数理解析研究所講究録 1455 巻 2005 年 106-119

On Frame Wavelet Sets and Some Related Topics

Xingde Dai and Yuanan Diao

ABSTRACT. A special type of frame wavelets in $L^2(\mathbb{R})$ or $L^2(\mathbb{R}^d)$ consists of those whose Fourier transforms are defined by set theoretic functions. The corresponding sets involved are called frame wavelet sets. The seemingly simple structure of the frame wavelets induced by frame wavelet sets turns to be quite rich and complicated. The study of such frame wavelets can serve as a platform for the study of more general frame wavelets. In this paper, we review many recent results on frame wavelet sets and results obtained through the use of frame wavelet sets. We also pose some open questions in these related topics.

1. Introduction

The concept of frames first appeared in the late 40's and early 50's [**19, 29, 30**]. The concept of frame wavelets is a simple combination of the concepts of wavelets and frames. Naturally, frame wavelets share a close relation with wavelets and the development and study of wavelet theory during the last two decades also brought much attention and interest to the study of frame wavelets. For recent development and work on frames and some related topics, see [**1, 16, 17, 18, 23, 24, 28**].

In general, we can define a frame on any given separable Hilbert space. Let $\mathcal{H}$ be such a space, then a family of elements $\{x_j : j \in \mathcal{J}\}$ in $\mathcal{H}$ is called a *frame for* $\mathcal{H}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that for each $f \in \mathcal{H}$ we have

$$A\|f\|^2 \leq \sum_{j\in\mathcal{J}} |\langle f, x_j\rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

A is called a lower frame bound and B is called an upper frame bound of the frame in this case. The supremum of all lower frame bounds and the infimum of all upper frame bounds are called the *optimal frame bounds* of the frame and will be denoted by A_0 and B_0 respectively throughout this paper. A frame is said to be a *tight frame* when $A_0 = B_0$ and a *normalized tight frame* when $A_0 = B_0 = 1$. By definition, any orthonormal basis in a Hilbert space is a normalized tight frame, but in general a normalized tight frame may not be an orthonormal basis.

Following [**14**], we may define a *frame wavelet* under a general setting. First, a *unitary system* is a set of unitary operators $\mathcal{U}$ acting on a Hilbert space $\mathcal{H}$ which contains the identity operator I of $\mathcal{B}(\mathcal{H})$. An element of $\mathcal{H}$ is called a *frame wavelet* (with respect to

The first author thanks Shinzo Kawamura for inviting him and providing him partial travel support to the conference. He also thanks Kichi-Suke Saito for organizing this good conference.

the system $\mathcal{U}$) if the family $\{Ux : U \in \mathcal{U}\}$ is a frame for $\mathcal{H}$. One can similarly define *tight frame wavelets* and *normalized tight frame wavelets*. However, for our purposes in this paper, we usually limit us to $\mathcal{H} = L^2(\mathbb{R})$ or $\mathcal{H} = L^2(\mathbb{R}^d)$. Depending on how the unitary system $\mathcal{U}$ is chosen, we may then obtain different frame wavelets.

In this paper, we are mainly interested in a special type of frame wavelets, namely those whose Fourier transforms are set theoretic functions. More precisely, let E be a Lebesgue measurable set of finite measure. Define $\psi \in L^2(\mathbb{R})$ by $\widehat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$, where $\widehat{\psi}$ is the Fourier transform of ψ. If ψ so defined is a frame wavelet for $L^2(\mathbb{R})$ (with respect to the specified unitary system $\mathcal{U}$), then the set E is called a *frame wavelet set* (for $L^2(\mathbb{R})$). Similarly, E is called a (normalized) *tight frame wavelet set* if ψ is a (normalized) tight frame wavelet. The case of $L^2(\mathbb{R}^d)$ can be similarly treated.

2. Frame Wavelets with Respect to Dilation and Translation Operators

We define the dilation and translation operators, D and T on $L^2(\mathbb{R})$ as follows.

$$(Df)(x) = \sqrt{2}f(2x)$$
$$(Tf)(x) = f(x-1),$$

for any $f \in L^2(\mathbb{R})$. D, T are both unitary operators, i.e., $\|Df\| = \|Tf\| = \|f\|$ for any $f \in L^2(\mathbb{R})$. Thus $\{D^nT^\ell : n, \ell \in \mathbb{Z}\} = \mathcal{U}(D,T)$ defines a unitary system. A frame wavelet for $\mathcal{H} = L^2(\mathbb{R})$ with respect to $\mathcal{U}(d,t)$ is simply a function $\psi \in L^2(\mathbb{R})$ such that

$$\{D^nT^\ell\psi : n, \ell \in \mathbb{Z}\} = \{2^{\frac{n}{2}}\psi(2^nx - \ell) : n, \ell \in \mathbb{Z}\} \tag{2.1}$$

is a frame of $L^2(\mathbb{R})$. In other word, there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{n,\ell\in\mathbb{Z}} |\langle f, D^nT^\ell\psi\rangle|^2 \le B\|f\|^2 \tag{2.2}$$

for all $f \in L^2(\mathbb{R})$. In the literature, the term frame wavelet for $L^2(\mathbb{R})$ usually refers to a frame wavelet under this setting when the unitary system $\mathcal{U}$ is not specified.

2.1. The Characterization Problem. Similar to the study of wavelets, an essential question in the study of frame wavelets is how to characterize them under the given unitary operating system. Under the setting of this section, the question is how to characterize a function $\psi \in L^2(\mathbb{R})$ that is a frame wavelet with respect to $\mathcal{U} = \{D^nT^\ell : n, \ell \in \mathbb{Z}\}$. This problem remains open. However, a characterization of a normalized tight frame wavelet has been obtained [**23**] and we state the result below.

