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SEVERAL REVERSE INEQUALITIES OF OPERATORS

大阪教育大学　藤井正俊 (Masatoshi Fujii)
Osaka Kyoiku University
大阪教育大学附属高等学校天王寺校舎　瀬尾祐貴 (Yuki Seo)
Tennoji Branch, Senior Highschool, Osaka Kyoiku University

ABSTRACT. In this report, we show reverse inequalities to Araki's inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if A and B are positive operators on a Hilbert space H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p)\|BAB\|^p \leq \|B^p A^p B^p\| \qquad \text{for all } 0 < p < 1,$$

where $K(m, M, p)$ is a generalized Kantorovich constant by Furuta.

1. Introduction

Let A and B be positive operators on a Hilbert space H. The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [8], Furuta showed that the Cordes inequality

(1) $$\|A^p B^p\| \leq \|AB\|^p \qquad \text{for } 0 < p < 1$$

is equivalent to the Löwner-Heinz inequality (cf.[14])

(2) $$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for } 0 < p < 1$$

(cf. [5]). In [1], Araki showed a trace inequality which entails the following inequality:

(3) $$\|B^p A^p B^p\| \leq \|BAB\|^p \quad \text{for } 0 < p < 1.$$

Moreover, it was shown in [6, 2] that the Cordes inequality (1) is equivalent to Araki's inequality (3).

On the other hand, Furuta [9] showed the following Kantorovich type inequalities: If A and B are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

(4) $$A \geq B \geq 0 \quad \text{implies} \quad K(m, M, p)A^p \geq B^p \quad \text{for } p > 1,$$

where a generalized Kantorovich constant $K(m, M, p)$ [3, 7, 11] is defined as

(5)
$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p \quad \text{for all real numbers } p.$$

We here cite Furuta's textbook [10] as a pertinent reference to Kantorovich inequalities.

Also, Yamazaki [16] showed the following difference type reverse inequalities of the Löwner-Heinz inequality: If A and B are positive operators with $0 < mI \leq B \leq MI$ for some scalars $m < M$, then

(6) $$A \geq B \geq 0 \quad \text{implies} \quad C(m, M, p) + A^p \geq B^p \quad \text{for } p > 1,$$

where the constant $C(m,M,p)$ [12, 16] is defined as

(7)
$$C(m,M,p)=(p-1)\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{p}{p-1}}+\frac{Mm^p-mM^p}{M-m}\quad \text{for all real numbers } p.$$

In this report, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If A and B are positive operators with $0<mI\le A\le MI$ for some scalars $m<M$, then the following inequalities hold

(8) $$K(m,M,p)\|BAB\|^p\le\|B^pA^pB^p\|\quad \text{for } 0<p<1,$$

(9) $$K(m^2,M^2,p)^{1/2}\|AB\|^p\le\|A^pB^p\|\quad \text{for } 0<p<1,$$

respectively. We moreover show that reverse inequalities (4), (8) and (9) are mutually equivalent.

2. Main Results

First of all, we present our main theorem which is a reverse inequality to Araki's inequality (3).

Theorem 1. *If A and B are positive operators on H such that $0<mI\le A\le MI$ for some scalars $m<M$, then for each $\alpha>0$*

(10) $$\|BAB\|^p\le\alpha\ \|B^pA^pB^p\|+\beta(m^p,M^p,\frac{1}{p},\alpha)\|B\|^{2p}\qquad \text{for all } 0<p<1,$$

or equivalently

(11) $$\|B^pA^pB^p\|^{\frac{1}{p}}\le\alpha\ \|BAB\|+\beta(m,M,p,\alpha)\|B\|^2\qquad \text{for all } p>1,$$

where

(12)
$$\beta(m,M,p,\alpha)=\begin{cases}\frac{p-1}{p}\left(\frac{M^p-m^p}{\alpha p(M-m)}\right)^{\frac{1}{p-1}}+\frac{\alpha(Mm^p-mM^p)}{M^p-m^p} & \text{if } \frac{M^p-m^p}{pM^{p-1}(M-m)}\le\alpha\le\frac{M^p-m^p}{pm^{p-1}(M-m)},\\ (1-\alpha)M & \text{if } 0<\alpha\le\frac{M^p-m^p}{pM^{p-1}(M-m)},\\ (1-\alpha)m & \text{if } \alpha\ge\frac{M^p-m^p}{pm^{p-1}(M-m)}.\end{cases}$$

If we choose α satisfying $\beta(m,M,p,\alpha)=0$ in Theorem 1, then we have the following ratio type reverse inequalities.

