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The Path-Connectivity of Wavelets and Frame Wavelets

Xingde Dai and Yuanan Diao

ABSTRACT. In this paper, we discuss a special topological property of the various sets of wavelets frame wavelets, namely the path-connectivity of these sets. We review and outline some recent results in this area and post many open questions.

1. Introduction

The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. The question concerning the path-connectedness of the set of all orthonormal wavelets was first raised in [**7**]. Similar questions were raised and studied in [**15, 16, 17**] about the sets of all MRA-wavelets, tight frame wavelets and MRA tight frame wavelets. These questions are hard questions and many of them remain unsolved at this time. In this paper, we will tell some successful stories in this area, post some un-solved problems, discuss the difficulties one faces in attacking these problems. We hope the discussions here will shed some lights for future studies.

A set $S \subset L^2(\mathbb{R})$ is said to be *path-connected* under the norm topology of $L^2(\mathbb{R})$ if for any two elements f, $g \in S$, there exists a mapping $\gamma : [0,1] \longrightarrow S$ such that $\gamma(t)$ is continuous in the norm of $L^2(\mathbb{R})$, $\gamma(0) = f$ and $\gamma(1) = g$.

Let T and D be the translation and dilation unitary operators acting on $L^2(\mathbb{R})$ defined by

$$(Tf)(t) = f(t-1) \text{ and } \quad (Df)(t) = \sqrt{2}f(2t), \forall f \in L^2(\mathbb{R})$$

and let $\mathcal{F}$ be the (unitary) Fourier-Plancherel transform which is uniquely defined by the following formula for each function f in the dense (in $L^2(\mathbb{R})$) set $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$

$$(\mathcal{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)dt.$$

$\mathcal{F}(f)$ is sometimes written as $\widehat{f}$ as well. Notice that

$$\begin{aligned}(\mathcal{F}T_\alpha f)(s) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t-\alpha)dt \\ &= e^{-is\alpha}(\mathcal{F}f)(s).\end{aligned}$$

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So $\mathcal{F}T_\alpha\mathcal{F}^{-1}g = e^{-is\alpha}g$. For any bounded linear operator A acting on $L^2(\mathbb{R})$, denote $\mathcal{F}A\mathcal{F}^{-1}$ by $\widehat{A}$. For any $h \in L^\infty(\mathbb{R})$, let M_h be the multiplication operator defined by $M_h(f) = hf$ for any $f \in L^2(\mathbb{R})$. Thus $\widehat{T_\alpha} = M_{e^{-i\alpha s}}$. Similarly,

$$\begin{aligned} (\mathcal{F}D^n f)(s) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ist}(\sqrt{2})^n f(2^n t)dt \\ &= (\sqrt{2})^{-n}\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i2^{-n}st}f(t)dt \\ &= (\sqrt{2})^{-n}(\mathcal{F}f)(2^{-n}s) = (D^{-n}\mathcal{F}f)(s). \end{aligned}$$

So $\widehat{D^n} = D^{-n} = D^{*n}$. In particular, $\widehat{D} = D^{-1} = D^*$.

DEFINITION 1.1. An *orthogonal wavelet* of $L^2(\mathbb{R})$ is a function $\psi(t)$ in $L^2(\mathbb{R})$ with unit norm such that $\{2^{\frac{n}{2}}\psi(2^n t - \ell) : n, \ell \in \mathbb{Z}\} = \{D^n T^\ell \psi : n, \ell \in \mathbb{Z}\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

The following open problem concerns the path-connectivity of the set of all orthonormal wavelets of $L^2(\mathbb{R})$.

PROBLEM 1.2. Let $\mathcal{W}$ be the set of all orthonormal wavelets of $L^2(\mathbb{R})$. Prove or disprove that $\mathcal{W}$ is path-connected.

Although not much can be said about the set $\mathcal{W}$ itself, there are many subclasses of orthonormal wavelets that are of interest, both for the theoretical study and application reasons. Much effort has been devoted to the study of these subclasses and some very nice results have been obtained.

2. The MRA Wavelets

The most important subclass of orthonormal wavelets in practical applications is probably the MRA wavelets and it turns out that this subclass is indeed path-connected, a very nice property. In this section, we will introduce the concept of *Multi-Resolution Analysis* (MRA), state the path-connectivity theorem of MRA wavelets and outline a proof of it.

DEFINITION 2.1. A *multi-resolution analysis* (MRA) of $L^2(\mathbb{R})$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following conditions.

(1) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;

(2) $\cap_{j\in\mathbb{Z}}V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}}V_j} = L^2(\mathbb{R})$;

(3) $\forall f \in L^2(\mathbb{R})$, $f \in V_j$ if and only if $f(2s) \in V_{j+1}$, $j \in \mathbb{Z}$;

(4) there exists $\phi \in V_0$ such that $\{\phi(\cdot - \ell) : \ell \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function ϕ in the above condition (4) is called a *scaling function* for the multi-resolution analysis. A function $\psi \in V_1 \ominus V_0$ is called an *MRA wavelet* if it is a wavelet of $L^2(\mathbb{R})$. By [**14**], every MRA produces an MRA wavelet so the set of MRA wavelets for any given MRA is non-empty. We will write $\mathcal{W}^M$ for the set of all MRA wavelets. We have the following theorem.

