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Perron-Frobenius Operators in Banach lattices

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Introductuion. This is an article about a work deeply related to that of Professor Kakutani who passed away in the summer of 2004 [8]. Our work is to give a Banach lattice version of the paper [5]. We give a generalization of the theory discussed in [5] and new kind of theorems concerning the orbit of a vector with respect to the iteration of a linear operator on a Banach lattice.

We have been interested in chaotic maps on a compact space. A map φ of a compact space X into itself X is said to be chaotic if φ satisfies the following conditions ([2: §1.8, Definition. 8]):

(1) The set of periodic points is dense in X.

(2) φ is one-sided topologically transitive.

(3) φ has sensitive dependence on initial conditions.

The above chaotic conditions are properties of the behavior of the orbits of a point in X with respect to the iteration of φ. In [4] and [5], Kawamura studied the properties of those chaotic maps on a measure space which was called a maps with n laps φ (MWnL for short) (Defintion in §2) and the behavior of the orbits of a probability density function on X. The study was extended to the case of states of von Neumann algebras on a Hilbert space associated with the measure space. The results were simple convergence theorems in contrast with the above three conditions and thus turned out to give another view point concerning chaotic maps.

Here, we study the Perron-Frobenious operator $A(\varphi)$ in L^1-space associated with each MWnL φ and the behavior of the orbit of a positive unit vector with respect to the iteration of $A(\varphi)$. Our main result is to find a subspace $\mathcal{M}$ of L^1-space and a subspace $\mathcal{N}$ of L^∞-space, which satisfies the following convergence property:

$$\lim_{n\to\infty} \|A(\varphi)^n f - e\|_1 = 0$$

for all positive unit vectors f in $\mathcal{M}$ and

$$\lim_{m\to\infty} \|A(\varphi)^m f - e\|_\infty = 0$$

for all positive unit vectors f in $\mathcal{N}$, where e is an $A(\varphi)$-invariant positive unit vector.

Before the discussion, we note that there symbols **N**, **Z** and **R** means the set of positive integers, the set of all integers and the set of all real numbers.

§1. A property of a sequence in an abstract L-space

A linear space $\mathcal{B}$ over the real field **R** is called a Banach lattice with respect to $(\|\cdot\|, \leqq)$, if $\mathcal{B}$ satisfies the following conditions ([6: II.8.1.Definition]):

(B-1) $\mathcal{B}$ is a lattice-ordered linear space with order $\leqq$.
(B-2) $\mathcal{B}$ is a Banach space with norm $\|\cdot\|$.
(B-3) $|x| \leqq |y|$ implies $\|x\| \leqq \|y\| (x, y \in \mathcal{B})$.

Here a linear operator on $\mathcal{B}$ means a linear operator of $\mathcal{B}$ into $\mathcal{B}$. A Banach lattice $\mathcal{B}$ with norm $\|\cdot\|$ is called an abstract A-space (AL-space for short)([5:II.8.1.Definition]) if $\mathcal{B}$ satisfies the following condition.

(L) $x, y \geqq 0$ implies $\|x + y\| = \|x\| + \|y\| \quad (x, y \in \mathcal{B})$.

It is well known that every AL-space $\mathcal{B}$ is isomorphic to $L^1(X, \mu)$ for a locally compact space X and a strictly positive Radon measure μ. This fact is due to Kakutani [1].

A Banach lattice $\mathcal{B}$ with norm $\|\cdot\|$ is called an abstract M-space (AM-space for short) ([5:II.7.1.Definition]) if $\mathcal{B}$ satisfies the following condition.

(M) $x, y \geqq 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad (x, y \in \mathcal{B})$.

For a subset $\mathcal{E}$ of a Banach lattice $\mathcal{B}$, we denote by $L(\mathcal{E})$ the linear span of $\mathcal{E}$ in $\mathcal{B}$. Moreover the closure of $L(\mathcal{E})$ in $\mathcal{B}$ is denote by $L^1(\mathcal{E})$ when $\mathcal{B}$ is an AL-space and $L^\infty(\mathcal{E})$ when $\mathcal{B}$ is an AM-space.

In the case where $\mathcal{B}$ is an AL-space with norm $\|\cdot\|_1$. We set

$$PUV(\mathcal{B}) = \{e \in \mathcal{B} |\ e \geq 0 \text{ and } \|e\|_1 = 1\}.$$

Namely, $PUV(\mathcal{B})$ is the set of all positive unit vectors in $\mathcal{B}$. Let A be a bounded linear operator on an it AL-space $\mathcal{B}$ and let e be a vector in $PUV(\mathcal{B})$ with the property $Ae = e$. Moreover let $\mathcal{E}$ be a sequence $\{e_i\}_{i=1}^{\infty}$ in $PUV(\mathcal{B})$. We say that $\mathcal{E}$ has Property(A, e) if $\mathcal{E}$ satisfies the following conditions:

(1) $e_1 = e$.
(2) For each e_i in $\mathcal{E}$, there exists $m \in \mathbf{N}$ such that $A^m e_i = e_1$.