Corollary 2. *If A and B are positive operators on H such that $0<mI\le A\le MI$ for some scalars $m<M$, then*

(13) $$K(m,M,p)\|BAB\|^p\le\|B^pA^pB^p\|\qquad \text{for } 0<p<1,$$

or equivalently

(14) $$\|BAB\|^p\le K(m,M,p)\|B^pA^pB^p\|\qquad \text{for } p>1,$$

where $K(m,M,p)$ is defined as (5) in §1.

If we put $\alpha = 1$ in Theorem 1, then we have the following difference type reverse inequalities.

Corollary 3. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$\|BAB\|^p - \|B^pA^pB^p\| \leq -C(m, M, p)\|B\|^{2p} \qquad \text{for } 0 < p < 1, \tag{15}$$

or equivalently

$$\|B^pA^pB^p\|^{\frac{1}{p}} - \|BAB\| \leq -C(m^p, M^p, \frac{1}{p})\|B\|^2 \qquad \text{for } p > 1, \tag{16}$$

where $C(m, M, p)$ is defined as (7) in §1.

As special cases of Corollary 2 and Corollary 3, we have the following corollary.

Corollary 4. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$\|B^2A^2B^2\| \leq \frac{(M+m)^2}{4Mm}\|BAB\|^2. \tag{17}$$

$$\|B^2A^2B^2\|^{\frac{1}{2}} - \|BAB\| \leq \frac{(M-m)^2}{4(M+m)}\|B\|^2. \tag{18}$$

$$\frac{2\sqrt[4]{Mm}}{\sqrt{M}+\sqrt{m}}\|BAB\|^{\frac{1}{2}} \leq \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \tag{19}$$

$$\|BAB\|^{\frac{1}{2}} - \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{(\sqrt{M}-\sqrt{m})^2}{4(\sqrt{M}+\sqrt{m})}\|B\|. \tag{20}$$

Since $\|X^*X\| = \|X\|^2$ for an operator X, we obtain the following reverse inequality to the Cordes inequality by Corollary 2.

Theorem 5. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$K(m^2, M^2, p)^{\frac{1}{2}}\|AB\|^p \leq \|A^pB^p\| \qquad \text{for all } 0 < p < 1, \tag{21}$$

or equivalently

$$\|A^pB^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}}\ \|AB\|^p \quad \text{for all } p > 1. \tag{22}$$

In particular,

$$\sqrt{\frac{2\sqrt{Mm}}{M+m}}\|AB\|^{\frac{1}{2}} \leq \|A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \tag{23}$$

and

$$\|A^2B^2\| \leq \frac{M^2+m^2}{2Mm}\|AB\|^2 \tag{24}$$

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.

Theorem 6. *For a given $p > 1$, the following are mutually equivalent: For all $A, B \geq 0$ and $0 < mI \leq A \leq MI$*

(A) $A \geq B \geq 0$ *implies* $K(m, M, p)A^p \geq B^p$.

(B) $\|A^pB^p\| \leq K(m^2, M^2, p)^{1/2}\|AB\|^p$.

(C) $\|B^pA^pB^p\| \leq K(m, M, p)\|BAB\|^p$.

(B') $K(m^2, M^2, 1/p)^{1/2}\|AB\|^p \leq \|A^pB^p\|$.

(C') $K(m, M, 1/p)\|BAB\|^p \leq \|B^pA^pB^p\|$.

3. Lemmas

We start with the following three lemmas before we give proofs of the results in §2.