THEOREM 2.2. *The set $\mathcal{W}^M$ of all MRA-wavelets is path-connected.*

2.1. Filters, Multipliers and Phases. This subsection concerns three important concepts needed for the proof of the connectivity of $\mathcal{W}^M$. For any MRA wavelet ψ and its corresponding scaling function ϕ, there exists a 2π periodic function $m(s)$ (called the *filter function*) satisfying the following conditions [**8**]

$$\widehat{\phi}(s) = m(s/2)\widehat{\phi}(s/2),$$
$$|m(s)|^2 + |m(s+\pi)|^2 = 1,$$
$$\widehat{\psi}(s) = e^{is/2}\overline{m(s/2+\pi)}\widehat{\phi}(s/2),$$

where the equalities in the above are all in the a.e. sense.

DEFINITION 2.3. A measurable function ν is called a *functional wavelet multiplier* if the Fourier inverse transform of $\nu\widehat{\psi}$ is an orthonormal wavelet whenever ψ is an orthonormal wavelet.

The functional wavelet multipliers can be characterized by the following theorem.

THEOREM 2.4. *A measurable function ν is a functional wavelet multiplier if and only if it is unimodular and $\nu(2t)/\nu(t)$ is a 2π periodic function a.e..*

Let ν be as in Theorem 2.4 and let M_ν be the multiplicative operator on $L^2(\mathbb{R})$, i.e., $M_\nu g(s) = \nu g(s)$ for any $g \in L^2(\mathbb{R})$. So if ψ is an orthonormal wavelet, then by theorem 2.4, $\mathcal{F}^{-1}M_\nu\mathcal{F}\psi$ is also an orthonormal wavelet. Let us write $\widehat{M_\nu}$ for $\mathcal{F}^{-1}M_\nu\mathcal{F}$. Then $\widehat{M_\nu}$ is a unitary operator and it maps $\mathcal{W}$ into itself. Since ν^{-1} is also a functional wavelet multiplier, it follows that $\widehat{M_\nu}$ is mapping $\mathcal{W}$ one-to-one and onto itself. In other word, ν induces an one-to-one and onto mapping $\widehat{M_\nu}$ from $\mathcal{W}$ to $\mathcal{W}$. Of course there are other mappings with such a property. For example, let M_* be the (unitary) operator defined by $M_*g(s) = g(-s)$ (for any $g \in L^2(\mathbb{R})$) and let $\widehat{M_*}$ be the unitary operator $\mathcal{F}^{-1}M_*\mathcal{F}$. Then it is obvious that $\widehat{M_*}$ is also an one-to-one and onto mapping from $\mathcal{W}$ to $\mathcal{W}$. Notice however that the operator M_* is *not* a functional multiplier. In general, we will call a unitary operator an *operator wavelet multiplier* if it is an one-to-one and onto mapping from $\mathcal{W}$ to $\mathcal{W}$. Thus the set of all wavelet multipliers is a subset of the set of all operator wavelet multipliers. A natural question is how to characterize an operator wavelet multiplier. However this question remains open up to date. We list it as the following (open) problem.

PROBLEM 2.5. Characterize the operator wavelet multipliers.

DEFINITION 2.6. Let $\psi(t)$ be a wavelet. Then $\widehat{\psi}(s) = e^{i\alpha(s)}|\widehat{\psi}(s)|$ for some real valued function $\alpha(s)$. $\alpha(s)$ is called the *phase function* (or just the phase) of ψ. If α is a linear function then we say that ψ has a *linear phase.*

It is an interesting (and important) question whether a wavelet ψ has linear phase [**2**], as such information would lead us to a better understanding of the structure of $\mathcal{W}$. The following theorem [**13**] gives a nice answer to this question and provided a key step in proving the path-connectivity of $\mathcal{W}^M$.

THEOREM 2.7. [**13**] *Let ψ be an MRA wavelet, then there exists a functional wavelet multiplier ν such that*

$$\widehat{\psi}(s) = e^{is/2}\nu(s)|\widehat{\psi}(s)|.$$

2.2. An Outline of the Proof of the Path-Connectivity of $\mathcal{W}^M$. Now, we will outline a proof of Theorem 2.2. Let ψ be a given MRA-wavelet. Let ψ_0 be defined by $\widehat{\psi}_0(s) = e^{is/2}|\widehat{\psi}(s)|$. It can be shown that ψ_0 is also an MRA wavelet with the same MRA as that of ψ. Let ϕ_0 and $m_0(s)$ be its corresponding scaling function and filter function. On the other hand, let ψ_1 be the well known Payley-Littlewood wavelet defined by

$$\widehat{\psi}_1(s) = e^{is/2}\frac{1}{\sqrt{2\pi}}\chi_E,$$

where χ_E is the characteristic function of the set $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$. It is known that ψ_1 is an MRA wavelet. Let $\phi_1(s)$ and $m_1(s)$ be its corresponding scaling function and filter function. We have

$$\phi_1(s) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi)},\ m_1(s) = \chi_{[-\frac{\pi}{2},\frac{\pi}{2})}.$$

We wish to construct a path connecting ψ to ψ_0 and a path connecting ψ_0 to ψ_1. This is done by constructing paths connecting their Fourier transforms first, then taking the inverse Fourier transform of these paths.

Step 1. By Theorem 2.7 and the definition of ψ_0, $\widehat{\psi}(s) = \nu(s)\widehat{\psi}_0(s)$ for some functional wavelet multiplier $\nu(s)$. In general, it can be shown that a function $\nu(s)$ is a functional wavelet multiplier if and only it can be written as $e^{i\alpha(s)}$ for some real valued function $\alpha(s)$ such that $\alpha(2s) - \alpha(s)$ is 2π-periodic. Therefore, for any $\lambda \in [0,1]$, $e^{i\lambda\alpha(s)}$ is also a wavelet multiplier, thus

$$\widehat{\psi}_\lambda(s) = e^{i\lambda\alpha(s)}\widehat{\psi}_0(s),\ \lambda \in [0,1]$$

defines a path connecting $\widehat{\psi}$ to $\widehat{\psi}_0$.