In this article, we are interested in AL-spaces $\mathcal{B}$ and linear operators A on $\mathcal{B}$ such that $A(PUV(\mathcal{B})) \subset PUV(\mathcal{B})$. Here we remark that the norm $\|A\|$ of those operators A is always 1. In general, a bounded linear operator A on a Banach space $\mathcal{B}$ is said to be contractive on $\mathcal{B}$ if $\|A\| \leqq 1$. The following is our first result.

Theorem 1.1. *Let $\mathcal{B}$ be an AL-space and let A be a bounded linear operator on $\mathcal{B}$ such that $A(PUV(\mathcal{B}))\subset PUV(\mathcal{B})$. Suppose that there exists an A-invarinat vector e in $PUV(\mathcal{B})$. If a sequence $\mathcal{E}= \{e_i\}_{i=1}^{\infty}$ in $PUV(\mathcal{B})$ has Property (A, e), then it follows that for any vector f in $PUV(\mathcal{B})\cap L^1(\mathcal{E})$, we have*

$$\lim_{m\to\infty} \|A^m f - e\|_1 = 0.$$

In order to prove Theorem 1.1, we need the following lemma.

Lemma 1.2. *Suppose $\mathcal{B}$ is an AL-space with norm $\|\cdot\|_1$ and f is a vector in $\mathcal{B}$. Let $\{e_i\}_{i=1}^{k}$ and $\{a_i\}_{i=1}^{k}$ be a family of vectors in $PUV(\mathcal{B})$ and a family of real numbers respectively. Then we have the following:*

(1) *If $f \geqq 0$, then it follows that*

$$\|f\|_1 - \left|\sum_{i=1}^{k} a_i\right| \leqq \|f\|_1 - \sum_{i=1}^{k} a_i \leqq \left\| f - \sum_{i=1}^{k} a_i e_i \right\|_1 .$$

(2) *If there exists a contractive linear operator A on $\mathcal{B}$ such that $A^m e_i = e_1$ $(i = 1, \ldots, k)$ for some $m \in \mathbf{N}$, it follows that*

$$\sum_{i=1}^{k} a_i - \|f\|_1 \leqq \left|\sum_{i=1}^{k} a_i\right| - \|f\|_1 \leqq \left\| f - \sum_{i=1}^{k} a_i e_i \right\|_1 .$$

Remark 1.3. Under the conditions (1) and (2) of Lemma 1.2, if $f = \sum_{i=1}^{k} a_i e_i \geqq 0$, then we have $\|f\|_1 = \left|\sum_{i=1}^{k} a_i\right|$.

Remark 1.4. In Theorem 1.1, if f is in $L(\mathcal{E})$, then we have $A^m f = e_1$ for some $m \in \mathbf{N}$. Indeed, for $f = \sum_{i=1}^{k} a_i e_i \geq 0$, we have $\|f\|_1 = \sum_{i=1}^{k} a_i = 1$ and thus

$$A^m f = \sum_{i=1}^{k} a_i A^m e_i = \sum_{i=1}^{k} a_i e_1 = \left(\sum_{i=1}^{k} a_i\right) e_1 = e_1$$

for a large number $m \in \mathbf{N}$.

Next we note how the sequences $\{e_i\}$ in $\mathcal{B}$ in Theorem 1.1 are constructed when a bounded linear operator A on $\mathcal{B}$ has an invariant positive unit vector e.

Proposition 1.5. *Let B be an AL-space. Suppose that A and $\{B_i\}_{i=1}^{n}$ are bounded linear operators on $\mathcal{B}$ satisfying the following conditions:*

(a) $Af \geqq 0$ *and* $B_i f \geqq 0$ for $f \in \mathcal{B}$ with $f \geqq 0$, $(i = 1, \ldots, n)$

(b) $\|Af\|_1 \leqq \|f\|_1$ *and* $\|B_i f\|_1 \leqq \|f\|_1$ $(i = 1, \ldots, n)$ *for* $f \in \mathcal{B}$ and $\|Af\|_1 = \|B_i f\|_1 = \|f\|_1$ *if* $f \geq 0$.

