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ON THE STRONG CONVERGENCE OF MODIFIED ISHIKAWA ITERATES WITH ERRORS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Hafiz Fukhar-ud-din and Wataru Takahashi
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
O-okayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail: hfdin@yahoo.com , wataru@is.titech.ac.jp

Abstract: Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n \geq 1\}$ such that $\lim_{n\to\infty} k_n = 1$. We prove that modified Mann and modified Ishikawa iterative schemes with errors converge strongly to a fixed point of T without assuming $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r \geq 1$.

1. INTRODUCTION

Let C be a nonempty subset of a normed space E. A mapping $T : C \to C$ is an asymptotically nonexpansive if there exists a sequence $\{k_n \geq 1\}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \ x, y \in C, \ n \geq 1.$$

In particular, if $k_n = 1$ for all $n \geq 1$, it becomes nonexpansive.

The class of asymptotically nonexpansive mappings which is a natural generalization of the important class of nonexpansive mappings, was introduced by Goebel and Kirk[2] in 1972 and they proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping on C has a fixed point. In 1978, Bose[1] obtained the first weak convergence result of Picard iterations of asymptotically nonexpansive mappings. Later, Górniki[3] improved the Bose's result. Schu[11] also introduced the following iterative schemes:

Let C be a nonempty convex subset of a normed space E and let $T : C \to C$ be a given mapping. Then
(1) $\{x_n\}$ given by:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is an appropriate sequence in $[0, 1]$, is known as a modified Mann iterative scheme.

(2) $\{x_n\}$ obtained by:

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some suitable sequences in $[0,1]$, is known as a modified Ishikawa iterative scheme.

Using (1.1), Schu[11] proved the following convergence theorem for asymptotically nonexpansive mappings.

Theorem S(Theorem 1.5[11]). Let C be a nonempty bounded closed convex subset of a Hilbert space H. Let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n \geq 1\}$ such that $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ such that $\epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. Then the modified Mann iterative scheme $\{x_n\}$ converges strongly to a fixed point of T.

Later, Rhoades [10] extended Theorem S to a uniformly convex Banach space and to the modified Ishikawa iterative scheme. In 1995, Liu [8] introduced the following modified Ishikawa iterative scheme $\{x_n\}$ with errors in C defined by:

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where$\{u_n\}$ and $\{v_n\}$ are two summable sequences in E and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in the interval $[0,1]$ with appropriate conditions. In particular, if we choose $\beta_n = 0$ and $v_n = 0$ in (1.3), it reduces to the modified Mann iterative scheme with errors.

In 1999, Huang[4] studied the modified Mann and the modified Ishikawa iterative schemes with errors introduced by Liu[8] and extended Theorems 1 and 2 of Rhoades [10].

Theorem H1(Theorem 1[4]). Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let T be a completely continuous asymptotically nonexpansive selfmapping of C with $\{k_n\} \geq 1$ such that $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r \geq 1$. Define $\{\alpha_n\}$ satisfying $0 < a_1 \leq \alpha_n \leq 1 - a_2$ for all n and some $a_1, a_2 \in (0,1)$. For any $x_1 \in C$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n$ for $n \geq 1$ where $\{u_n\}$ is a sequence in C satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$. Then $\{x_n\}$ converges strongly to some fixed point of T.

Theorem H2(Theorem 2[4]). Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Let T be a completely continuous asymptotically nonexpansive selfmapping of C with $\{k_n\} \geq 1$ such that $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r \geq 1$. Let $\{x_n\}$ be as given in (1.3) with $\{\alpha_n\}, \{\beta_n\}$ satisfying $0 < a_1 \leq \alpha_n \leq 1 - a_2$ for all $n \geq 1$ and $\limsup_{n\to\infty}\beta_n \leq b$ for some constants $a_1, a_2 \in (0,1), b \in [0,1)$ and $\{u_n\}, \{v_n\}$ are two sequences in C satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$. Then $\{x_n\}$ converges strongly to some fixed point of T.