Step 2. Notice that the filter function for ψ_1 is $\chi_E(s)$ where $E = [-\frac{\pi}{2}, \frac{\pi}{2})$. Let m_0 be the filter function for ψ_0. For any $\lambda \in [0,1]$, define a new function $m_\lambda(s)$ on $[-\pi,\pi)$ as follows.

$$\begin{cases} m_\lambda(s) &= (1-\lambda)m(s), & s \in [-\frac{\pi}{2},\frac{\pi}{2})\backslash(1-\lambda)[-\pi,\pi) \\ m_\lambda(s) &= 1, & s \in (1-\lambda)[-\pi,\pi) \\ m_\lambda(s) &= \sqrt{1-m_\lambda^2(s+\pi)}, & s \in [-\pi,-\frac{\pi}{2}) \\ m_\lambda(s) &= \sqrt{1-m_\lambda^2(s-\pi)}, & s \in [\frac{\pi}{2},\pi). \end{cases}$$

Extending $m_\lambda(s)$ to a 2π periodic function on $\mathbb{R}$ yields a filter function, which will in turn define a scaling function $\phi_\lambda(s)$ by the relation equation $\widehat{\phi}_\lambda(s) = m_\lambda(s/2)\widehat{\phi}_\lambda(s/2)$. The corresponding wavelet function $\psi_\lambda(s)$ will then define the path connecting ψ_0 to ψ_1.

Interested readers can find detailed proofs of Theorems 2.2, 2.4 and 2.7 in [**13**], as well as some related discussions. A proof to Theorem 2.2 was obtained by D. Han and S. Lu, and by R. Liang and X. Dai independently at about the same time. Theorem 2.4 was first obtained by Q. Gu [**10**]. These results were collected in [**17**].

3. The FMRA Wavelets

It is known that there exist orthonormal wavelets which are not MRA wavelets. Naturally, we would like to extend the results of last section to more general cases. One way to do so is to generalize the MRA wavelets to a larger set in a natural way so that some earlier results and methods may be applied. The *Frame Multi-Resolution Analysis* (FMRA) is one of such choices. More specifically, an FMRA of $L^2(\mathbb{R})$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following conditions.

(1) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;

(2) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;

(3) $\forall f \in L^2(\mathbb{R})$, $f \in V_j$ if and only if $f(2s) \in V_{j+1}$, $j \in \mathbb{Z}$;

(4′) there exists $\phi \in V_0$ such that for every function f in V_0 we have

$$\sum_{\ell\in\mathbb{Z}} \langle f, T^\ell \phi \rangle T^\ell \phi$$

converges to f in norm.

Notice that the first three conditions above are identical to those defining an MRA. The function ϕ in (4′) is called a *frame scaling function* for the frame multi-resolution analysis. If $\psi \in V_1 \ominus V_0$ is a wavelet, then ψ will be called an *FMRA wavelet.*

It is clear from the definition that MRA wavelets are also FMRA wavelets, but not vice versa. In fact, it is known that there exist FMRA wavelets which are not an MRA wavelets [**5**]. Let $\mathcal{W}^F$ be the set of all FMRA wavelets and we would like to know whether $\mathcal{W}^F$ is path-connected. Unfortunately, even though the structure of an FMRA seems very close to that of an MRA, results such as Theorems 2.2, 2.4 and 2.7 simply do not exist in the case of FMRA. Thus the following remains a challenging open question.

PROBLEM 3.1. Prove or disprove that $\mathcal{W}^F$ is path-connected.

4. The s-elementary Wavelets

We will now consider a different set of wavelets other than the MRA wavelets. In a sense this is the set of simplest wavelets since the Fourier transforms of the wavelets in this set are simple set theoretic functions.

DEFINITION 4.1. Let $E \subset \mathbb{R}$ be a measurable set of finite measure. If $\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_E) = \psi_E$ is an orthonormal wavelet, then ψ_E is called an s-elementary wavelet and E is called a *wavelet set.* Furthermore, the set of all s-elementary wavelets is denoted by $\mathcal{W}^s$.

Notice that there are s-elementary wavelets that are not MRA wavelets [**7**]. Therefore, $\mathcal{W}^s$ is not a subset of $\mathcal{W}^M$.

Let E be a (Lebesgue) measurable set. We say that $x, y \in E$ are $\overset{\delta}{\sim}$ *equivalent* if $x = 2^n y$ for some integer n. The δ-*index* of a point x in E is the number of elements in its $\overset{\delta}{\sim}$ equivalent class and is denoted by $\delta_E(x)$. Let $E(\delta, k) = \{x \in E : \delta_E(x) = k\}$. Then E is the disjoint union of the sets $E(\delta, k)$ and each $E(\delta, k)$ is also measurable. Similarly, we say that $x, y \in E$ are $\overset{\tau}{\sim}$ *equivalent* if $x = y + 2n\pi$ for some integer n. The τ-*index* of a point x in E is the number of elements in its $\stackrel{\tau}{\sim}$ equivalent class and is denoted by $\tau_E(x)$. Let $E(\tau,k)=\{x\in E:\ \tau_E(x)=k\}$. Then E is the disjoint union of the sets $E(\tau,k)$ and each $E(\tau,k)$ is measurable. Each $E(\delta,k)$ (resp. $E(\tau,k)$) can be further decomposed into k disjoint copies $E^{(j)}(\delta,k)$ (resp. $E^{(j)}(\tau,k)$) such that each $E^{(j)}(\delta,k)$ (resp. $E^{(j)}(\tau,k)$) is measurable and $E^{(j)}(\delta,k)=(E^{(j)}(\delta,k))(\delta,1)$ (resp. $E^{(j)}(\tau,k)=(E^{(j)}(\tau,k))(\delta,1)$), though these decompositions are not unique in general. The following theorem characterizes a wavelet set.