(c) $AB_i = I$ $(i = 1, \ldots, n)$, *where I is the identity map of $\mathcal{B}$.*

(d) *There exists A-invariant vector e in $PUV(\mathcal{B})$.*

Moreover let

$$\mathcal{E} = \cup_{k=1}^{\infty} \{e_{i_1, i_2, \ldots, i_k} : i_1 = 1, i_2, \cdots, i_k \in \{1, \cdots, n\}\}$$

be the at most countable set in $\mathcal{B}$ defined by the following induction:
(i) $e_1 = e$.
(ii)$e_{1, i_2, \ldots, i_k} = B_{i_k} B_{i_{k-1}} \cdots B_{i_2} e_1$.
Then $\mathcal{E}$ has the following properties:
(1) $e_{1, i_2, \ldots, i_k} \in PUV(\mathcal{B})$.
(2) $Ae_1 = e_1$.
(3) $A^{k+n-1} e_{1, i_2, \ldots, i_k} = e_1$ *for all non-negative integers n.*

Hereafter, for an A-invariant vector e, we denote by $\mathcal{E}(e)$ the set $\mathcal{E}$ defined in Propositin 1.5. The following is our second result.

Theorem 1.6. *Let $\mathcal{B}$ be an AL-space and A be a bounded linear operator on $\mathcal{B}$. Moreover let $\mathcal{C}$ be a linear subspace of $\mathcal{B}$, which is an AM-space with norm $\|\cdot\|_\infty$ such that*

$$\|f\|_1 \leqq \|f\|_\infty \quad (f \in \mathcal{C}).$$

Suppose that $\mathcal{B}$, A and $\mathcal{C}$ satisfy the following conditions:
(1)$A(PUV(\mathcal{B})) \subset PUV(\mathcal{B})$.
(2) *There exists an A-invarinat vector e in $PUV(\mathcal{B})$.*
(3)$A(\mathcal{C}) \subset \mathcal{C}$.
(4) *The operator A is a contraction on $\mathcal{C}$ with respect to the norm $\|\cdot\|_\infty$,*
(5)*A sequence $\mathcal{E} = \{e_i\}_{i=1}^{\infty}$ in $PUV(\mathcal{B})$ has Property (A, e) and is contained in $\mathcal{C}$.*
Then for any vector f in $PUV(\mathcal{B}) \cap L^\infty(\mathcal{E})$, it follows that

$$\lim_{m \to \infty} \|A^m f - e\|_\infty = 0.$$

§2. Chaotic maps and the behavior of the orbit of probability density function

Let (X, μ) be a σ-finite measure space. A measurable map φ of X into X is called a map with n laps (MWnL for short (cf.[5. Definition 2.1])) if there exist n measurable subsets $\{X_i\}_{i=1}^n$ of X such that

(i) $\bigcup_{i=1}^n X_i = X$, $\mu(X_i \cap X_j) = 0$ for $i \neq j$ and $\mu(X_i) > 0$ for all i.

(ii) Each restriction φ_i of φ to X_i is a non-singular map of X_i onto X.

In the case where φ is an MWnL on X, since each map φ_i of X_i onto X is non-singular, we have two Radon-Nikodym derivatives $\frac{d\mu \circ \varphi_i}{d\mu}$ and $\frac{d\mu \circ \varphi_i^{-1}}{d\mu}$ such that

(iii) $\frac{d\mu \circ \varphi_i}{d\mu}(x) \neq 0$ for a.a. x in X_i and $\frac{d\mu \circ \varphi_i^{-1}}{d\mu}(x) \neq 0$ for a.a. x in X.

(iv) $\frac{d\mu \circ \varphi_i}{d\mu}(\varphi_i^{-1}(x))\frac{d\mu \circ \varphi_i^{-1}}{d\mu}(x) = 1$ for a.a. x in X and $\frac{d\mu \circ \varphi_i^{-1}}{d\mu}(\varphi_i(x))\frac{d\mu \circ \varphi_i}{d\mu}(x) = 1$ for a.a. x in X_i.

For a measure space (X, μ), two Banach spaces $L^1(X, \mu)$ ($L^1(X)$ for short) and $L^\infty(X, \mu)$ ($L^\infty(X)$ for short) with usual norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are an it AL-space and an it AM-space respectively. Here we denote by $PDF(X)$ instead of $PUV(L^1(X))$. Namely

$$PDF(X) = \{f \in L^1(X) | f \geq 0 \text{ and } \int_X f(x) d\mu = 1\}.$$

For an MWnL φ on X, we consider the Perron-Frobenius operator $A(\varphi)$. The operator $A(\varphi)$ on $L^1(X)$ is defined by

$$(A(\varphi)f)(x) = \sum_{i=1}^{n} \frac{d\mu \circ \varphi_i^{-1}}{d\mu}(x) f(\varphi_i^{-1}(x)) \quad (x \in X).$$

Our purpose is to analyze the orbit $\{A(\varphi)^n f\}_{n=1}^\infty$ for a function $f \in PDF(X)$ by using the results in §1. In the present paper, in addition to $A(\varphi)$, we need other linear operators $B(\varphi)_i$ $(i = 1, \ldots, n)$ which are defined by

$$(B(\varphi)_i f)(x) = \frac{d\mu \circ \varphi_i}{d\mu}(x) f(\varphi_i(x)) \chi_{X_i}(x) \quad (x \in X).$$

Then we have the following result.