Let A be a positive operator on a Hilbert space H and x a unit vector in H. Then it follows from Hölder-McCarthy inequality that

$$(Ax, x) \leq (A^px, x)^{\frac{1}{p}} \qquad \text{for all } p > 1. \tag{25}$$

By using the Mond-Pečarić method [12, 13], we have the following reverse inequality of (25) [15, 4]:

Lemma 7. *If A is a positive operator on H such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$*

$$(A^px, x)^{\frac{1}{p}} \leq \alpha(Ax, x) + \beta(m, M, p, \alpha) \qquad \text{for all } p > 1 \tag{26}$$

holds for every unit vector $x \in H$, where $\beta(m, M, p, \alpha)$ is defined as (12) in Theorem 1.

Proof. For the sake of reader's convenience, we give a proof. Put $\beta = \beta(m, M, p, \alpha)$ and $f(t) = (at+b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p - m^p}{M-m}$ and $b = \frac{Mm^p - mM^p}{M-m}$, then we have $f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{1}{a}(\frac{\alpha p}{a})^{\frac{p}{1-p}} - \frac{b}{a}$. If $m \leq t_0 \leq M$, then we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2}(at+b)^{\frac{1}{p}-2} < 0$ and the condition $m \leq t_0 \leq M$ is equivalent to the condition

$$\frac{M^p - m^p}{pM^{p-1}(M-m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M-m)}.$$

If $M \leq t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)m$ for $t_0 = m$ if $t_0 \leq m$. Hence it follows that

$$(at+b)^{\frac{1}{p}} - \alpha t \leq \beta \qquad \text{for all } t \in [m, M].$$

Since t^p is convex for $p > 1$, it follows that $t^p \leq at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \leq aA + b$ and hence $(A^px, x) \leq a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

$$\begin{aligned}(A^px, x)^{\frac{1}{p}} - \alpha(Ax, x) &\leq (a(Ax, x) + b)^{\frac{1}{p}} - \alpha(Ax, x) \\ &\leq \max_{m \leq t \leq M} f(t) = \beta(m, M, p, \alpha).\end{aligned}$$

□

By Lemma 7, we have the following estimates of both the difference and the ratio in the inequality (25).

Lemma 8. *If A is a positive operator on H such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $p > 1$*

$$(A^p x, x)^{\frac{1}{p}} \leq K(m, M, p)^{\frac{1}{p}}(Ax, x) \tag{27}$$

and

$$(A^p x, x)^{\frac{1}{p}} - (Ax, x) \leq -C(m^p, M^p, \frac{1}{p}) \tag{28}$$

hold for every unit vector $x \in H$, where $K(m, M, p)$ is defined as (5) in §1 and $C(m, M, p)$ is defined as (7) in §1.

Proof. If we choose α satisfying $\beta(m, M, p, \alpha) = 0$ in Lemma 7, then we have $\alpha = K(m, M, p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 7, then we have $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. □

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

We summarize some important properties of a generalized Kantorovich constant [3, 11].

Lemma 9. *Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.*

(i) $K(m, M, p) = K(M, m, p)$ *for all* $p \in \mathbb{R}$.
(ii) $K(m, M, p) = K(m, M, 1 - p)$ *for all* $p \in \mathbb{R}$.
(iii) $K(m, M, 0) = K(m, M, 1) = 1$ *for all* $p \in \mathbb{R}$.
(iv) $K(m, M, p)$ *is increasing for* $p > \frac{1}{2}$ *and decreasing for* $p < \frac{1}{2}$.
(v) $K(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} = K(m^p, M^p, \frac{r}{p})^{-\frac{1}{r}}$ *for* $pr \neq 0$.

4. Proof of results

Based on Lemmas in the preceding section, we give proofs of the results mentioned in the second section.