THEOREM 2.1. *Let ψ be a fixed wavelet and let $C_\psi(\mathcal{U}(D,T))$ be the local commutant at ψ, that is, $C_\psi(\mathcal{U}(D,T))$ is the set of all bounded linear operators T acting on $L^2(\mathbb{R})$ such that $UT - TU = 0$ for any $U \in \mathcal{U}(D,T)$. Then $f \in L^2(\mathbb{R})$ is a normalized tight frame wavelet if and only if there is a co-isometry $A \in C_\psi(\mathcal{U}(D,T))$ (i.e., A^* is an isometry) such that $f = A\psi$.*

Furthermore, the following theorem provides a sufficient condition for a frame wavelet.

THEOREM 2.2. [24] *For $\psi \in L^2(\mathbb{R})$, define*

$$\underline{S}_\psi = \operatorname{ess\,inf}_{\xi\in\mathbb{R}} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2, \ \overline{S}_\psi = \operatorname{ess\,sup}_{\xi\in\mathbb{R}} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2$$

and

$$\beta_\psi(m) = \operatorname{ess\,sup}_{\xi\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\sum_{j=0}^{\infty} \widehat{\psi}(2^j 2^k\xi)\overline{\widehat{\psi}(2^j(2^k\xi + 2m\pi))}|.$$

If

$$A_\psi = \underline{S}_\psi - \sum_{q\in 2\mathbb{Z}+1} \left[\beta_\psi(q)\beta_\psi(-q)\right]^{\frac{1}{2}} > 0,$$

and

$$B_\psi = \overline{S}_\psi + \sum_{q\in 2\mathbb{Z}+1} \left[\beta_\psi(q)\beta_\psi(-q)\right]^{\frac{1}{2}} < \infty,$$

then ψ is a frame wavelet with A_ψ as a lower frame bound and B_ψ as an upper frame bound.

Note that A_ψ and B_ψ in the above theorem are not necessarily the optimal bounds in general.

2.2. Frame Wavelet Sets with Respect to Dilation and Translation Operators. In an attempt to better understand the frame wavelets, we then turn our attention to study a much simpler subclass of frame wavelets whose Fourier transforms are simply set theoretic functions. More specifically, let E be a Lebesgue measurable set of finite measure. Define $\psi \in L^2(\mathbb{R})$ by $\widehat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$, where $\widehat{\psi}$ is the Fourier transform of ψ. If ψ so defined is a frame wavelet for $L^2(\mathbb{R})$ (with respect to the dilation and translation operators), then the set E is called a *frame wavelet set* (for $L^2(\mathbb{R})$). Similarly, E is called a (normalized) *tight frame wavelet set* if ψ is a (normalized) tight frame wavelet. We wish to characterize (tight, normalized tight) frame wavelet sets. Toward this direction, a characterization of the tight frame wavelet sets has been successfully obtained (this would include the normalized tight frame wavelet sets as a special case), and some fairly useful necessary conditions and sufficient conditions for frame wavelet sets are also obtained. However, the characterization of frame wavelet sets in general remains an open question. Before we state these results, we will need to introduce some terms and definitions.

Let $\mathcal{F}$ be the Fourier-Plancherel transform on $\mathcal{H} = L^2(\mathbb{R})$: if $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$(\mathcal{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt = \widehat{f}(s),$$

and

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s)ds.$$

For a bounded linear operator S on $L^2(\mathbb{R})$, we will denote $\mathcal{F}S\mathcal{F}^{-1}$ by $\widehat{S}$. We have $\widehat{D} = D^{-1}$ and $\widehat{T}f = e^{is} \cdot f$. Notice that $\mathcal{F}$ is a unitary operator and that (2.2) is equivalent to

$$A\|f\|^2 \le \sum_{n,\ell\in\mathbb{Z}} |\langle f, \widehat{D}^n\widehat{T}^\ell\widehat{\psi}\rangle|^2 \le B\|f\|^2, \ \forall f \in L^2(\mathbb{R}). \tag{2.3}$$

Let E be a measurable set. We say that $x, y \in E$ are $\overset{\delta}{\sim}$ *equivalent* if $x = 2^n y$ for some integer n. The *δ-index* of a point x in E is the number of elements in its $\overset{\delta}{\sim}$ equivalent class and is denoted by $\delta_E(x)$. Let $E(\delta, k) = \{x \in E : \delta_E(x) = k\}$. Then E is the disjoint union of the sets $E(\delta, k)$. Let

$$\delta(E) = \bigcup_{n\in\mathbb{Z}} 2^{-n}\Big(E \cap \big([-2^{n+1}\pi, -2^n\pi) \cup [2^n\pi, 2^{n+1}\pi)\big)\Big).$$

The above is a disjoint union if and only if $E = E(\delta, 1)$. Similarly, we say that $x, y \in E$ are $\overset{\tau}{\sim}$ *equivalent* if $x = y + 2n\pi$ for some integer n. The *τ-index* of a point x in E is the number of elements in its $\overset{\tau}{\sim}$ equivalent class and is denoted by $\tau_E(x)$. Let $E(\tau, k) = \{x \in E : \tau_E(x) = k\}$. Then E is the disjoint union of the sets $E(\tau, k)$. Define

$$\tau(E) = \bigcup_{n\in\mathbb{Z}} \Big(E \cap [2n\pi, 2(n+1)\pi) - 2n\pi\Big).$$

Again, this is a disjoint union if and only if $E = E(\tau, 1)$. If E is of finite measure, then $E(\tau, \infty)$ is of zero measure. Each $E(\delta, k)$ (resp. $E(\tau, k)$) can be further decomposed into k disjoint copies $E^{(j)}(\delta, k)$ (resp. $E^{(j)}(\tau, k)$). But these decompositions are not unique in general. However, under any given such decomposition, we will define $\Delta(E) = \cup_{k\in\mathbb{Z}} E^{(1)}(\delta, k)$.