Proposition 2.1. *Let φ be an MWnL on X. Then the operators $A(\varphi)$ and $\{B(\varphi)_i\}_{i=1}^n$ satisfy the following conditions.*

(a) $A(\varphi)f \geqq 0$, $B(\varphi)_i f \geqq 0$ $(i = 1, \ldots, n)$ *for all f in $L^1(X)$ with $f \geqq 0$.*

(b-1) $\|A(\varphi)f\|_1 \leqq \|f\|_1$ *for all f in $L^1(X)$ and* $\|A(\varphi)f\|_1 = \|f\|_1$ *if $f \geqq 0$.*

(b-2) $\|B(\varphi)_i f\|_1 = \|f\|_1$ $(i = 1, \ldots, n)$ *for all f in $L^1(X)$.*

(c) $AB_i = I$ $(i = 1, \ldots, n)$.

Using Theorem 1.1, Proposition 1.5 and the above proposition, we have the following theorem.

Theorem 2.2. *Let φ be an MWnL on X. Suppose that there exists an $A(\varphi)$-invariant vector e in $PDF(X)$ and $\mathcal{E}(e)$ is the sequence defined in Proposition* 1.3. *Then, for any vector f in $PDF(X) \cap L^1(\mathcal{E}(e))$, we have*

$$\lim_{m\to\infty} \|A(\varphi)^m f - e\|_1 = 0.$$

Moreover, suppose that $\mu(X) = 1$ and e belongs to $L^\infty(X)$. Then, for any vector f in $PDF(X) \cap L^\infty(\mathcal{E}(e))$, we have

$$\lim_{m\to\infty} \|A(\varphi)^m f - e\|_\infty = 0.$$

Now let φ be an MWnL on a probability measure space (X, μ). As in the case of measure preserving bijectve transformation on X, a map φ is said to be *strong-mixing* if

$$\lim_{k\to\infty} \mu(\varphi^{-k}(E) \cap F) = \mu(E)\mu(F)$$

for each pair of measurable sets E and F. Moreover, in the same manner as in [6: Lemma 6.11], we can see that this is equivalent to that, for any η in $L^1(X)$ and any f in $L^\infty(X)$, it follows that

$$\lim_{k\to\infty} \int_X f(\varphi^k(x))\eta(x)d\mu = \int_X f(x)\mu \int_X \eta(x)\mu.$$

This equation can be derived by the conclusion of Theorem 2.2, in which e is the case where $e(x) = 1$ $(x \in X)$ and $L(\mathcal{E}(e)) = L^1(X)$. Namely we have the following corollary.

Corollary 2.3. *Let φ be an MWnL on X such that the constant function $e(x) = 1$ is φ- invariant and $L(\mathcal{E}(e))$ coincides with the whole space $L^1(X)$. Then φ is strong-mixing.*

§3. Example of the case of tent map

Let τ be the tent map on the unit interval $X = [0,1]$ with the Lebesgue measure, that is, $\tau(x) = 1 - |1-2x|$. Then τ is an MW2L with $\tau_1(x) = 2x$ on $X_1 = [0, \frac{1}{2}]$ and $\tau_2(x) = 2 - 2x$ on $X_2 = [\frac{1}{2}, 1]$. Since $\tau_1^{-1}(x) = \frac{1}{2}x$ and $\tau_2^{-1}(x) = 1 - \frac{1}{2}x$, we have

$$\begin{aligned}(A(\tau)f)(x) \quad &= \frac{d\mu \circ \tau_1^{-1}}{d\mu}(x) f(\tau_1^{-1}(x)) + \frac{d\mu \circ \tau_2^{-1}}{d\mu}(x) f(\tau_2^{-1}(x)) \\ &= \tfrac{1}{2}\{f(\tfrac{x}{2}) + f(1 - \tfrac{x}{2}))\}\end{aligned}$$

and

$$(B(\tau)_1 f)(x) = 2f(2x)\chi_{[0,\frac{1}{2}]}(x) \qquad (B(\tau)_2 f)(x) = 2f(2-2x)\chi_{[\frac{1}{2},1]}(x).$$

Let $e = 1 = \chi_{[0,1]}$. Then $A(\tau)e = e$. Now we put

$$e_1 = e \text{ and } e_{1,i_2,\dots,i_k} = B(\tau)_{i_k} B(\tau)_{i_{k-1}} \cdots B(\tau)_{i_2} e_1$$

for $i_2, \dots, i_k \in \{1,2\}$. Then we have
$e_{1,1} = 2\chi_{[0,\frac{1}{2}]}$, $e_{1,2} = 2\chi_{[\frac{1}{2},1]}$, $e_{1,1,1} = 4\chi_{[0,\frac{1}{4}]}$, $e_{1,1,2} = 4\chi_{[\frac{3}{4},1]}$, $e_{1,2,1} = 4\chi_{[\frac{1}{4},\frac{2}{4}]}$, $e_{1,2,2} = 4\chi_{[\frac{2}{4},\frac{3}{4}]}, \cdots$.

