cdot\|_1$ and $\|\cdot\|_4$ and $\gamma_{\mathbf{a}}$. These results are shown by using a similar method in the case of $\mu_{\mathbf{a}}$.

It is well-known that $\|\cdot\|_4$ is measurable for canonical Gauss cylindrical measures. And we do not know whether $\|\cdot\|_1$ is measurable for canonical Gauss cylindrical measures. We have that this two norms $\|\cdot\|_1$ and $\|\cdot\|_4$ are (D)measurable with respect to $\gamma_{\mathbf{a}}$ which is a Gauss cylindrical measure in the sense of Baxendale.

Theorem 4.3 *$\|\cdot\|_1$ is $\gamma_{\mathbf{a}}$-(D)measurable.*

Theorem 4.4 *$\|\cdot\|_4$ is $\gamma_{\mathbf{a}}$-(D)measurable.*

The notion of measurable norms by Gross and the notion of measurable norms by D.F.L. are equivalent for canonical Gauss cylindrical measures. We show the existence of a Gauss cylindrical measure in the the sense of Baxendale that the notions of these two measurability do not coincide.

Theorem 4.5 *$\|\cdot\|_2$ is $\gamma_{\mathbf{a}}$-(D)measurable.*

Proof. Let E be the completion of ℓ^2 with respect to the norm $\|\cdot\|_2$, and j the inclusion map of ℓ^2 into E. Denote by j' the dual operator of j. Let $(\cdot,\cdot)_E$ be the natural pairing $E' \times E \to \mathbb{R}$.

$$E' \xrightarrow{j'} (\ell^2)' \simeq \ell^2 \xrightarrow{j} E$$

To prove that the norm $\|\cdot\|_2$ is $\gamma_{\mathbf{a}}$-(D)measurable, it suffices to show that $j(\mu_{\mathbf{a}})$, the image of $\gamma_{\mathbf{a}}$ under the map j, is σ-additive on $(E, \mathcal{C}_E)$.
Claim. $\mathbf{a}$ vanishes on $j'(E')$.
Proof of claim. Suppose $y \in E'$ is given. We have to show that $(\mathbf{a}, j'(y)) = 0$. Since $(\mathbf{a}, e_\alpha) = 0$ for all $e_\alpha \in \mathcal{J} \setminus \{e_n\}_{n=1,2,\ldots}$, we may assume $j'(y)$ is of the form $\sum_{n=1}^N A_n e_n$, where $A_1, A_2, \ldots, A_N \in \mathbb{R}$.

Now define a sequence $\{x^m\}_{m=1,2,\ldots}$ in ℓ^2 by

$$\begin{aligned} x^1 &= e_1, \\ x^2 &= e_1 + e_2 + e_3, \\ &\vdots \\ x^m &= e_1 + \ldots + e_{2m-1}, \\ &\vdots \end{aligned}$$

Then $\langle e_k, x^m \rangle = 1$ for $m \geq (k+1)/2$, so $\langle j'(y), x^m \rangle = \sum_{n=1}^{N} A_n$ for all $m \geq N$. Moreover, since $\langle j'(y), x^m \rangle = (y, j(x^m))_E$, we obtain $(y, j(x^m))_E = \sum_{n=1}^{N} A_n$ for all $m \geq N$. Therefore,

$$\lim_{m \to \infty} (y, j(x^m))_E = \sum_{n=1}^{N} A_n. \tag{1}$$

But by constructions of $\{\beta_m\}$ and U_2, we know that $\beta_{2m-1} x^m \in U_2$, so that $\|x^m\|_2 \leq 1/\beta_{2m-1}$. The assumption $\beta_{2m-1} \to \infty$ as $m \to \infty$ implies $\|x^m\|_2 \to 0$ as $m \to \infty$. Therefore,

$$\lim_{m \to \infty} j(x^m) = 0 \text{ in } E. \tag{2}$$

Thus by (17) and (18) we deduce that $\sum_{n=1}^{N} A_n = 0$, and hence $(\mathbf{a}, j'(y)) = \sum_{n=1}^{N} A_n = 0$. This completes the proof of our claim.

Let i be the canonical map of $(\ell^2)^*$ into $(E')^*$. Then our claim implies $i(\mathbf{a}) = 0$, so that $i(\gamma_{\mathbf{a}}^*)$ is the Dirac measure $\gamma_0 = \delta_0$ on $(E')^*$. Therefore, $j(\gamma_{\mathbf{a}})$ is extendable to δ_0 on E, so it is σ-additive on $(E, \mathcal{C}_E)$. □

Theorem 4.6 *$\|\cdot\|_2$ is not $\gamma_{\mathbf{a}}$-(G)measurable.*

Proof. It suffices to show that there exists a number $\varepsilon_0 > 0$ such that for every $G \in FD(\ell^2)$ there exists $F \in FD(\ell^2)$ satisfying $F \perp G$ and $\gamma_{\mathbf{a}}(\varepsilon_0 U_2 \cap F + F^{\perp}) < 1 - \varepsilon_0$.

Let G be an arbitrary finite dimensional subspace of ℓ^2, and $\{\xi^j\}_{j=1,2,\ldots,n}$ be a CONS of G. Then each ξ^j is of the form $\xi^j = \sum_{i=1}^{\infty} \alpha_i^j e_i$ where $\alpha_i^j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots$. Then we have the following matrix A:

$$A = \begin{pmatrix} \alpha_1^1 & \ldots & \alpha_n^1 & \ldots & \alpha_{n+m}^1 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ \alpha_1^n & \ldots & \alpha_n^n & \ldots & \alpha_{n+m}^n \end{pmatrix},$$

where m is chosen such that rank $A = n$. Suppose $N > n + m$. Then the next equation has its solution in $\mathbb{R}^{n+m}$.