We are now ready to state the characterization theorem for the tight frame wavelet sets.

THEOREM 2.3. [10] *Let E be a Lebesgue measurable set with finite measure. Then E is a tight frame wavelet set if and only if $E = E(\tau, 1) = E(\delta, k)$ for some $k \geq 1$ and $\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}$. In particular, E is a normalized tight frame wavelet set if and only if $E = E(\tau, 1) = E(\delta, 1)$ and $\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}$.*

EXAMPLE 2.4. Let $E = [-\pi, -\frac{\pi}{4}) \cup [\frac{\pi}{8}, \frac{\pi}{2})$. Then $E = E(\tau, 1)$ and $E = E(\delta, 2)$, hence E is a tight frame wavelet set of optimal frame bound 2.

In the case of frame wavelet sets, we have the following sufficient condition.

THEOREM 2.5. [10] *Let E be a Lebesgue measurable set with finite measure. Then E is a frame wavelet set if (i) $\cup_{n\in\mathbb{Z}} 2^n E(\tau, 1) = \mathbb{R}$ and (ii) There exists $M > 0$ such that $\mu(E(\delta, m)) = 0$ and $\mu(E(\tau, m)) = 0$ for any $m > M$ (where μ is the Lebesgue measure). Furthermore, in this case, the lower optimal frame bound is at least 1, and the upper optimal frame bound is at most $M^{5/2}$.*

EXAMPLE 2.6. Let $E = [-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup [\frac{\pi}{4}, \pi)$, then E is a frame wavelet set with lower bound at least one and upper bound at most $4\sqrt{2}$ since $E(\tau, 1) = [-\pi, -\frac{\pi}{4}) \cup [\frac{\pi}{4}, \frac{\pi}{2})$ satisfies condition (i) of Theorem 2.5 and $E(\delta, m) = E(\tau, m) = \emptyset$ for $m > 2$. In fact, in this case we can show that the lower bound is exactly one since $E(\delta, 1) \cap E(\tau, 1) = [-\pi, -\frac{3\pi}{4}) \neq \emptyset$.

On the other hand, we have a necessary condition stated in the following theorem which is "very close" to above sufficient condition. And we do not have a right candidate for a possible statement as if and only if conditions.

THEOREM 2.7. [10] *Let E be a Lebesgue measurable set with finite measure. If E is a frame wavelet set, then (i) $\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}$ and (ii) There exists $M > 0$ such that $\mu(E(\delta, m)) = 0$ and $\mu(E(\tau, m)) = 0$ for any $m > M$.*

Although Theorems 2.5 and 2.7 are very useful tools in identifying frame wavelet sets and non frame wavelet sets, they do not provide if and only if conditions for a frame wavelet set, as shown by the following examples.

EXAMPLE 2.8. Let $E = [-\pi, -\frac{\pi}{2}) \cup [\pi, 2\pi)$, Then E is not a frame wavelet set. We leave this to our reader to verify as an exercise. Notice that Theorem 2.7 fails to detect this set.

EXAMPLE 2.9. Let $E = [-3\pi, -\pi) \cup [\pi, 2\pi)$, then E does not satisfy the conditions in Theorem 2.5 since $E_1 = E(\tau, 1) = [-2\pi, -\pi)$ so $\cup_{n\in\mathbb{Z}} 2^n E_1 \neq \mathbb{R}$. However, one can prove that E is indeed a frame wavelet set [10]. Thus, Theorem 2.5 fails to detect this set.

REMARK 2.10. The results derived in this subsection are independent of Theorems 2.1 and 2.2 since the approaches used are totally different. One can verify that examples 2.6 and 2.9 above do not satisfy the conditions given in 2.2.

It turns out that Theorem 2.5 can be greatly improved, however we will need to introduce some new terms before we do so. Let E be a Lebesgue measurable set such that $E(\delta, m) = E(\tau, m) = \emptyset$ for $m > M$ and let $\Omega = \cup_{k\in\mathbb{Z}} 2^k E$. Define $E_1 = E \cap (\cup_{k\in\mathbb{Z}} 2^k E(\tau, 1))$, $\overline{E}_1 = E \backslash E_1$, $E_2 = \overline{E}_1 \cap (\cup_{k\in\mathbb{Z}} 2^k \overline{E}_1(\tau, 1))$, $\overline{E}_2 = \overline{E}_1 \backslash E_2$, $E_3 = \overline{E}_2 \cap (\cup_{k\in\mathbb{Z}} 2^k \overline{E}_2(\tau, 1))$, ... In general, once E_n is defined, we will define $\overline{E}_n = \overline{E}_{n-1} \backslash E_n$ and then define $E_{n+1} = \overline{E}_n \cap (\cup_{k\in\mathbb{Z}} 2^k \overline{E}_n(\tau, 1))$. Let $\Omega_j = \cup_{k\in\mathbb{Z}} 2^k E_j$. By the definition, $\Omega_i = \cup_{k\in\mathbb{Z}} 2^k \overline{E}_{i-1}(\tau, 1)$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. The set $C(E) = \bigcup_{1\leq j\leq n} \Delta(\overline{E}_{j-1}(\tau, 1))$ is called a *core* of the set E.