Since $\mathcal{E}(e) = \bigcup_{k=1}^{\infty}\{e_{1,i_2,\dots,i_k} | i_2, i_k \in \{1,2\}\}$, then we have

$$\mathcal{E}(e) = \bigcup_{k=1}^{\infty}\left\{2^k \chi_{[\frac{i-1}{2^k},\frac{i}{2^k}]} | i = 1, 2, \cdots, 2^k\right\}$$

and

$$L(\mathcal{E}(e)) = \bigcup_{k=1}^{\infty}\left\{\sum_{i=1}^{2^k} a_i \chi_{[\frac{i-1}{2^k},\frac{i}{2^k}]} | a_i \in \mathbf{R}\right\}, \qquad L^1(\mathcal{E}(e)) = L^1([0,1]).$$

Therefore by Theorem 2.2, we have

$$\lim_{m\to\infty} \|A(\tau)^m f - \chi_{[0,1]}\|_1 = 0$$

for all f in $PDF([0,1])$. Namely we have the following proposition.

Proposition 3.1. *Let τ be the tent map on the unit interval $X = [0,1]$ with the Lebesgue measure and $e(x) = 1$ $(x \in [0,1])$. Then e is τ-invarinat and it follows that*

$$L^1(\mathcal{E}(e)) = L^1([0,1]).$$

Now we consider the Banach space $L^\infty(\mathcal{E}(e))$. We denote by $C([0,1])$ the Banach space of all continuous functions on $[0,1]$ with the norm $||\cdot||_\infty$. Since every function in $C([0,1])$ can be approximated by the functions in $L(\mathcal{E}(e))$, it follows that

$$C([0,1]) \subset L^\infty(\mathcal{E}(e)).$$

On the other hand, we have known that $L^\infty(\mathcal{E}(e))$ is a commutative C*-algebra, so it is isometrically isomorphic to $C(\Omega)$, where $C(\Omega)$ is the Banach space of all continuous functions on a compact space Ω. This is denoted by $L^\infty(\mathcal{E}(e)) \cong C(\Omega)$ and we can prove that $\Omega = \prod_{i=1}^\infty \{0,1\}$. Moreover we denote by $P([0,1])$ the set of all polynomials on $[0,1]$. Then we have the following proposition.

Proposition 3.2. *Let φ be the tent map τ on the unit interval $X = [0,1]$ with the Lebesgue measure. Then we have the following*:

(1)$P([0,1]) \subset C([0,1]) \subset L^\infty(\mathcal{E}(e)) \subset L^\infty([0,1]) \subset L^1([0,1])$,

(2)$L^\infty(\mathcal{E}(e)) \cong C(\Omega)$,

(3) $\lim_{m\to\infty} ||A(\tau)^m f - \chi_{[0,1]}||_\infty = 0$ for all f in $L^\infty([0,1]) \cap PDF([0,1])$.

Remark 3.3. (i) For the probability density function $f(x) = 2x$ on $[0,1]$ we have $(A(\tau)f)(x) = \chi_{[0,1]}$ and thus $A(\tau)^m f = \chi_{[0,1]}$ for all $m \geq 2$.

(ii) For $f(x) = 3x^2$, we have

$$(A(\tau)^m f)(x) = \frac{3x^2}{4^m} - \frac{3x}{4^{m-1}\cdot 2} + \frac{2\cdot 4^{m-1}+1}{2\cdot 4^{m-1}}.$$

Thus $\lim_{m\to\infty}(A(\tau)^m f)(x) = 1$ (uniformly on $[0,1]$).

(iii) For any positive continuous function f on $[0,1]$, the sequence $\{A(\tau)^m f\}_{m=1}^\infty$ converges to $\chi_{[0,1]}$ uniformly on $[0,1]$.

Remark 3.4. Though any function f in $PDF([0,1]) \cap C([0,1])$, the sequence $\{A(\tau)^m f\}_{m=1}^\infty$ converging to $\chi_{[0,1]}$ uniformly on $[0,1]$, there exists a function f in $PDF([0,1])$ such that $\{A(\tau)^m f\}_{m=1}^\infty$ does not converge to $\chi_{[0,1]}$ uniformly on $[0,1]$. The following is such an example. First we arrange the set $Q(2) = \bigcup_{k=1}^\infty \{\frac{j}{2^k} | j = 0,\ldots,2^k\}$ in an order by using a suitable way, that is, we consider it as a sequence $\{r_m\}_{m=1}^\infty$ of mutually distinct numbers. Let

$$J_m = [r_m - \frac{1}{2^{m+2}}, r_m + \frac{1}{2^{m+2}}] \cap [0,1] \quad \text{and} \quad J = \bigcup_{m=1}^\infty J_m.$$