Proof of Theorem 1.
For every unit vector $x \in H$, it follows that

$$\begin{aligned}
&((BAB)^p x, x) \\
\leq\ & (BABx, x)^p \quad \text{by Hölder-McCarthy inequality and } 0 < p < 1 \\
=\ & \left((A^p)^{\frac{1}{p}} \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right)^p \|Bx\|^{2p} \\
\leq\ & \left(\alpha(A^p \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}) + \beta(m^p, M^p, \frac{1}{p}, \alpha)\right) \|Bx\|^{2p} \quad \text{by Lemma 7} \\
=\ & \alpha(A^p Bx, Bx)\|Bx\|^{2p-2} + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p} \\
=\ & \alpha \left(B^p A^p B^p \frac{B^{1-p}x}{\|B^{1-p}x\|}, \frac{B^{1-p}x}{\|B^{1-p}x\|}\right) \|Bx\|^{2p-2}\|B^{1-p}x\|^2 + \beta(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p}
\end{aligned}$$

and

$$\begin{aligned}\|Bx\|^{2p-2}\|B^{1-p}x\|^2 &= (B^2x,x)^{p-1}(B^{2-2p}x,x)\\ &\le (B^2x,x)^{p-1}(B^2x,x)^{1-p}=1 \quad \text{by } 0<1-p<1.\end{aligned}$$

By combining two inequalities above, we have

$$\begin{aligned}\|BAB\|^p &= \|(BAB)^p\|\\ &\le \alpha\|B^pA^pB^p\| + \beta(m^p,M^p,\frac{1}{p},\alpha)\|B\|^{2p}\end{aligned}$$

and hence we have the desired inequality (10).

Next, we show (10)$\Longrightarrow$(11). For $p>1$, since $0<\frac{1}{p}<1$, it follows from (10) that

$$\|BAB\|^{\frac{1}{p}} \le \alpha\ \|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\| + \beta(m^{\frac{1}{p}},M^{\frac{1}{p}},p,\alpha)\|B\|^{\frac{2}{p}}.$$

By replacing A by A^p and B by B^p in the above inequality respectively, we have

$$\|B^pA^pB^p\|^{\frac{1}{p}} \le \alpha\ \|BAB\| + \beta(m,M,p,\alpha)\|B^p\|^{\frac{2}{p}},$$

and so we have the desired inequality (11). Similarly we can show (11)$\Longrightarrow$(10). Therefore (10) is equivalent to (11). □

Proof of Corollary 2.

For $p>1$, if we put $\beta(m,M,p,\alpha)=0$ in Theorem 1, then it follows that

$$\frac{p-1}{p}\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{1}{p-1}} + \alpha^{\frac{p}{p-1}}\frac{(Mm^p-mM^p)}{M^p-m^p}=0$$

and hence

$$\alpha^{\frac{p}{p-1}} = -\frac{p-1}{p}\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{1}{p-1}}\frac{M^p-m^p}{Mm^p-mM^p}.$$

Therefore, we have

$$\begin{aligned}\alpha^p &= \frac{M^p-m^p}{p(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^{p-1}\\ &= K(m,M,p)\end{aligned}$$

and we obtain the desired inequality (14).

For $0<p<1$, since $1/p>1$, it follows from (14) that

$$\|BAB\|^{\frac{1}{p}} \le K(m,M,\frac{1}{p})\|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\|.$$

By replacing A and B by A^p and B^p respectively, then we have

$$\|B^pA^pB^p\|^{\frac{1}{p}} \le K(m^p,M^p,\frac{1}{p})\|BAB\|.$$

Hence it follows from Lemma 9 that

$$\begin{aligned}\|B^pA^pB^p\| &\le K(m^p,M^p,\frac{1}{p})^p\|BAB\|^p\\ &\le K(m,M,p)^{-1}\|BAB\|^p,\end{aligned}$$

and we have the desired inequality (13). Similarly we have the implication (13)$\Longrightarrow$(14).