THEOREM 4.2. [**7**] *A measurable set $E\subset\mathbb{R}$ is a wavelet set if and only if $E=E(\delta,1)=E(\tau,1)$, $\cup_{k\in\mathbb{Z}}2^kE=\mathbb{R}$ and $\cup_{k\in\mathbb{Z}}(E+2k\pi)=\mathbb{R}$.*

Based on the above characterization, D. Speegle [**16**] was able to prove the following theorem.

THEOREM 4.3. *The set of s-elementary wavelets is path-connected.*

Theorem 4.3 is proved through direct construction of the path. The proof is quite technical and complicated. Interested reader please refer to [**16**]. One may wonder if Theorem 4.3 may be used as a tool in proving the path-connectivity of $\mathcal{W}$, since a wavelet set is not only the support set of the Fourier transform of an s-elementary wavelet, but it is a "minimum" support set in the sense that any essential smaller subset of it cannot be the support of the Fourier transform of any orthonormal wavelet [**12**]. However, it is not clear that wavelet sets are really "minimum" in the sense that the support of the Fourier transform of any orthonormal wavelet must contain a wavelet set. Because of its apparent importance, we list this as the following open question.

PROBLEM 4.4. Let ψ be any orthonormal wavelet. Does the support of $\widehat{\psi}$ have to contain a wavelet set?

Notice that the set $\mathcal{W}^M\cap\mathcal{W}^s$ is not empty. We also propose the following problem.

PROBLEM 4.5. Is the set $\mathcal{W}^M\cap\mathcal{W}^s$ path-connected?

Similarly, following the definition of FMRA wavelets, one may define the s-elementary FMRA wavelets. The set of all such wavelets is simply the set $\mathcal{W}^F\cap\mathcal{W}^s$. In [**5**], it is shown that there exist s-elementary FMRA wavelets, i.e., $\mathcal{W}^F\cap\mathcal{W}^s\neq\emptyset$. Thus the following is a valid question.

PROBLEM 4.6. Is the set $\mathcal{W}^F\cap\mathcal{W}^s$ path-connected?

Let E, F be two wavelet sets and let ψ_E, ψ_F be their corresponding s-elementary wavelets. We will say that ψ_E, ψ_F are connected by a *direct path* if there exists a path connecting ψ_E, ψ_F such that each point on the path is an s-elementary wavelet ψ_S for some wavelet set $S\subset E\cup F$. The following is still an open question.

PROBLEM 4.7. Given any two s-elementary wavelets, is there always a direct path connecting them?

Interested readers may refer to [**1**] for more general discussions on this issue.

5. The s-elementary Frame Wavelets

A function $\psi \in L^2(\mathbb{R})$ is called a *frame wavelet* for $L^2(\mathbb{R})$ if there exist two positive constants $0 < a \leq b$ such that for any $f \in L^2(\mathbb{R})$,

$$a\|f\|^2 \leq \sum_{n,\ell \in \mathbb{Z}} |\langle f, D^n T^\ell \psi \rangle|^2 \leq b\|f\|^2. \tag{5.1}$$

A number a with this property is called a *lower frame bound* of ψ and a number b with this property is called an *upper frame bound* of ψ. The supremum of all such numbers a is called the *optimal lower frame bound of* ψ and is denoted by a_0. Similarly, the infimum of of all such numbers b is called the *optimal upper frame bound of* ψ and is denoted by b_0. If $a_0 = b_0$, then ψ is called a *tight frame wavelet*. Furthermore, if $a_0 = b_0 = 1$, then ψ is called a *normalized tight frame wavelet*. Let E be a Lebesgue measurable set of finite measure and χ_E be the corresponding characteristic function. If the function $\psi_E \in L^2(\mathbb{R})$ defined by $\widehat{\psi_E} = \frac{1}{\sqrt{2\pi}}\chi_E$ is a frame wavelet, a tight frame wavelet or a normalized tight frame wavelet for $L^2(\mathbb{R})$, then the set E is called a *frame wavelet set*, a *tight frame wavelet set* or a *normalized tight frame wavelet set* for $L^2(\mathbb{R})$ respectively. The corresponding function ψ_E is called an *s-elementary*, a *tight s-elementary* or a *normalized tight s-elementary* frame wavelet. The set of all frame wavelets is denoted by $\mathcal{W}_f$ and the set of all s-elementary frame wavelets is denoted by $\mathcal{W}_f^s$. Furthermore, the set of all s-elementary tight frame wavelets with optimal frame bound k is denoted by $\mathcal{W}_f^s(k)$. By a result from [**3**], the optimal frame bound of an s-elementary tight frame wavelet is an integer, i.e., $k \in \mathbb{N}$ for each $\mathcal{W}_f^s(k)$. In particular, $\mathcal{W}_f^s(1)$ is the set of all s-elementary normalized tight frame wavelets. The following problem is still open at this time.