Then we have $0 < \mu(J) \leqq \frac{1}{2}$, where μ is the Lebesgue measure on $[0,1]$. Let $f = \frac{1}{m([0,1]\setminus J)}(\chi_{[0,1]} - \chi_J)$. Then f belongs to $PDF([0,1])$ and thus we have

$$\lim_{m\to\infty} ||A(\tau)^m f - \chi_{[0,1]}||_1 = 0.$$

Now let m be a positive integer. Then, for each $i_1, \dots, i_m \in \{1,2\}$ and each $r_p \in Q(2)$, there exists $\delta > 0$ such that $\tau_{i_m}^{-1}(\tau_{i_{m-1}}^{-1}(\cdots(\tau_{i_1}^{-1}([r_p - \delta, r_p + \delta]))\cdots)) \subset J_q$ where $q = \tau_{i_m}^{-1}(\tau_{i_{m-1}}^{-1}(\cdots(\tau_{i_1}^{-1}(r_p)))$. Thus we have $(A(\tau)^m f)(x) = 0$ for $x \in [r_p - \delta, r_p + \delta]$, that is,

$$\|A(\tau)^m f - \chi_{[0,1]}\|_\infty = 1$$

for all m.

As mentioned above, $C([0,1])$ is embedded in $C(\Omega)$. Here we show a Banach subspace of $C(\Omega)$ which is isometric isomorphism to $C([0,1])$. Let p be the map of Ω onto $[0,1]$ defined by

$$p(\omega) = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i} \qquad (\omega = (\omega_i)_{i=1} \in \Omega)$$

We denote by $C_p(\Omega)$ the set of all functions f in $C(\Omega)$ with $f(\omega) = f(\omega')$ if $p(\omega) = p(\omega')$. Let Φ be the map of $C([0,1])$ into $C_p(\Omega)$ defined by

$$\Phi(f)(\omega) = f(p(\omega)) \qquad (f \in C([0,1]).$$

Then Φ is an isometric isomorphism. Hence we have the following proposition.

Proposition 3.5. *Let φ be the tent map τ on the unit interval $X = [0,1]$ with the Lebesgue measure. Then we have*

$$C([0,1]) \cong C_p(\Omega) \subset C(\Omega) \cong L^\infty(\mathcal{E}(e)) \subset L^\infty([0,1]).$$

§4. Example of the other cases

First we show an example that e is not bounded and $L^1(\mathcal{E}(e)) = L^1([0,1])$.

Example 4.1. Let λ be the logistic map on the unit interval $X = [0,1]$ with the Lebesgue measure, that is, $\lambda(x) = 4x(1-x)$. Then λ is an MW2L with $\lambda_1(x) = 4x(1-x)$ on $X_1 = [0, \frac{1}{2}]$ and $\lambda_2(x) = 4x(1-x)$ on $X_2 = [\frac{1}{2}, 1]$, too. Since $\lambda_1^{-1}(x) = \frac{1-\sqrt{1-x}}{2}$ and $\lambda_2^{-1}(x) = \frac{1+\sqrt{1-x}}{2}$, we have

$$\begin{aligned}(A(\lambda)f)(x) &= \tfrac{d\mu\circ\lambda_1^{-1}}{d\mu}(x) f(\lambda_1^{-1}(x)) + \tfrac{d\mu\circ\lambda_2^{-1}}{d\mu}(x) f(\lambda_2^{-1}(x)) \\ &= \tfrac{1}{4\sqrt{1-x}}\left(f\left(\tfrac{1-\sqrt{1-x}}{2}\right) + f\left(\tfrac{1+\sqrt{1-x}}{2}\right)\right)\end{aligned}$$

and

$$(B(\lambda)_1 f)(x) = (4-8x) f(4x(1-x))\chi_{[0,\frac{1}{2}]}(x), \qquad (B(\lambda)_2 f)(x) = (8x-4) f(4x(1-x))\chi_{[\frac{1}{2},1]}(x).$$

Let $e(x) = \frac{1}{\pi\sqrt{x(1-x)}}$. Then $A(\lambda)e = e$. Now we set

$$e_1 = e \quad \text{and} \quad e_{1,i_2,\dots,i_k} = B(\lambda)_{i_k} B(\lambda)_{i_{k-1}} \cdots B(\lambda)_{i_2} e_1$$

for $i_2, \dots, i_k \in \{1,2\}$.