□

Proof of Corollary 3.
If we put $\alpha = 1$ in Theorem 1, then it follows that

$$\begin{aligned}\beta(m^p, M^p, \frac{1}{p}, 1) &= \frac{\frac{1}{p}-1}{\frac{1}{p}}\left(\frac{M-m}{\frac{1}{p}(M^p-m^p)}\right)^{\frac{\frac{1}{p}}{\frac{1}{p}-1}} + \frac{M^p m - m^p M}{M-m}\\ &= (1-p)\left(\frac{p(M-m)}{M^p-m^p}\right)^{\frac{p}{1-p}} + \frac{M^p m - m^p M}{M-m}\\ &= -C(m, M, p).\end{aligned}$$

Similarly it follows that $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. Hence we have the equivalence (15) $\Longleftrightarrow$ (16) □

Proof of Corollary 4.
In Corollary 2 and 3, we have only to put $p = 2$ and $p = 1/2$. □

Proof of Theorem 5
By Corollary 2, it follows that

$$K(m, M, p)\|A^{\frac{1}{2}}B\|^{2p} \le \|A^{\frac{p}{2}}B^p\|^2.$$

By replacing A by A^2, we have

$$K(m^2, M^2, p)\|AB\|^{2p} \le \|A^pB^p\|^2.$$

Therefore we have (21). Similarly, we have the equivalence (21)$\Longleftrightarrow$(22). □

Proof of Theorem 6
The proof is divided into three parts, namely the equivalence $(A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (A)$, $(B) \Longleftrightarrow (B')$ and $(C) \Longleftrightarrow (C')$.
$(A) \Longrightarrow (B)$. It follows that

$$\begin{aligned}(A) &\Longleftrightarrow \|A^{-\frac{1}{2}}B^{\frac{1}{2}}\| \le 1 \rightarrow \|A^{-\frac{p}{2}}B^{\frac{p}{2}}\|^2 \le K(m, M, p)\\ &\Longleftrightarrow \|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \le 1 \rightarrow \|A^{\frac{p}{2}}B^{\frac{p}{2}}\|^2 \le K(M^{-1}, m^{-1}, p) = K(m, M, p)\\ &\Longleftrightarrow \|AB\| \le 1 \rightarrow \|A^pB^p\| \le K(m^2, M^2, p).\end{aligned}$$

If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that

$$\|A^pB_1^p\| \le K(m^2, M^2, p)^{\frac{1}{2}} \Longleftrightarrow \|A^pB^p\| \le K(m^2, M^2, p)^{\frac{1}{2}}\|AB\|^p.$$

$(B) \Longrightarrow (C)$. If we replace A by $A^{\frac{1}{2}}$ in (A), then it follows that

$$\|A^{\frac{p}{2}}B^p\| \le K(m, M, p)^{\frac{1}{2}}\|A^{\frac{1}{2}}B\|^p.$$

Square both sides, we have

$$\|B^pA^pB^p\| \le K(m, M, p)\|BAB\|^p.$$

$(C) \Longrightarrow (A)$. If we replace B by $B^{\frac{1}{2}}$ and A by A^{-1} in (C), then it follows that

$$\|B^{\frac{p}{2}}A^{-p}B^{\frac{p}{2}}\| \le K(M^{-1}, m^{-1}, p)\|B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\|^p.$$

By rearranging it, we have

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \le K(m, M, p)\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\|^p.$$

Since $A \geq B \geq 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq 1$ that

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \leq K(m,M,p)$$

and hence

$$B^p \leq K(m,M,p)A^p.$$

$(B) \Longleftrightarrow (B')$: If we replace A and B by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ in (B) respectively, then it follows that

$$\begin{aligned}(B) &\Longleftrightarrow \|AB\| \leq K(m^{\frac{2}{p}},M^{\frac{2}{p}},p)^{\frac{1}{2}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\|^p \\ &\Longleftrightarrow \|AB\|^{\frac{1}{p}} \leq K(m^{\frac{2}{p}},M^{\frac{2}{p}},p)^{\frac{1}{2p}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \\ &\Longleftrightarrow K(m^2,M^2,p)^{\frac{1}{2}}\|AB\|^{\frac{1}{p}} \leq \|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \quad \text{by Lemma 9} \\ &\Longleftrightarrow (B')\end{aligned}$$

Similarly we have $(C) \Longleftrightarrow (C')$ and so the proof is complete. □

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