In [13], Xu also introduced error terms in the Mann and Ishikawa iterative schemes which appear to be more satisfactory. For the nonempty convex subset C of a normed space E, and a given mapping $T : C \to C$, the modified Ishikawa iterative scheme $\{x_n\}$ with errors in the sense of Xu is given by:

$$\begin{cases} x_1 \in C, \\ y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for all $n \geq 1$ and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C with the guarantee that it always lies in C. It becomes the Mann iterative scheme with errors, if we choose $b'_n = 0 = c'_n$.

Xu's iterative schemes with errors are always well-defined and the occurence of errors is also in random.

Moreover observe that if $b_n + c_n = \alpha_n$ and $b'_n + c'_n = \beta_n, u'_n = c_n(u_n - T^n y_n)$ and $v'_n = c'_n(v_n - T^n x_n)$ in (1.4), we obtain

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v'_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + u'_n, \quad n \geq 1. \end{cases}$$

Thus, if $\{T^n x_n\}$ and $\{T^n y_n\}$ are bounded (in particular if C is bounded) and $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty$, then (1.4) takes the form of (1.3). In this paper, C will be taken bounded so that (1.3) is contained in (1.4).

Recently, Theorem S, Theorem H1, Theorem H2 and the results of Rhoades in [10], have been obtained for unbounded domain C provided that $F(T) = \{x \in C : Tx = x\} \neq \phi$ (for example, see [9]). Some authors have also extended these results for the mappings including asymptotically nonexpansive mappings as a subclass. For details; see [5-7].

In this paper, we extend and improve Theorem S, Theorem H1, Theorem H2 and the results of Rhoades[10] and Xu and Noor[12], by showing that condition $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r \geq 1$ is superfluous in these and hence, in similar type results in the literature.

It is also worth mentioning that our calculations of the proof are comparatively simple and shorter than those done by Huang[4], Rhoades[10] and Schu [11].

In the sequel, we shall need the following lemma.

Lemma L([14]). Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda) g(\|x-y\|)$$

for all $x, y \in B_r[0]$, where $B_r[0] = \{x \in E : \|x\| \leq r\}$ and $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ for all $\lambda \in [0,1]$.

2. STRONG CONVERGENCE

We begin with the following lemma.

Lemma 2.1. Let C be a nonempty bounded closed convex subset of a normed space E and let $T : C \to C$ be a uniformly λ-Lipschitzian mapping. Define a sequence

$\{x_n\}$ as in (1.4) with $\{u_n\}, \{v_n\}$ sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ satisfying

$$\begin{cases} a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \quad \text{for all } n \geq 1, \\ \lim_{n\to\infty} c_n = 0 = \lim_{n\to\infty} c'_n. \end{cases}$$

Then

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0 \text{ implies } \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Proof. Since C is bounded, we can choose $M > 0$ such that $M = \max\{\sup_{n\geq 1} \|x_n - u_n\|, \sup_{n\geq 1} \|x_n - v_n\|\} < \infty$.
Denote $\|x_n - T^n x_n\|$ by d_n. Then we have

$$\begin{aligned} \|x_n - x_{n+1}\| &= \|x_n - (a_n x_n + b_n T^n y_n + c_n u_n)\| \\ &= \|b_n(x_n - T^n y_n) + c_n(x_n - u_n)\| \\ &\leq \|x_n - T^n x_n\| + \|T^n x_n - T^n y_n\| + c_n \|x_n - u_n\| \\ &\leq d_n + \lambda \|x_n - y_n\| + c_n M \\ &= d_n + \lambda \left\|b'_n(x_n - T^n x_n) + c'_n(x_n - v_n)\right\| + c_n M \\ &\leq d_n + \lambda b'_n \|x_n - T^n x_n\| + (\lambda c'_n + c_n)M \\ &\leq (1+\lambda)d_n + (\lambda c'_n + c_n)M. \end{aligned} \tag{2.1}$$