THEOREM 2.11. [9] *Let E be a Lebesgue measurable set such that $E(\delta, m) = E(\tau, m) = \emptyset$ for $m > M$. If $\mathbb{R} = \cup_{1\leq j\leq n} \Omega_j$ for some $n \geq 1$, then E is a frame wavelet set.*

We close this subsection with the following open question:

PROBLEM 2.12. Find the characterization of a frame wavelet set. That is, find the if and only conditions for a frame wavelet set.

2.3. Frame Wavelets with Frame Set Support in the Frequency Domain. An application of the frame wavelet sets is the construction of various frame wavelets whose Fourier transforms are supported by frame wavelet sets. This turns out to be quite fruitful in the sense that we are able to construct many frame wavelets that have not been constructed by conventional methods. A few results toward this direction are listed in the following theorems.

THEOREM 2.13. [9] *Let E be a frame wavelet set satisfying the conditions of Theorem 2.11, i.e., $E(\delta, m) = E(\tau, m) = \emptyset$ for $m > M$, and $\mathbb{R} = \cup_{1\leq j\leq n} \Omega_j$ for some n. If the support of $\widehat{\psi}$ is contained in E and there exists a constant $a > 0$ such that $|\widehat{\psi}(\xi)| \geq a$ a.e. on a core of E, then ψ is a frame wavelet.*

Theorem 2.13 provides a very flexible means for constructing frame wavelets, so long as we can find a core of the frame wavelet set. However, it remains an open question whether a frame wavelet set always has a core or not.

PROBLEM 2.14. Prove or disprove: there exist coreless frame sets.

On the other hand, the following theorem provides us a different method of constructing frame wavelets without having to rely on a core of the frame wavelet set.

THEOREM 2.15. [9] *Let E be a frame set, then $\psi \in L^2(\mathbb{R})$ is a frame wavelet if $\widehat{\psi}$ is bounded,* $\text{supp}(\widehat{\psi}) = E$, $|\widehat{\psi}| \geq a > 0$ *on E for some constant $a > 0$ and $\widehat{\psi}(s) = \widehat{\psi}(2^k s)$ whenever s and $2^k s$ are both in E for any integer k.*

The last result of this section concerns the measure of the support of the frequency domain frame wavelets. This is stated as the following theorem.

THEOREM 2.16. [9] *Let $\alpha > 0$ be any given constant, then there exist frequency domain frame wavelets with support of measure α.*

EXAMPLE 2.17. Let $E = [-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup [\frac{\pi}{4}, \pi)$ so that $E(\tau, 1) = [-\pi, -\frac{\pi}{4}) \cup [\frac{\pi}{4}, \frac{\pi}{2})$. Define ψ by letting

$$\widehat{\psi} = \chi_{[-\pi,-\pi/2)} + 2\chi_{[-3\pi/2,-\pi)} + \chi_{[\pi/4,\pi/2)} + 2\chi_{[\pi/2,\pi)}.$$

One can verify that $A_\psi = 1 - 4 = -3$ so Theorem 2.2 does not apply. However, according to Theorem 2.13, ψ is a frame wavelet.

Many examples can be constructed in this way, see [9]. It is an interesting question whether there are other ways to verify frame wavelets so constructed. We list this as an open problem to end this section.

PROBLEM 2.18. Find alternative ways to verify whether a function with its support on a frame wavelet set is a frequency frame wavelet.

2.4. Frame Wavelet Sets in $\mathbb{R}^d$. Most results in Section 2.2 can be extended to the high dimensional cases, although some care needs to be taken at some details. We will outline a few results in the following.

Let A be a real expansive matrix (i.e., all eigenvalues of A have absolute value greater than 1). We define a unitary operator D_A (called an *A-dilation operator*) acting on $L^2(\mathbb{R}^d)$ by

$$(D_A f)(t) = |\det A|^{\frac{1}{2}} f(At), \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d. \tag{2.4}$$

In an analogous fashion, a vector s in $\mathbb{R}^d$ induces a unitary *translation operator* T^s defined by

$$(T^s f)(t) = f(t - s), \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d.$$

If we let $\mathcal{H} = L^2(\mathbb{R}^d)$ and let $\mathcal{U}$ be the unitary system $\{D_A^n T^s : n \in \mathbb{Z}, s \in \mathbb{Z}^d\}$, then again we can speak of frame wavelets of $\mathcal{H}$ with respect to $\mathcal{U}$. Let E be a Lebesgue measurable set of finite measure. E is called a frame wavelet set if $\frac{1}{(2\pi)^{d/2}} \chi_E$ is a frequency domain frame wavelet, i.e., its inverse Fourier transform is a frame wavelet for $L^2(\mathbb{R}^d)$ under the system $\mathcal{U} = \{D_A^n T^s : n \in \mathbb{Z}, s \in \mathbb{Z}^d\}$. Here, the Fourier transform is defined by

$$(\mathcal{F} f)(s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(s \circ t)} f(t) dm, \tag{2.5}$$

where $s \circ t$ denotes the real inner product. The terms *tight frame wavelet sets* and *normalized tight frame wavelet sets* can be similarly defined as we did before. Also, the sets $E(\tau, k)$ and $E(\delta, k)$ can be defined along similar lines. The following theorems are quoted from [12].