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} -\alpha^1_{2N+1} \\ \vdots \\ -\alpha^n_{2N+1} \end{pmatrix}. \tag{3}$$

By the construction we know that $\alpha^j_i \to 0$ as $i \to \infty$ for $j = 1, 2, \ldots, n$. Therefore, for every $\delta > 0$, we may choose a positive integer $N\,(> n+m)$, N sufficiently large, such that the equation (19) has a solution $x_1 = \eta_1, \ldots, x_{n+m} = \eta_{n+m}$ satisfying

$$\max_{1 \le l \le n+m} |\eta_l| < \delta. \tag{4}$$

Now choose a number $\delta > 0$ in (20) such that

$$\frac{\eta_1 + \eta_2 + \ldots + \eta_{n+m} + 1}{(\eta_1)^2 + (\eta_2)^2 + \ldots + (\eta_{n+m})^2 + 1} > \frac{1}{2}. \tag{5}$$

Let $\tau = \eta_1 e_1 + \ldots + \eta_{n+m} e_{n+m} + e_{2N+1}$ and F be the one dimensional subspace of ℓ^2 generated by τ. $\langle \tau, \xi^j \rangle = 0$ for $j = 1, 2, \ldots, n$, so that $F \perp G$.

Put $\phi = \frac{\tau}{|\tau|}$, where $|\cdot|$ is the Hilbert norm of ℓ^2, then

$$(\mathbf{a}, \phi) = \frac{(\mathbf{a}, \tau)}{|\tau|} = \frac{\eta_1 + \eta_2 + \ldots + \eta_{n+m} + 1}{({\eta_1}^2 + \ldots + {\eta_{n+m}}^2 + 1)^{\frac{1}{2}}}.$$

Let $\varepsilon_1 = \int_{(\mathbf{a},\phi)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$ then $0 < \varepsilon_1 < \frac{1}{2}$. Put $\varepsilon_0 = \min(\varepsilon_1, \frac{1}{6})$.

Now we show that $(\mathbf{a}, \phi)\phi \notin \varepsilon_0 U_2$. Suppose this is not the case. Then $(\mathbf{a}, \phi)\phi \in \varepsilon_0 U_2$ implies that $(\mathbf{a}, \phi)\phi$ may be expressed as $(\mathbf{a}, \phi)\phi = X + Y$, where $X \in \varepsilon_0 \Gamma_2$ and $Y \in \varepsilon_0 B_1$. Since $X, Y \in \ell^2$, X and Y are of the form $X = \sum_{i=1}^{\infty} X_i e_i$ and $Y = \sum_{i=1}^{\infty} Y_i e_i$, where $X_i, Y_i \in \mathbb{R}$ for $i = 1, 2, \ldots$. Then $(\mathbf{a}, \phi)\phi = \sum_{i=1}^{\infty} (X_i + Y_i) e_i$ and by (21) we have

$$X_{2N} + Y_{2N} = 0$$

and

$$X_{2N+1} + Y_{2N+1} = \frac{\eta_1 + \eta_2 + \ldots + \eta_{n+m} + 1}{{\eta_1}^2 + \ldots + {\eta_{n+m}}^2 + 1} > \frac{1}{2}.$$

The fact $X \in \varepsilon_0 \Gamma_2$ implies that $X_{2N} = X_{2N+1}$. Therefore,

$$\left| \frac{\eta_1 + \eta_2 + \ldots + \eta_{n+m} + 1}{{\eta_1}^2 + \ldots + {\eta_{n+m}}^2 + 1} - Y_{2N+1} \right| = |X_{2N+1}| = |Y_{2N}| < \varepsilon_0 \le \frac{1}{6}$$

On the other hand by (21),

$$\left| \frac{\eta_1 + \eta_2 + \ldots + \eta_{n+m} + 1}{{\eta_1}^2 + \ldots + {\eta_{n+m}}^2 + 1} - Y_{2N+1} \right| > \frac{1}{2} - |Y_{2N+1}| \ge \frac{1}{3},$$

and since N is sufficiently large, we reach a contradiction. Therefore we have $(\mathbf{a},\phi)\phi \notin \varepsilon_0 U_2$.

By $(\mathbf{a},\phi)\phi \notin \varepsilon_0 U_2$, we obtain $q\phi \notin \varepsilon_0 U_2$ for any $q \geq (\mathbf{a},\phi)$. Since $(\gamma_{\mathbf{a}})_\phi = \gamma_{\mathbf{a}} \circ \phi^{-1}$,

$$\begin{aligned}
\gamma_{\mathbf{a}}(\varepsilon_0 U_2 \cap F + F^{\perp}) &= (\gamma_{\mathbf{a}})_\phi(\varepsilon_0 U_2 \cap F) \\
&\leq 1 - 2\int_{(\mathbf{a},\phi)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\
&= 1 - 2\varepsilon_1 \\
&< 1 - \varepsilon_1 \\
&\leq 1 - \varepsilon_0,
\end{aligned}$$

and the proof is complete. □

We conclude that Gauss cylindrical measures of Baxendale contain not only the canonical Gauss cylindrical measure and γ^t but also another cylindrical measure. And they do not have a particular property that conditions (i), (ii), (iii), (v) and (vi) in Theorem 3.1 are equivalent. This property characterizes the Gauss cylindrical measures in the original sense.

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