PROBLEM 5.1. Is the set $\mathcal{W}_f$ path-connected?

Again, as a first step, we will look into the simple cases of the s-elementary frame wavelets.

5.1. The Characterization of Frame Wavelet Sets. Before we consider the path-connectivity of the sets $\mathcal{W}_f^s$ or $\mathcal{W}_f^s(k)$, let us take a look at the characterization of the frame wavelet sets and tight frame wavelet sets, since this will determine the properties of the corresponding frame wavelets and tight frame wavelets. Also, it is intuitive to expect that such characterization would play an essential role in proving the path-connectivity of $\mathcal{W}_f^s$ or $\mathcal{W}_f^s(k)$. The following theorem is quoted from [**3**], which characterizes the tight frame wavelet sets (hence the s-elementary tight frame wavelets).

THEOREM 5.2. *Let E be a Lebesgue measurable set with finite measure. Then E is a tight frame wavelet set if and only if $E = E(\tau, 1) = E(\delta, k)$ for some $k \geq 1$ and $\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}$. In particular, E is a normalized tight frame wavelet set if and only if $E = E(\tau, 1) = E(\delta, 1)$ and $\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}$.*

However, the characterization of frame wavelet sets is still an open question. Only some partial results are known about the frame wavelet sets.

5.2. The Normalized Tight s-elementary Frame Wavelets. The following theorem is proved in [**6**].

THEOREM 5.3. *The set $\mathcal{W}_f^s(1)$ is path-connected.*

As expected, the proof of this theorem is indeed based on the characterization of normalized tight frame wavelet sets. In fact, this result also holds in the higher dimensions. Please refer to [**6**] for its proof.

5.3. The Tight s-elementary Frame Wavelets. The path-connectivity of tight s-elementary frame wavelets at this time remains an open question, despite Theorem 5.2. However we do strongly feel that this is true. We state it as the following conjecture.

CONJECTURE 5.4. The set $\mathcal{W}_f^s(k)$ of all tight s-elementary frame wavelets with optimal frame bound k is path-connected.

5.4. The s-elementary Frame Wavelets. As the characterization of frame wavelet sets remains an open question, one probably would not have expected to see the following result concerning the path-connectivity of the set $\mathcal{W}_f^s$, since a proof to it would seem to have to heavily depend on the characterization.

THEOREM 5.5. [4] *The set $\mathcal{W}_f^s$ is path-connected.*

The proof of Theorem 5.5 involves some basic techniques the authors developed in dealing with the frame wavelet problems. We will provide a rather detailed proof here to help our reader to gain some feeling about the nature and the difficulties of this problem. The basic idea is that for a given frame wavelet set E, we prove that there is a continuous path of the form χ_{W_t} connecting χ_E to χ_F, where each W_t is a frame wavelet set and F is a normalized tight frame set. This implies that each s-elementary frame wavelet is connected by a continuous path (of s-elementary frame wavelets) to a normalized tight s-elementary frame wavelet. This then leads to our result by Theorem 5.3. We will first need some new concepts and a few lemmas.

For a measurable set E with finite measure and for any $f \in L^2(\mathbb{R})$, define:

$$(H_E f)(s) = \sum_{n,\ell\in\mathbb{Z}} \langle f, \widehat{D}^n\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E\rangle \widehat{D}^n\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E(s). \tag{5.2}$$

A set E is called a *Bessel set* if $H_E f$ converges in norm unconditionally for each $f \in L^2(\mathbb{R})$ and $\langle H_E f, f\rangle \le B\|f\|^2$ for some constant $B > 0$. On the other hand, E is called a *basic set* if there exists $M > 0$ such that $\mu(E(\delta, m)) = \mu(E(\tau, m)) = 0$ for all $m > M$ (where μ is the Lebesgue measure). Theorem 1 of [**3**] implies the following lemma.

LEMMA 5.6. *A set E is Bessel if and only if it is a basic set. Moreover, if $\mu(E(\delta, m)) = \mu(E(\tau, m)) = 0$ for all $m > M$ (where μ is the Lebesgue measure), then $\langle H_E f, f\rangle \le M^{5/2}\|f\|^2$ for any $f \in L^2(\mathbb{R})$.*

The following lemma can also be obtained using similar arguments in the proof of Theorem 2 in [**3**].

LEMMA 5.7. *Let E be a basic set. Assume that $\Omega = \cup_{k\in\mathbb{Z}}2^kE(\tau, 1) = \cup_{k\in\mathbb{Z}}2^kE$. Then*

$$\langle H_E f, f\rangle \ge \|f\|^2, \forall f \in L^2(\Omega).$$

Lemma 3 below is obtained by combining Lemma 5.6 and Lemma 5.7.

LEMMA 5.8. *Let E be a basic set and $E(\tau, m) = E(\delta, m) = \emptyset, \forall m > M$. Let F be a measurable set such that $E \subset \cup_{k\in\mathbb{Z}} 2^k F$ and $F = F(\tau, 1)$. Then*

$$\langle H_E f, f\rangle \leq M^{5/2}\langle H_F f, f\rangle, \quad \forall f \in L^2(\mathbb{R}).$$

For any $E \subset \mathbb{R}$, let $\tau(E) = \bigcup_{\mathbb{Z}}(E + 2k\pi)$. Be careful not to confuse $\tau(E)$ with $\tau_E(x)$, the translation index of x in E. We say that two sets E and F are 2π-translation disjoint if $\tau(E) \cap \tau(F) = \emptyset$. The following lemma is obtained from Lemma 5 of [**3**].