THEOREM 2.19. *Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with finite measure. Then E is a tight frame wavelet set if and only if $E = E(\tau, 1) = E(\delta, k)$ for some $k \geq 1$ and $\cup_{n\in\mathbb{Z}}(A')^n E = \mathbb{R}^d$ (A' is the transpose of A). It follows that if E is a tight frame wavelet set, then the corresponding optimal frame bound is an integer.*

THEOREM 2.20. *If $E = E(\tau, 1)$, $\cup_{n\in\mathbb{Z}}(A')^n E(\tau, 1) = \mathbb{R}^d$ and there exist $1 \leq k_1 \leq k_2$ such that $\mu(E(\delta, m)) = 0$ for $m < k_1$ and $m > k_2$, $\mu(E(\delta, k_1)) \neq 0$ and $\mu(E(\delta, k_2)) \neq 0$, then E is a frame wavelet set with k_1 as a lower bound and k_2 as an upper bound.*

THEOREM 2.21. *Let E be a Lebesgue measurable set with finite measure. If E is a frame wavelet set, then (i) $\cup_{n\in\mathbb{Z}}(A')^n E = \mathbb{R}^d$ and (ii) There exists $M > 0$ such that $\mu(E(\delta, m)) = 0$ and $\mu(E(\tau, m)) = 0$ for any $m > M$. On the other hand, if $E = E(\delta, 1)$ and $\mu(E(\tau, k) \neq 0$ for some $k > 1$, then E is not a frame wavelet set.*

We end this section with the following question.

QUESTION 2.22. If we drop the condition that A is expansive, then can we still obtain results similar to the above under some other weaker conditions?

3. Wavelets with Frame Multiresolution Analysis

A classical way of constructing wavelets is through the use of multiresolution analysis (MRA) method. This method is based on an expansive matrix A with integer entries and requires $|\det(A)| - 1$ functions to generate a multiple A-dilation wavelet basis for $L^2(\mathbb{R}^d)$ although single function A-dilation wavelets also exist for any expansive dilation matrix A [**15**]. The usual well-known and most widely applied A-dilation wavelets are MRA wavelets. Unfortunately, single function MRA A-dilation wavelets do not always exist. For example, in the case that A is a matrix with integer entries, a single function MRA A-dilation wavelet exists if and only if $|\det(A)| = 2$ [**20, 26**]. So, if A has integer entries and $|\det(A)| > 2$, then there are no single function MRA A-dilation wavelets (though there exist multi-function MRA wavelets), even if $A = 2I$ when $d \geq 2$. For the matrices A with non-integer entries, it is not clear whether or when MRA A-dilation wavelets exist. However, under some conditions similar to, those in the MRA, there exist MRA-like single function A-dilation wavelets. One such approach is to use the concept of normalized tight frames. This section is devoted to the A-dilation wavelets constructed using this approach which is called a *frame multiresolution analysis (FMRA)*, a natural generalization of MRA and was introduced in [**3**]. For more general multiresolution analysis and wavelets, see [1]. For more related topics and works, see [**2, 6, 13**].

3.1. Definition of Frame Multiresolution Analysis (FMRA).

DEFINITION 3.1. A frame multiresolution analysis associated with a real expansive matrix A (or in short, an A-dilation FMRA) is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying the following conditions:

(1) $V_j \subset V_{j+1}$, $\forall j \in \mathbb{Z}$;
(2) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;
(3) $f \in V_j$ if and only if $f(As) \in V_{j+1}$, $j \in \mathbb{Z}$;
(4) There exists a function ϕ in V_0 such that $\{\phi(x - \ell) : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for V_0.

The function ϕ in (4) is called a frame scaling function for the A-dilation FMRA. A function ψ in $V_{-1} \ominus V_0$ is called an A-dilation FMRA wavelet (or just an FMRA wavelet with the understanding that it is associated with the given expansive matrix A) if it is an A-dilation orthonormal wavelet for $L^2(\mathbb{R}^d)$.

If, in the above definition, we replace "normalized tight frame" by "orthonormal basis" in (4), then we obtain the standard definition for a multiresolution analysis.

3.2. The Existence of FMRA Wavelets. A measurable set $E \subset \mathbb{R}^d$ is called an *A-dilation wavelet set* if $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ is an A-dilation wavelet. $E \subset \mathbb{R}^d$ is called an *A-dilation FMRA wavelet set* if $\mathcal{F}^{-1}(\frac{1}{\sqrt{\mu(E)}}\chi_E)$ is an A-dilation FMRA wavelet.

THEOREM 3.2. [11] *For every expansive matrix A with integer entries, there exists an A-dilation FMRA wavelet set. It follows that FMRA Wavelets always exist for any expansive matrix A with integer entries.*

In the case that A does not necessarily have integer entries, we have

THEOREM 3.3. [11] *Let $A = diag(a_1, \ldots, a_d)$ be a diagonal expansive matrix such that $|a_i| \geq 2$ for some i. Then there exists an A-dilation FMRA wavelet set. In particular, if $d = 1$ and $A = a$, then there exists an a-dilation FMRA wavelet set for $L^2(\mathbb{R})$ if and only if $|a| \geq 2$.*

The following theorem gives a necessary condition for an expansive matrix to admit an A-dilation FMRA wavelet set.

THEOREM 3.4. [11] *Let A be an expansive matrix. If there exists an A-dilation FMRA wavelet set E, then $|\det(A) \geq 2$.*

The proofs of the above theorems are actually quite technical and lengthy, see [11] for the details. The following examples will give the reader some idea how such A-dilation FMRA wavelet sets are constructed.

EXAMPLE 3.5. Let $A^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\Omega = [-\pi, \pi) \times [-\pi, \pi)$, then $E = A^t\Omega \setminus \Omega$ is an A-dilation FMRA wavelet set as shown in Figure 1.