Then we have

$e_{1,1} = 2e\chi_{[0,\frac{1}{2}]}$, $e_{1,2} = 2e\chi_{[\frac{1}{2},1]}$, $e_{1,1,1} = 4e\chi_{[0,\lambda_1^{-1}(\frac{1}{2})]}$, $e_{1,1,2} = 4e\chi_{[\lambda_2^{-1}(\frac{1}{2}),1]}$, $e_{1,2,1} = 4e\chi_{[\lambda_1^{-1}(\frac{1}{2}),\frac{1}{2}]}$, $e_{1,2,2} = 4e\chi_{[\frac{1}{2},\lambda_2^{-1}(\frac{1}{2})]}$, and so on.

Moreover inductively we can get each $e_{1,i_2,\dots,i_k}$ for $i_2, \dots, i_k \in \{1,2\}$ and we have

$$\mathcal{E}(e) = \bigcup_{k=1}^{\infty} \{e_{1,i_2,\dots,i_k} | i_2, \dots, i_k \in \{1,2\}\}.$$

The set $\{(\lambda_{i_k}^{-1} \circ \cdots \circ \lambda_{i_1}^{-1})(0) | i_1, \dots, i_k \in \{1,2\}\}$ consists of $2^{k-1}+1$ points in [0,1] and is arranged as $\{x_i\}_{i=1}^{2^{k-1}+1}$ with $0 = x_1 < x_2 < \cdots < x_{2^{k-1}} < x_{2^{k-1}+1} = 1$.

Then we have

$$\mathcal{E}(e) = \bigcup_{k=1}^{\infty} \{2^{k-1} e\chi_{[x_i,x_{i+1}]} | i = 1, 2, \cdots, 2^{k-1}\}$$

and

$$L(\mathcal{E}(e)) = \bigcup_{k=1}^{\infty} \left\{ \sum_{i=1}^{2^{k-1}} a_i e\chi_{[x_i,x_{i+1}]} | a_i \in \mathbf{R} \right\}, \qquad L^1(\mathcal{E}(e)) = L^1([0,1]).$$

Therefore by Theorem 2.2, we have

$$\lim_{m\to\infty} \|A(\lambda)^m f - e\|_1 = 0$$

for all $f \in PDF([0,1])$. Since the function e is not bounded, for any bounded function f in $PDF([0,1])$, the sequence $\{A(\lambda)^m f\}$ cannot converge uniformly to e on X, though it converges to e in the sense of $\sigma(L^1([0,1]), L^\infty([0,1]))$-topology.

Remark 4.2. The tent map τ and the Logistic map λ are topologically conjugate by the conjugacy $h(x) = \sin^2(\pi x/2)$(cf. [3: Theorem 3.24]). However, by Example 2.4 and 2.5, we can see that the behavior of covergence of orbits with rerpect to each map has dissimilar phenomina.

The following are two examples of $L^1(\mathcal{E}(e))$ associated with well-known maps on a totally disconected compact set X. The Banach space $L^1(\mathcal{E}(e))$ associated with one map is the whole space $L^1([0,1])$ and the other $L^1(\mathcal{E}(e))$ is one-dimensional.

Example 4.3. Let $X = \sum_{\mathbf{N}} = \prod_{m\in\mathbf{N}} \{0,1\}$ and σ_+ be the one-sided shift map of X onto X, that is, $y = \sigma^+(x)$, where $x = (x_m)_{m\in\mathbf{N}}, y = (y_m)_{m\in\mathbf{N}}$ and

$y_m = x_{m+1}$ $(m \in \mathbf{N})$. Let μ be the canonical measure on X with $\mu(X) = 1$ and $\mu(E) = \frac{1}{2^k}$ for each cylinder sets E of the form,

$$E = E(i_1, \ldots, i_k | c_1, \ldots, c_k) = \{x = (x_m)_{m \in \mathbf{N}} | x_{i_1} = c_1, \cdots, x_{i_k} = c_k\},$$

where $\{i_1, \ldots, i_k\}$ are mutually distinct natural numbers and $\{c_1, \ldots, c_k\}$ are numbers in $\{0, 1\}$. Let $X_1 = E(1|1)$, $X_2 = E(1|0)$ and σ_1^+, σ_2^+ be the restrictions of σ^+ to X_1 and X_2 respectively. Then $(\sigma_1^+)^{-1}$(resp. $(\sigma_2^+)^{-1}$) is the map defined by $y = (\sigma_1^+)^{-1}(x)$ (resp. $y = (\sigma_2^+)^{-1}(x)$), where $x = (x_m)_{m \in \mathbf{N}}$ and $y_1 = 1$ (resp. $y_1 = 0$), $y_m = x_{m-1}$ for $m \geq 2$. Therefore we have

$$\frac{d\mu \circ (\sigma_1^+)^{-1}}{d\mu}(x) = \frac{d\mu \circ (\sigma_2^+)^{-1}}{d\mu}(x) = \frac{1}{2} \quad (x \in X).$$

Thus we have

$$(A(\sigma^+)f)(x) = \frac{1}{2}(f((\sigma_1^+)^{-1}(x)) + f((\sigma_2^+)^{-1}(x)))$$

and

$$(B(\sigma^+)_1 f)(x) = 2f(\sigma_1^+(x))\chi_{E(1|1)}(x) \quad \text{and} \quad (B(\sigma^+)_2 f)(x) = 2f(\sigma_2^+(x))\chi_{E(1|0)}(x).$$

Let $e(x) = \chi_X(x), (x \in X)$. Then $A(\sigma^+)e = e$ and inductively we have

$$e_{1,i_2,\ldots,i_k} = B(\sigma^+)_{i_k} \cdots B(\sigma^+)_{i_2} e = 2^{k-1}\chi_{E(1,2,\ldots,k-1|p_k,p_{k-1},\ldots,p_2)}$$

where $p_\ell = 1$ if $i_\ell = 1$, and $p_\ell = 0$ if $i_\ell = 2$, for $\ell = 2, \ldots, k$. Thus we have

$$\mathcal{E}(e) = \{\chi_X\} \cup (\bigcup_{k=1}^{\infty}\{2^k\chi_{E(1,2,\ldots,k|q_1,q_2,\ldots q_k)} | q_i \in \{0,1\}\}),$$

where

$$L(\mathcal{E}(e)) = \bigcup_{k=1}^{\infty}\left\{\sum_{i=1}^{2^k} a_i\chi_{E_i} | a_i \in \mathbf{R} \text{ and } E_i \text{ is of the form } E(1, \ldots, k|q_1, \ldots, q_k), (q_1, \ldots, q_k \in \{0,1\}\right\})$$

Therefore $L^1(\mathcal{E}(e)) = L^1(\sum_{\mathbf{N}})$ and we have

$$\| \lim_{k \to \infty} A(\sigma^+)^k f - e\|_1 = 0$$

for all f in $PDF(\sum_{\mathbf{N}})$. Moreover we have $L^\infty(\mathcal{E})= L^\infty(\sum_{\mathbf{N}})$ and

$$\| \lim_{k \to \infty} A(\sigma^+)^k f - e\|_\infty = 0$$

for all f in $L^\infty(\sum_{\mathbf{N}}) \cup PDF(\sum_{\mathbf{N}})$.

Example 4.4. Let $X = \sum_{\mathbf{Z}} = \prod_{m\in\mathbf{Z}}\{0,1\}$ and σ be the two-sided shift map of X onto X, that is, $y = \sigma(x)$, where $x = (x_m)_{m\in\mathbf{Z}}, y = (y_m)_{m\in\mathbf{Z}}$ and $y_m = x_{m+1}$ $(m \in \mathbf{Z})$. Let μ be the canonical measure on X, which satisfies the same property as μ in Example 2.6. Namely, for

$$E = E(i_1,\ldots,i_k|c_1,\ldots,c_k) = \{x = (x_m)_{m\in\mathbf{Z}}|x_{i_1} = c_1,\cdots,x_{i_k} = c_k\}$$

it follows that $\mu(E) = \frac{1}{2^k}$. Since σ is a homeomorphism of X onto itself, it is a bi-measurable map of X onto itself. Hence σ is an MW1L on X with $B(\sigma)_1 = A(\sigma^{-1})$ with $\frac{d\mu\circ\sigma^{-1}}{d\mu}(x) = 1$ and

$$(A(\sigma)f)(x) = \frac{d\mu\circ\sigma^{-1}}{d\mu}(x)f(\sigma^{-1}(x)) = f(\sigma^{-1}(x)).$$

Set $e = \chi_X$. Then e is a unique $A(\sigma)$-invariant vector in $L^1(\sum_{\mathbf{Z}})$ and the set $\mathcal{E}(e)$ defined in Proposition 1.5 consists of only one vector e. Hence $L^1(\mathcal{E}(e))$ is the one-dimensional space generated by e and $PDF(\sum_{\mathbf{Z}}) \cap L^1(\mathcal{E}(e)) = \{e\}$. Thus the following convergency is guaranteed for only $f = e$.

$$\lim_{m\to\infty} \|A(\sigma)^m f - e\|_1 = 0.$$

In fact, we can find easily a vector f in $L^1(\sum_{\mathbf{Z}})$ such that $\{A(\sigma)^m f\}_{m=1}^{\infty}$ does not converge to e in the $\|\cdot\|_1$-topology. Namely, put $f = 2\chi_{E(0|0)}$, where $E(0|0) = \{x = (x_k) \in X|x_0 = 0\}$. Then we have

$$A(\sigma)^m f = 2\chi_{E(-m|0)} \quad \text{and} \quad \|2\chi_{E(-m|0)} - \chi_X\|_1 = 1$$

for all m in $\mathbf{N}$.

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