LEMMA 5.9. *If E and F are 2π-translation disjoint basic sets, then*

$$H_{E\cup F} f = H_E f + H_F f, \forall f \in L^2(\mathbb{R}).$$

It is well-known that if $\widehat{\psi}_E = \frac{1}{\sqrt{2\pi}}\chi_E$, then ψ_E is a frame wavelet with frame bounds $0 < a \leq b$ if and only if

$$a\|f\|^2 \leq \sum_{n,\ell\in\mathbb{Z}} |\langle f, \widehat{D}^n\widehat{T}^\ell \frac{1}{\sqrt{2\pi}}\chi_E\rangle|^2 \leq b\|f\|^2. \tag{5.3}$$

Thus, ψ_E is a frame wavelet with frame bounds $0 < a \leq b$ if and only if

$$a\|f\|^2 \leq \langle H_E f, f\rangle \leq b\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{5.4}$$

Now let E be a frame wavelet set and ψ_E be its corresponding s-elementary frame wavelet. E is a Bessel set hence a basic set by Lemma 5.6. So there is a number $M > 0$ such that $E(\tau, m) = E(\delta, m) = \emptyset, \forall m > M$. Thus $B = M^{5/2}$ is an upper frame bound of ψ_E. Let $a > 0$ be a lower frame bound of ψ_E. So we have $a\|f\|^2 \leq \langle H_E f, f\rangle \leq M^{5/2}\|f\|^2$ for all $f \in L^2(\mathbb{R})$. Let m_0 be a positive integer large enough so that $M/2^{m_0} < \frac{1}{4}$. Let

$$F = [-\frac{2\pi}{2^{m_0+1}}, -\frac{\pi}{2^{m_0+1}}) \cup [\frac{\pi}{2^{m_0+1}}, \frac{2\pi}{2^{m_0+1}}).$$

By Theorem 5.2, F is a normalized tight frame wavelet set. It is easy to see that $E \cup F$ is a basic set and every measurable subset of $E \cup F$ is a basic set.

For any $s \in E$, there is a unique integer $k(s)$ such that $s/2^{k(s)} \in F$. Thus $h(s) = s/2^{k(s)}$ defines a mapping from E to F. One can prove that the image of each measurable subset in E under h is measurable. Furthermore, if E' is a subset of $E \cap \mathbb{R}\backslash[-\pi, \pi]$, then $\mu(h(E')) < \frac{1}{2^{m_0+1}}\mu(E')$. Define

$$\begin{aligned}
F_t^0 &= [-\frac{2\pi}{2^{m_0+1}}, -\frac{(2-t)\pi}{2^{m_0+1}}] \cup [\frac{\pi}{2^{m_0+1}}, \frac{(1+t)\pi}{2^{m_0+1}}] \\
F_t^1 &= h(\tau(F_t^0) \cap (E\backslash F_t^0)), \\
F_t^2 &= h(\tau(F_t^1) \cap (E\backslash F_t^1)), \\
&\cdots \\
F_t^n &= h(\tau(F_t^{n-1}) \cap (E\backslash F_t^{n-1})), \\
&\cdots \\
F_t &= \bigcup_{k\geq 0} F_t^k, t \in [0, 1].
\end{aligned}$$

Notice that the set F_t is a measurable subset of F, hence it is a basic set. Let $E_t = \tau(F_t) \cap E$. It is clear that any point in $\tau(E_t)$ must be in $\tau(F_t)$ hence cannot be in $\tau(E \backslash E_t)$. So the sets E_t and $E \backslash E_t$ are 2π-translation disjoint. By Lemma 5.9 we have

$$H_E f = H_{E_t} f + H_{E\backslash E_t} f.$$

Hence

$$\langle H_E f, f\rangle = \langle H_{E_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle \geq a\|f\|^2. \tag{5.5}$$

Similarly,

$$H_{F_t \cup (E\backslash E_t)} f = H_{F_t} f + H_{E\backslash E_t} f$$

since F_t and $E \backslash E_t$ are also 2π-translation disjoint. It follows that

$$\langle H_{F_t \cup (E\backslash E_t)} f, f\rangle = \langle H_{F_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle. \tag{5.6}$$

Notice that $F_t = F_t(\tau, 1)$ since $F_t \subset F$ and $F = F(\tau, 1)$. Let $x \in E_t = E \cap \tau(F_t)$. If $x \notin F_t$, then $x \in \tau(F_t^n) \cup (E \backslash F_t^n)$ for some $n \geq 0$. So $h(x) \in F_t^{n+1} \subset F_t$. Hence we have

$$E_t \subset \bigcup_{k\in\mathbb{Z}} 2^k F_t. \tag{5.7}$$

By Lemma 5.8 we have

$$\langle H_{F_t} f, f\rangle \geq M^{-\frac{5}{2}} \langle H_{E_t} f, f\rangle. \tag{5.8}$$

Now define $W_t = F_t \cup (E \backslash E_t)$. Since $W_t \subset F \cup E$, it is a basic set. By Lemma 5.6, there is a positive number B (independent of t) such that

$$\langle H_{W_t} f, f\rangle \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{5.9}$$