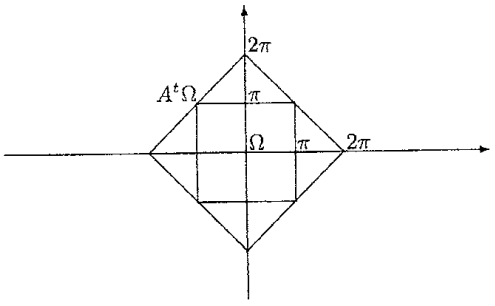

FIGURE 1. The set $E = A^t\Omega \setminus \Omega$

EXAMPLE 3.6. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix}$ where $a > 1$. Define two sequences $\{c_n\}$ and $\{b_n\}$ such that $c_{-1} = 0$, $c_n = (1 - \frac{1}{2^{2n}})\frac{2\pi}{3} + \frac{\pi}{2^{2n+1}}$ if $n \geq 0$, $b_{-1} = \pi$ and $b_n = \frac{2\pi}{3} + \frac{\pi}{3 \cdot 2^{2n+2}}$ if $n \geq 0$. Notice that $\{c_n\}$ is increasing, $\{b_n\}$ is decreasing and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = \frac{2\pi}{3}.$$

Let P_j be the rectangle with corners $(c_{j-1}, \frac{\pi}{a^{2j+1}})$, $(c_j, \frac{\pi}{a^{2j+1}})$, $(c_{j-1}, -\frac{\pi}{a^{2j+1}})$ and $(c_j, -\frac{\pi}{a^{2j+1}})$ where $j \geq 0$. Let Q_j be the rectangle with corners $(b_{j-1}, \frac{\pi}{a^{2j+2}})$, $(b_j, \frac{\pi}{a^{2j+2}})$, $(b_{j-1}, -\frac{\pi}{a^{2j+2}})$ and $(b_j, -\frac{\pi}{a^{2j+2}})$ where $j \geq 0$. Also, let $-P_j$, $-Q_j$ be the mirror images of P_j, Q_j through the y-axis respectively. Let $K = \cup_{j\geq 0}(P_j \cup Q_j \cup (-P_j) \cup (-Q_j))$. See Figure 2. Then $E = A^t K \setminus K$ is an A-dilation FMRA wavelet set. See Figure 3.

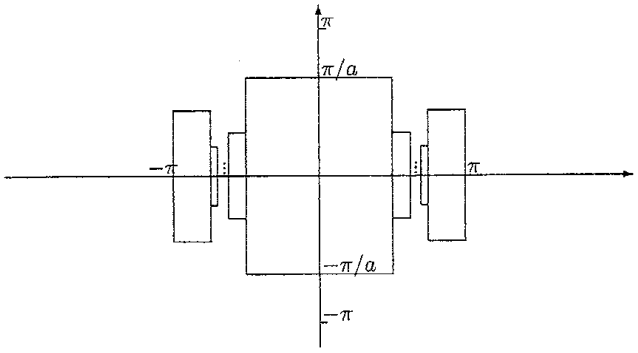

FIGURE 2. The set K.

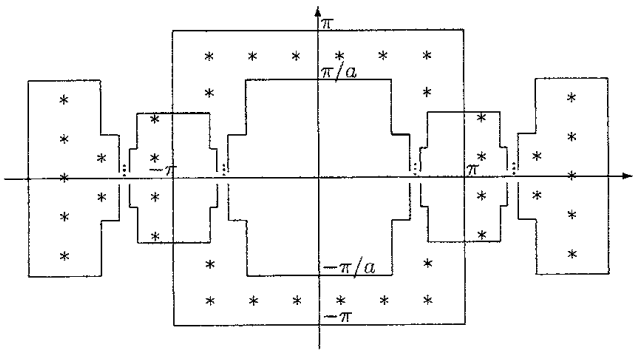

FIGURE 3. The set $E = A^K \setminus K$ (marked with $*$'s).

Several interesting problems remain open in this direction of study and we list them below.

PROBLEM 3.7. Is the condition $|\det(A)| \geq 2$ also a necessary condition for A to allow an A-dilation FMRA wavelet? In order to disprove this, one would have to construct an expansive matrix such that $|\det(A)| < 2$ and there exists an A-dilation FMRA wavelet.

PROBLEM 3.8. Is the condition $|\det(A)| \geq 2$ also a sufficient condition for A to allow an A-dilation FMRA wavelet?

The above problem is actually quite hard. One may want to study some special cases such as the following problem to gain some knowledge first.

PROBLEM 3.9. Prove or disprove that the expansive matrix $A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ has an A-dilation FMRA wavelet set.

4. Weyl-Heisenberg Frame Wavelet Sets

Let a, b be two fixed positive constants and let T_a and M_b be the *translation operator by* a and *modulation operator by* b respectively, i.e., $T_a g(t) = g(t-a)$ and $M_b g(t) = e^{ibt} g(t)$ for any $g \in L^2(\mathbb{R})$. Notice that T_a and M_b are both unitary operators. If we let $\mathcal{U}(a,b) = \{M_b^m T_a^n : m,n \in \mathbb{Z}\} = \{M_{mb} T_{na} : m,n \in \mathbb{Z}\}$, then we may speak of frame wavelets under the unitary system $CU(a,b)$. A frame wavelet under this setting is called a *Weyl-Heisenberg frame wavelet*, or a *Gabor frame wavelet*. A measurable set $E \subset \mathbb{R}$ is called a Weyl-Heisenberg frame wavelet set for (a,b) if the function $g = \chi_E$ is a Weyl-Heisenberg frame wavelet under the unitary system $\mathcal{U}(a,b)$.

It is known that if $ab > 2\pi$, then there are no Weyl-Heisenberg frame wavelets under the unitary system $\mathcal{U}(a,b)$. On the other hand, for any $a > 0$, $b > 0$ such that $ab \leq 2\pi$, there always exists a function $g \in L^2(\mathbb{R})$ such that g is a Weyl-Heisenberg frame wavelet under the unitary system $\mathcal{U}(a,b)$ [8]. However, in general, for any given $a > 0$, $b > 0$ with $ab \leq 2\pi$, characterizing the Weyl-Heisenberg frame wavelets under the unitary system $\mathcal{U}(a,b)$ is a very difficult problem. So again, it is natural for us to try some simpler cases first. In this section, we will take a look at the Weyl-Heisenberg frame wavelet sets under the special case $a = 2\pi$ and $b = 1$. We will show that the study of this simple case can actually lead us to some rather interesting results.