We also have

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|Tx_{n+1} - T^{n+1}x_{n+1}\right\| \\ &\leq d_{n+1} + \lambda \|x_{n+1} - T^n x_{n+1}\| \\ &= d_{n+1} + \lambda \|(x_{n+1} - x_n) + (x_n - T^n x_n) + (T^n x_n - T^n x_{n+1})\| \\ &\leq d_{n+1} + \lambda [\|x_{n+1} - x_n\| + \|x_n - T^n x_n\| + \lambda \|x_n - x_{n+1}\|] \\ &= d_{n+1} + \lambda d_n + \lambda(\lambda+1)\|x_{n+1} - x_n\|. \end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2) and then applying lim sup on both sides of the new inequality, we obtain that

$$\limsup_{n\to\infty} \|x_{n+1} - Tx_{n+1}\| \leq 0$$

and hence

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof.

Now we prove our main theorem.

Theorem 2.1. Suppose that E is a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Let $T : C \to C$ be completely continuous and asymptotically nonexpansive mapping with sequence $\{k_n \geq 1\}$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{x_n\}$ be the iterative scheme given in (1.4) where the sequences $\{a_n\}, \{b_n\}, \{c_n\}$, $\{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ satisfy $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ for all $n \geq 1$, $0 < \delta \leq b_n \leq 1 - \delta$ for some $\delta \in (0,1)$, $\limsup_{n\to\infty} b'_n < 1$, $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty$ and $\{u_n\}, \{v_n\}$ are sequences in C. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the same fixed point of T.

Proof. Let $F(T)$ denote the set of fixed points of T. By Goebel and Kirk[2], we have $F(T) \neq \phi$. Since C is bounded, for any $p \in F(T)$, we have that $\{ x_n-p, T^n y_n - p\} \subset B_r[0] \cap C$ for some $r > 0$. We denote by M, the maximum of $\sup_{n\geq 1} \|x_n - p\|^2$, $\sup_{n\geq 1}\left(\|u_n - x_n\|^2 + 2\|b_n(T^n y_n - p) + (1-b_n)(x_n - p)\| \|u_n - x_n\|\right)$ and $\sup_{n\geq 1}(\|v_n - x_n\|^2 + 2\|b'_n(T^n x_n - p) + (1-b'_n)(x_n-p)\| \|v_n - x_n\|)$.

From Lemma L and (1.4), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|b_n(T^n y_n - p) + (1-b_n)(x_n - p) + c_n(u_n - x_n)\|^2 \\
&\leq \|b_n(T^n y_n - p) + (1-b_n)(x_n - p)\|^2 + c_n M \\
&\leq b_n \|T^n y_n - p\|^2 + (1-b_n)\|x_n - p\|^2 \\
&\quad - w_2(b_n) g(\|x_n - T^n y_n\|) + c_n M \\
&\leq b_n k_n^2 \|y_n - p\|^2 + (1-b_n)\|x_n - p\|^2 \\
&\quad - w_2(b_n) g(\|x_n - T^n y_n\|) + c_n M \\
&= b_n k_n^2 \left\|b'_n(T^n x_n - p) + (1-b'_n)(x_n - p) + c'_n(v_n - x_n)\right\|^2 \\
&\quad + (1-b_n)\|x_n - p\|^2 - w_2(b_n) g(\|x_n - T^n y_n\|) + c_n M \\
&\leq \left[b_n b'_n k_n^4 + b_n k_n^2(1-b'_n) + (1-b_n)\right]\|x_n - p\|^2 \\
&\quad - w_2(b_n) g(\|x_n - T^n y_n\|) + c_n M + c'_n k_n^2 M \\
&\leq \left[b_n b'_n k_n^4 + b_n k_n^4(1-b'_n) + k_n^4(1-b_n)\right]\|x_n - p\|^2 \\
&\quad - w_2(b_n) g(\|x_n - T^n y_n\|) + c_n M + c'_n k_n^2 M \\
&\leq \|x_n - p\|^2 + M\left[(k_n^4 - 1) - \frac{\delta^2}{2M} g(\|x_n - T^n y_n\|)\right] \\
&\quad - \frac{\delta^2}{2} g(\|x_n - T^n y_n\|) + c_n M + c'_n k_n^2 M \,. \qquad (2.3)
\end{aligned}$$

Transposing the terms in (2.3), we have

$$\begin{aligned}
\frac{\delta^2}{2} g(\|x_n - T^n y_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (c_n + c'_n k_n^2)M \\
&\quad + M\left[(k_n^4 - 1) - \frac{\delta^2}{2M} g(\|x_n - T^n y_n\|)\right]. \qquad (2.4)
\end{aligned}$$

Denote $\sigma = \inf_{n\geq 1}\|x_n - T^n y_n\|$ and claim $\sigma = 0$. If $\sigma > 0$, then by the definition of g, we have $g(\|x_n - T^n y_n\|) \geq g(\sigma) > 0$. From (2.4), it follows that

$$\begin{aligned}
\frac{\delta^2}{2} g(\sigma) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (c_n + c'_n k_n^2)M \\
&\quad + M\left[(k_n^4 - 1) - \frac{\delta^2}{2M} g(\sigma)\right]. \qquad (2.5)
\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$ and $\frac{\delta^2}{2M} g(\sigma) > 0$, there exists $n_0 \geq 1$ such that $k_n^4 - 1 < \frac{\delta^2}{2M} g(\sigma)$ for all $n \geq n_0$. Hence (2.5) reduces to

$$\frac{\delta^2}{2} g(\sigma) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + c_n M + c'_n k_n^2 M, \quad n \geq n_0.$$

Let $m \geq n_0$ be any positive integer. Summing up the terms from n_0 to m in the above inequality, we have

$$\begin{aligned}\sum_{n=n_0}^{m} \frac{\delta^2}{2} g(\sigma) &\leq \|x_{n_0} - p\|^2 - \|x_{m+1} - p\|^2 + M \sum_{n=n_0}^{m} (c_n + c'_n k_n^2) \\ &\leq \|x_{n_0} - p\|^2 + M \sum_{n=n_0}^{m} (c_n + k_n^2 c'_n). \qquad (2.6)\end{aligned}$$

When $m \to \infty$ in (2.6), we get

$$\infty \leq \|x_{n_0} - p\|^2 + M \sum_{n=n_0}^{\infty} (c_n + k_n^2 c'_n) < \infty.$$

This is a contradiction. Hence $\sigma = 0$.

By the definition of σ, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n_j \to \infty} \left\|x_{n_j} - T^{n_j} y_{n_j}\right\| = 0.$$

From

$$\begin{aligned}\left\|x_{n_j} - T^{n_j} x_{n_j}\right\| &\leq \left\|x_{n_j} - T^{n_j} y_{n_j}\right\| + \left\|T^{n_j} x_{n_j} - T^{n_j} y_{n_j}\right\| \\ &\leq \left\|x_{n_j} - T^{n_j} y_{n_j}\right\| + k_{n_j} \left\|x_{n_j} - y_{n_j}\right\| \\ &\leq \left\|x_{n_j} - T^{n_j} y_{n_j}\right\| + b'_{n_j} k_{n_j} \left\|x_{n_j} - T^{n_j} x_{n_j}\right\| + c'_{n_j} k_{n_j} \left\|x_{n_j} - v_{n_j}\right\|,\end{aligned}$$

we have

$$\left\|x_{n_j} - T^{n_j} x_{n_j}\right\| \leq \frac{1}{1 - b'_{n_j} k_{n_j}} \left(\left\|x_{n_j} - T^{n_j} y_{n_j}\right\| + k_{n_j} c'_{n_j} \left\|x_{n_j} - v_{n_j}\right\| \right).$$

Taking lim sup on both sides, we have

$$\lim_{n_j \to \infty} \left\|x_{n_j} - T^{n_j} x_{n_j}\right\| = 0.$$

Using Lemma 2.1, we have

$$\lim_{n_j \to \infty} \left\|x_{n_j} - T x_{n_j}\right\| = 0. \qquad (2.7)$$

Since T is completely continuous and $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_{n_j}\}$ such that $\{T x_{n_i}\}$ converges. Thus from (2.7), $\{x_{n_i}\}$ converges. Let $\lim_{n_i \to \infty} x_{n_i} = q$. Now from continuity of T and (2.7) we have $Tq = q$.

From (1.4), it follows that

$$\|y_{n_i} - q\| \leq (a'_{n_i} + b'_{n_i} k_{n_i}) \|x_{n_i} - q\| + c'_{n_i} \|v_{n_i} - q\| \to 0 \qquad (2.8)$$

as $n_i \to \infty$. Further, this implies that

$$\|T^{n_i} y_{n_i} - q\| \to 0 \qquad (2.9)$$

as $n_i \to \infty$. Using (2.8) and (2.9) in inequality (2.3) with $x_n = x_{n_i}$, and $p = q$, we have

$$\|x_{n_i+1} - q\|^2 \leq \|x_{n_i} - q\|^2 + M\left[(k_{n_i}^4 - 1) - \frac{\delta^2}{2M} g\left(\|x_{n_i} - T^{n_i} y_{n_i}\|\right)\right]$$
$$-\frac{\delta^2}{2} g\left(\|x_{n_i} - T^{n_i} y_{n_i}\|\right) + c_{n_i} M + k_{n_i}^2 c'_{n_i} M \to 0$$

as $n_i \to \infty$ and hence

$$x_{n_i+1} \to q \text{ as } n_i \to \infty.$$

Inductively, we obtain $x_{n_i+m} \to q$ as $n_i \to \infty$ for $m = 0, 1, 2, 3, ...$, which gives that $\{x_n\}$ converges to q.

Finally from the inequality

$$\|y_n - q\| \leq (a'_n + b'_n k_n) \|x_n - q\| + c'_n \|v_n - q\|,$$

we deduce that $y_n \to q$ as $n \to \infty$. This completes the proof.

Taking $b'_n = 0 = c'_n$ in Theorem 2.2, we have the following result for the modified Mann iterative scheme with errors.

Corollary 2.1. Suppose that E is a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}, k_n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1, \sum_{n=1}^{\infty} c_n < \infty$ and $0 < \delta \leq b_n \leq 1 - \delta$ for some $\delta \in (0, 1)$. For an initial value $x_1 \in C$, define

$$x_{n+1} = a_n x_n + b_n T^n x + c_n u_n, \quad n \geq 1,$$

where $\{u_n\}$ is a sequence in C. Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 2.1. Theorem 2.1 unifies the proofs of Mann-type and Ishikawa-type convergence results in the current literature.

Remark 2.2. Theorem 2.1 extends and improve Theorem H1 and Theorem H2, Theorems 1 and 2 in [10], Theorem S, Theorems 2.2 and 2.3 in [12] in the following different ways:

(i) Mann and Ishikawa iteration schemes in [10-12], Mann and Ishikawa iterative scheme with errors (in the sense of Liu[8]) used by Huang [4] are extended to the Mann and Ishikawa iterative scheme with errors in the sense of Xu[13].

(ii) The assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ imposed on the sequence $\{k_n\}, k_n \geq 1$, in [10-12] is removed.

(iii) The Hilbert space in [11] is replaced by a uniformly convex Banach space.

Finally, we state the following open question.

Open Question: Can we remove $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r \geq 1$ for the weak convergence of Ishikawa iterates of an asymptotically nonexpansive mapping T with associated sequence $k_n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and under the same iteration parameters used in Theorem 2.2.

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