On the other hand, (5.5), (5.6) and (5.8) imply that

$$\begin{aligned}
\langle H_{W_t} f, f\rangle &= \langle H_{F_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle \\
&\geq M^{-\frac{5}{2}} \langle H_{E_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle \\
&> M^{-\frac{5}{2}} \big(\langle H_{E_t} f, f\rangle + \langle H_{E\backslash E_t} f, f\rangle\big) \\
&\geq aM^{-\frac{5}{2}} \|f\|^2.
\end{aligned}$$

Therefore, W_t is a frame wavelet set for each $t \in [0, 1]$. It is easy to verify that $W_0 = E$ (since $F_0 = E_0 = \emptyset$) and $W_1 = F \cup (E \backslash \tau(F))$. Notice that F, $E \backslash \tau(F)$ are 2π-translation disjoint. Thus, by Lemma 5.7, for any measurable subset G of $E \backslash \tau(F)$, $F \cup G$ is a frame set since $\cup_{k\in\mathbb{Z}} 2^k F = \mathbb{R}$. In particular, if we let $G_t = (-\tan(\frac{\pi}{2}t), \tan(\frac{\pi}{2}t)) \cap (E \backslash \tau(F))$, then $F \cup G_t$ is a frame set. We leave it to our reader to verify that the mapping $t \longrightarrow \chi_{F\cup G_t}$ is continuous in norm. Since $G_0 = \emptyset$ and $G_1 = E \backslash \tau(F)$, this defines a continuous path from χ_F to χ_{W_1}. Therefore, to complete the proof of Theorem 5.5, it suffices to show that the mapping $t \longrightarrow \chi_{W_t}$ is continuous in norm. We will achieve this in a few steps.

Step 1: We first show that the mapping $t \to \chi_{F_t}$ is continuous in norm. For $0 \leq t \leq 1$, we have $\mu(F_t^0) \leq \pi/2^{m_0}$. By the property of E, for a point $s \in F_t^0$, the set $\{s+2k\pi : k \in \mathbb{Z}\} \cap E$ has at most M points. This implies that

$$\mu(\tau(F_t^0) \cap (E \backslash F_t^0)) \leq M\mu(F_t^0). \tag{5.10}$$

Since $\tau(F_t^0)\cap(E\backslash F_t^0)\subset\mathbb{R}\backslash[-\pi,\pi]$, it follows from (5.10) that

$$\begin{aligned}\mu(F_t^1) &\leq \frac{1}{2^{m_0+1}}\mu(\tau(F_t^0)\cap(E\backslash F_t^0))\\ &\leq \frac{M}{2^{m_0+1}}\mu(F_t^0)\leq\frac{1}{4}\mu(F_t^0).\end{aligned}$$

By induction, we have

$$\mu(F_t^n)\leq\frac{M}{2^{m_0+1}}\mu(F_t^{n-1})\leq\frac{1}{4^n}\mu(F_t^0). \tag{5.11}$$

Therefore, the convergence of $\chi_{\cup_{0\leq k\leq n}F_t^k}$ to χ_{F_t} is uniform with respect to $t\in[0,1]$. $\forall\epsilon>0$, choose $N>0$ large enough such that $\pi/4^N<\epsilon/4$, then for any $t\in[0,1]$, we have

$$\begin{aligned}|\chi_{\cup_{0\leq k\leq N}F_t^k}-\chi_{F_t}| &\leq \sum_{k>N}\frac{1}{4^k}\mu(F_t^0)\\ &\leq \frac{\mu(F_t^0)}{4^N}<\frac{\pi}{4^N}<\frac{\epsilon}{4},\end{aligned}$$

since $\mu(F_t^0)\leq\pi$ for any t. If the mapping $t\rightarrow\chi_{F_t^n}$ is continuous in norm for each n, then $\chi_{\cup_{0\leq k\leq N}F_t^k}$ is uniformly continuous on $[0,1]$. Thus, there exists $\delta(\epsilon)>0$ such that $|\chi_{\cup_{0\leq k\leq N}F_{t_2}^k}-\chi_{\cup_{0\leq k\leq N}F_{t_1}^k}|<\epsilon/2$ whenever $|t_2-t_1|<\delta(\epsilon)$. It follows that

$$\begin{aligned}|\chi_{F_{t_2}}-\chi_{F_{t_1}}| &\leq |\chi_{\cup_{0\leq k\leq N}F_{t_2}^k}-\chi_{\cup_{0\leq k\leq N}F_{t_1}^k}|\\ &+ |\chi_{\cup_{0\leq k\leq N}F_{t_2}^k}-\chi_{F_{t_2}}|+|\chi_{\cup_{0\leq k\leq N}F_{t_1}^k}-\chi_{F_{t_1}}|\\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\epsilon.\end{aligned}$$

That is, χ_{F_t} is also uniformly continuous on $[0,1]$. Therefore, it suffices for us to prove that the mapping $t\rightarrow\chi_{F_t^n}$ is continuous in norm for each n. We will prove this by induction. Clearly, the mapping $t\rightarrow\chi_{F_t^0}$ is continuous. Assume that it is true for n. We will show that it is true for $n+1$. For this purpose, we write $K\Delta L=(K\backslash L)\cup(L\backslash K)$ for any sets K and L, and let $D_t^n=\tau(F_t^n)\cap(E\backslash F_t^n)$. For any $t,t'\in[0,1]$, we claim that $D_t^n\Delta D_{t'}^n\subset\tau(F_t^n\Delta F_{t'}^n)\cap E$. Let $s\in D_t^n\Delta D_{t'}^n$. We can assume that $s\in D_t^n\backslash D_{t'}^n$. Then there is an integer k such that $s+2k\pi\in F_t^n$. However $s\notin F_t^n$. It follows that $k\neq0$. Thus $s\notin F_{t'}^n$, for otherwise we would have both s and $s+2k\pi\in F_{t'}^n\cup F_t^n\subset F\subset[-\pi,\pi)$ which is impossible since $k\neq0$. Therefore $s\in E\backslash F_{t'}^n$. Since $s\notin D_{t'}^n=\tau(F_{t'}^n)\cap(E\backslash F_{t'}^n)$, it follows that $s\notin\tau(F_{t'}^n)$. Hence $s+2k\pi\in F_t^n\Delta F_{t'}^n$ and therefore $s\in\tau(F_t^n\Delta F_{t'}^n)\cap E$, as expected.

We now have

$$F_t^{n+1}\Delta F_{t'}^{n+1}\subset h(D_t^n\Delta D_{t'}^n)\subset h(\tau(F_t^n\Delta F_{t'}^n)\cap E). \tag{5.12}$$

Therefore,

$$\begin{aligned}\mu(F_t^{n+1}\Delta F_{t'}^{n+1}) &\leq \mu(h((F_t^n\Delta F_{t'}^n)^+\cap E))\\ &\leq \frac{M}{2^{m_0+1}}\mu(F_t^n\Delta F_{t'}^n).\end{aligned} \tag{5.13}$$

(5.13) implies that the mapping $t\rightarrow\chi_{F_t^{n+1}}$ is continuous since the mapping $t\rightarrow\chi_{F_t^n}$ is. This completes the proof that the mapping $t\rightarrow\chi_{F_t^n}$ is continuous in norm for all n. Hence the mapping $t\rightarrow\chi_{F_t}$ is continuous, as claimed.

Step 2: We now show that the mapping $t \to \chi_{E_t}$ is also continuous. In fact, this follows from the inclusion $E_t \Delta E_{t'} \subset \tau(F_t \Delta F_{t'}) \cap E$, which implies that

$$\mu(E_t \Delta E_{t'}) \le \mu(\tau(F_t \Delta F_{t'}) \cap E) \le M\mu(F_t \Delta F_{t'}).$$

Step 3: Finally, the continuity of $t \to \chi_{W_t}$ follows from the continuity of the mappings $t \to \chi_{F_t}$ and $t \to \chi_{E \setminus E_t}$ and the fact that $F_t \cap (E \setminus E_t) = \emptyset$. This completes our proof of Theorem 5.5.

PROBLEM 5.10. Are any two s-elementary frame wavelets directly path-connected?

5.5. Connectivity with Direct Path. Notice that the proof of Theorem 5.5 depends on Theorem 5.3. Thus for any two given s-elementary frame wavelets ψ_{E_1} and ψ_{E_2}, the connecting path constructed using the approach outlined in the proof of Theorem 5.5 will use sets outside $E_1 \cup E_2$. In other word, it cannot be determined from the above section whether any two elements in $\mathcal{W}_f^s$ are connected by a direct path. This remains an open question.

PROBLEM 5.11. Given any two s-elementary frame wavelets ψ_{E_1} and ψ_{E_2}, is there always a direct path connecting them?

In fact, there are a few more questions one can ask here.

PROBLEM 5.12. Given any two s-elementary tight frame wavelets ψ_{E_1} and ψ_{E_2} (of the same optimal frame bound k), is there always a direct path within the set $\mathcal{W}_f^s(k)$ connecting them?

PROBLEM 5.13. Given any two s-elementary tight frame wavelets ψ_{E_1} and ψ_{E_2} (possibly with different optimal frame bounds), is there always a direct path within the set $\mathcal{W}_f^s$ connecting them?

5.6. Uniform Connectivity. Let ψ be a fixed orthonormal wavelet. The local commutant [7] at ψ is the set:

$$C_\psi(D,T) = \{A \in B(L^2(\mathbb{R})) : AD^nT^m\psi = D^nT^mA\psi\}.$$

For each frame wavelet η, there is a unique operator $U_\eta \in C_\psi(D,T)$ such that $U_\eta\psi = \eta$, U_η^* is injective and has closed range. Moreover, η is an orthonormal wavelet if and only if U_η is unitary, while η is a normalized tight frame wavelet if and only if U_η^* is an isometry [11].

Two frame wavelets η_0 and η_1 are said to be *uniformly path-connected* if there is a path of frame wavelets $\{\eta_t : t \in [0,1]\}$ such that U_{η_t} is a continuous path in the operator norm (and hence $\{\eta_t : t \in [0,1]\}$ is a continuous path in L^2-norm). The uniform connectivity for certain classes of wavelets is related to the interpolation theory of wavelets and was investigated in several papers [7, 9]. We may ask whether the path-connectedness of s-elementary frame wavelets can be strengthened to uniform path-connectedness. The answer to this question is no. In fact, the following theorem has more to say on this issue [4].

THEOREM 5.14. *None of the following sets is uniformly path-connected:*

(i) The set $\mathcal{W}_f$ of all frame wavelets;

(ii) The set $\mathcal{W}_f(1)$ of all normalized tight frame wavelets;

(iii) The set $\mathcal{W}_f^s$ of all s-elementary frame wavelets.

Given that all the answers above are negative, we wonder if there is any subset of $\mathcal{W}_f$ that is uniformly path-connected. And if there is, what structure such a set may have. We post this as the following question.

PROBLEM 5.15. Prove or disprove the existence of a uniformly path-connected subset of $\mathcal{W}_f$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE, CHARLOTTE, NC 28223, USA