4.1. The Characterization of a Special Kind of Weyl-Heisenberg Frame Wavelet Set.

Let $n_1 < n_2 < \cdots < n_k$ be k positive integers and let E be the set

$$\cup_{j=1}^{k} \big([0, 2\pi) + 2\pi n_j\big). \tag{4.1}$$

The following problem is proposed in [7].

PROBLEM 4.1. Characterize the Weyl-Heisenberg frame wavelet set E as defined above (for $(2\pi, 1)$). In other word, we need to find a necessary and sufficient condition on the integers $0 < n_1 < n_2 < \cdots < n_k$ such that the corresponding set E is a Weyl-Heisenberg frame wavelet set for $(2\pi, 1)$.

Although Problem 4.1 is still an open question, it was observed that there is an equivalence relation between this problem and the following well-known open problem in complex analysis [4, 5, 27]:

PROBLEM 4.2. Classify the (positive) integer sets $\{n_1 < n_2 < \cdots < n_k\}$ such that the polynomial $p(z) = \sum_{1 \leq j \leq k} z^{n_j}$ does not have any unit roots.

In other word, the following theorem holds.

THEOREM 4.3. [7] *Let $0 < n_1 < n_2 < \cdots < n_k$ be k positive integers, then the set*

$$E = \cup_{j=1}^{k} \big([0, 2\pi) + 2\pi n_j\big) \tag{4.2}$$

is a Weyl-Heisenberg frame wavelet set for $(2\pi, 1)$ *if and only if the polynomial* $p(z) = \sum_{j=1}^{k} z^{n_j}$ *has no unit roots.*

A key tool used in the proof of the above theorem (as well as the theorems in the next subsection) is the Zak transformation, whose general definition can be found in [**25**]. The following is a slight variation of it since we have used the set $[0, 2\pi]$ instead of $[0, 1]$.

DEFINITION 4.4. The *Zak transform* of a function $f \in L^2(\mathbb{R})$ is defined as

$$Zf(t, w) = \sqrt{2\pi} \sum_{n \in \mathbb{Z}} f(t + 2\pi n) e^{inw}, \ \forall\ t,\ w \in [0, 2\pi). \tag{4.3}$$

The usefulness of the Zak transformation can be seen from the following lemma.

LEMMA 4.5. [**25**] *The Zak transform is a unitary map from* $L^2(\mathbb{R})$ *onto* $L^2(Q)$, *where* $Q = [0, 2\pi] \times [0, 2\pi]$ *and the inner product on* $L^2(Q)$ *is defined by*

$$\langle f(t, w), g(t, w) \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(t, w) \overline{g(t, w)} dt dw.$$

4.2. The Characterization of More General Weyl-Heisenberg Frame Wavelet Sets. Following the ideas in [**7**], we can establish some equivalence relations between the characterization problem of a Weyl-Heisenberg frame wavelet set of a more general form to several classification problems of as stated in Problem 4.2. For the sake of simplicity, we will only consider a set E that is the union of finitely many disjoint (finite) intervals. Such a set is called a *basic support set* in [**21**].

LEMMA 4.6. [**21**] *Let* E *be a basic support set, that is,* $E = \bigcup_{i=1}^{m} A_i$ *for some finite and non-overlapping intervals* $\{A_i\}_{i=1}^{m}$, *then there exists a finite sequence of disjoint intervals* $\{E_i\}_{i=1}^{k}$ *with* $E_i \subset [0, 2\pi)$, *and an integer sequence* $\{n_{ij}\}_{j=1}^{j_i}$ *for each* i, *such that*

$$E = \bigcup_{i=1}^{k} F_i, \text{ where } \mathrm{F_i} = \bigcup_{\mathrm{j=1}}^{\mathrm{j_i}} (\mathrm{E_i} + 2\pi \mathrm{n_{ij}}).$$

The sequence $\{E_i\}_{i=1}^{k}$ associated with the set E (defined in the above lemma) will be called the 2π-*translation generators* of E. Notice that $\bigcup_{j=1}^{j_i} (E_i + 2\pi n_{ij})$ is simply the pre-image of the function $\tau_{2\pi}$ restricted to E. We will call the sequence $\{n_{ij}\}_{j=1}^{j_i}$ the *step-widths* of the corresponding generator E_i. We then have the following theorem.

THEOREM 4.7. [**21**] *Let* E *be a basic support set with* $\{E_i\}_{i=1}^{k}$ *as its* 2π-*translation generators and let* $\{n_{ij}\}_{j=1}^{j_i}$ *be the step-widths of* E_i. *Then* E *is a Weyl-Heisenberg frame wavelet set if and only if none of the equations* $\sum_{j=1}^{j_i} z^{n_{ij}} = 0$ *has unit zeros.*

In fact, the above result can be extended to various functions whose support is a basic support set, see [**21**] for more details.

4.3. From Weyl-Heisenberg Frame Wavelet Sets to Some Infinite Quadratic Forms. Finally, we demonstrate how the Weyl-Heisenberg frame wavelet sets can be used to bridge a connection between some complex polynomials and some quadratic forms of infinite dimension. Only a rough outline will be given with one example given at the end. Interested readers please see [**22**] for the details.

Let $\{x_n\}$ be a real sequence in $\ell^2(\mathbb{Z})$, i.e., $\{x_n\}$ is a real valued sequence such that the series $\sum_{n\in\mathbb{Z}} x_n^2$ is convergent. Let $\{a_{ij}\}_{i,j\in\mathbb{Z}}$ be a sequence of real numbers with $a_{ij} = a_{ji}$. We can formally write $A = \{a_{ij}\}$ and think of A as an infinite dimensional symmetric matrix. Similarly, we will write $x = \{x_n\}$ and xAx^t for the formal sum $\sum_{i,j\in\mathbb{Z}} a_{ij}x_ix_j$. If this formal sum is convergent for all $x = \{x_n\} \in \ell^2(\mathbb{Z})$, then we will call xAx^t an *infinite quadratic form*. Notice that it is easy to come up with examples of A such that xAx^t is not defined for some x (that is, the series $\sum_{i,j\in\mathbb{Z}} a_{ij}x_ix_j$ is not convergent). We say that an infinite quadratic form xAx^t is strongly positive definite if there exists a constant $c > 0$ such that $xAx^t \geq c\|x\|^2$ for all $x \in \ell^2(\mathbb{Z})$, where $\|x\|^2 = \sum_{n\in\mathbb{Z}} x_n^2$. Similarly, one can define negative definiteness. First, we have the following theorem.

THEOREM 4.8. [22] *Let $0 = n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ given integers, and $a_0, a_1, a_2, \cdots, a_k$ be $k+1$ given nonzero real numbers, then if the infinite dimensional quadratic form*

$$\sum_{n\in\mathbb{Z}} (a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2$$

is strongly positive definite such that

$$c_1\|x\|^2 \leq \sum_{n\in\mathbb{Z}} (a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2 \leq c_2\|x\|^2$$

for some positive constants $c_1 \leq c_2$, then the function $g = \sum_{j=0}^k a_j\chi_{F_j}$ is a Weyl-Heisenberg frame wavelet for $(2\pi, 1)$ with c_1 as a lower frame bound and c_2 as an upper frame bound. Conversely, if the function $g = \sum_{j=0}^k a_j\chi_{F_j}$ is a Weyl-Heisenberg frame wavelet for $(2\pi, 1)$ with c_1 as a lower frame bound and c_2 as an upper frame bound, then

$$c_1\|x\|^2 \leq \sum_{n\in\mathbb{Z}} (a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2 \leq c_2\|x\|^2$$

for any $x \in \ell^2(\mathbb{Z})$.

This theorem then leads to the following when it is combined with the results from the last subsection.

THEOREM 4.9. [22] *Let $0 = n_0 < n_1 < n_2 < \cdots < n_k$ be $k+1$ given integers, a_0, a_1, $a_2, \cdots, a_k$ be $k+1$ given nonzero real numbers, and A be the symmetrical infinite matrix corresponding to the infinite quadratic form*

$$\sum_{n\in\mathbb{Z}} (a_0x_n + a_1x_{n+n_1} + a_2x_{n+n_2} + \cdots + a_kx_{n+n_k})^2.$$

Let $\min_{z\in\mathbb{T}} |a_0 + a_1z^{n_1} + \cdots + a_kz^{n_k}|^2 = C_1$ and $\max_{z\in\mathbb{T}} |a_0 + a_1z^{n_1} + \cdots + a_kz^{n_k}|^2 = C_2$, then the eigenvalues of any main diagonal block of A are bounded between C_1 and C_2.

We end this paper with the following example.

EXAMPLE 4.10. Let $p(z) = 2+3z^2+4z^3$. We have $min_{z\in\mathbb{T}}|p(z)|^2 = 1$, $max_{z\in\mathbb{T}}|p(z)|^2 = 81$. Thus the infinite quadratic form $\sum_{n\in\mathbb{Z}}(2x_n + 3x_{n+2} + 4x_{n+3})^2$ is strongly positive

definite. The corresponding symmetrical infinite matrix is

$$B = \begin{pmatrix} \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \\ \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 8 & 6 & 12 & 29 & 12 & 6 & 8 & 0 & \cdot \\ \cdot & & \cdot & & & \cdot & & & \cdot & & & & \cdot & & \end{pmatrix}.$$

Any eigenvalue λ of any main diagonal block of B must satisfy

$$1 \leq \lambda \leq 81.$$

A few main diagonal blocks of the above infinite matrix along with their eigenvalues are listed below.

$\begin{pmatrix} 29 & 12 \\ 12 & 29 \end{pmatrix}$, $\lambda_1 = 17, \lambda_2 = 41.$

$\begin{pmatrix} 29 & 12 & 6 \\ 12 & 29 & 12 \\ 6 & 12 & 29 \end{pmatrix}$, $\lambda_1 \approx 14.8, \lambda_2 = 23, \lambda_3 \approx 49.2$

$\begin{pmatrix} 29 & 12 & 6 & 8 \\ 12 & 29 & 12 & 6 \\ 6 & 12 & 29 & 12 \\ 8 & 6 & 12 & 29 \end{pmatrix}$, $\lambda_1 \approx 12.7, \lambda_2 \approx 20.9, \lambda_3 \approx 25.3, \lambda_4 \approx 57.1$

$\begin{pmatrix} 29 & 12 & 6 & 8 & 0 \\ 12 & 29 & 12 & 6 & 8 \\ 6 & 12 & 29 & 12 & 6 \\ 8 & 6 & 12 & 29 & 12 \\ 0 & 8 & 6 & 12 & 29 \end{pmatrix}$, $\lambda_1 \approx 9.8, \lambda_2 \approx 20.7, \lambda_3 = 21, \lambda_4 = 31, \lambda_5 \approx 62.5$

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Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA