

Property M and the fixed point property for nonexpansive mappings

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1. PRELIMINARIES

Let N_a be the set of all absolute normalized norms $\|\cdot\|$ on \mathbb{C}^2 , that is, $\|(z, w)\| = \||z|, |w|\|$ and $\|(1, 0)\| = \|(0, 1)\| = 1$, and Ψ the family of the convex (continuous) functions on the unit interval $[0, 1]$ with

$$(1) \quad \psi(0) = \psi(1) = 1 \text{ and } \max\{1 - t, t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1)$$

By (Bonsall and Duncan [3], see also [16]), If, for some $\|\cdot\| \in N_a$, a convex function ψ is defined by

$$\psi(t) = \|(1 - t, t)\| \quad (0 \leq t \leq 1), \tag{1}$$

then the function ψ is convex (continuous) and satisfies (1) and conversely, if, for any $\psi \in \Psi$, a norm $\|\cdot\|$ is define by

$$\|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z| + |w|}\right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0), \end{cases} \tag{2}$$

then $\|\cdot\|_\psi$ is an absolute normalized norm on \mathbb{C}^2 and satisfies (4). The ℓ_p -norms $\|\cdot\|_p$ are typical such examples and for any $\|\cdot\| \in N_a$ we have

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1 \tag{3}$$

([3]). By (4) the convex functions corresponding to the ℓ_p -norms are given by

$$\psi_p(t) = \begin{cases} \{(1 - t)^p + t^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1 - t, t\} & \text{if } p = \infty. \end{cases} \tag{4}$$

Let X and Y be Banach spaces and let $\psi \in \Psi$. The ψ -direct sum $X \oplus_\psi Y$ of X and Y is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi, \tag{5}$$

where the $\|(\cdot, \cdot)\|_\psi$ term in the right hand side is an element of N_a with the corresponding convex function ψ . The following monotone properties were proved.

Proposition 1 ([3]). Let X, Y Banach spaces and $\psi \in \Psi$. And let $(x, y), (z, w) \in X \oplus_\psi Y$. The following hold:

- (i) $\|x\| \leq \|z\|$ and $\|y\| \leq \|w\|$ implies $\|(x, y)\|_\psi \leq \|(z, w)\|_\psi$.
- (ii) $\|x\| < \|z\|$ and $\|y\| < \|w\|$ implies, $\|(x, y)\|_\psi < \|(z, w)\|_\psi$.

Proposition 2 ([17]). Let X, Y Banach spaces and $\psi \in \Psi$. And let $(x, y), (z, w) \in X \oplus_\psi Y$. The following hold:

- (i) $\|x\| \leq \|z\|$ and $\|y\| < \|w\|$, or, $\|x\| < \|z\|$ and $\|y\| \leq \|w\|$ implies $\|(x, y)\|_\psi < \|(z, w)\|_\psi$.
- (ii) For $t \in (0, 1)$, $\psi(t) > \psi_\infty(t)$ holds.

Proposition 3 ([10]). Let X, Y Banach spaces and $\psi \in \Psi$. And let $(x, y), (z, w) \in X \oplus_\psi Y$. The following hold:

- (i) Let $\|x\| < \|z\|$ and $\|y\| = \|w\|$. $\|(z, w)\|_\psi = \|(x, y)\|_\psi$ if and only if $\|(z, w)\|_\psi = \|w\|$.
- (ii) Let $\|x\| = \|z\|$ and $\|y\| < \|w\|$. $\|(z, w)\|_\psi = \|(x, y)\|_\psi$ if and only if $\|(z, w)\|_\psi = \|z\|$.

Let Y be a Banach space and P a projection on Y . P is called an $L(M)$ -projection if $x = \|Px\| + \|(id_Y - P)x\|(\max\{\|x\|, \|(id_Y - P)x\|\})$ for all $x \in Y$, respectively. Let $X^\perp \subset X^{***}$ be annihilator of X , i.e. $X^\perp = \{w \in X^{***} : w(x) = 0, \forall x \in X\}$. A closed subspace $X \subset Y$ is called an $L(M)$ -summand on Y if X is the range of an $L(M)$ -projection on Y . It is said that X is an $L(M)$ -embedded Banach space if there exists a closed subspace $X_s \subset X^{**}$ such that $X^{**} = X \oplus_1 X_s (X^{***} = X^* \oplus_1 X^\perp)$. (Cf. [8].) For some $\psi \in \Psi$, we shall introduce $\psi(\psi^*)$ -embedded Banach space if there exists a closed subspace $X_s \subset X^{**}$ such that $X^{**} = X \oplus_\psi X_s (X^{***} = X^* \oplus_\psi X^\perp)$.

Let $\{x_n\}$ is a sequence of a Banach space X , it is said that $\{x_n\}$ spans an asymptotically isometric copy of ℓ_1 if there exists a nonincreasing sequence $\{\delta_n\} \subset [0, 1)$ tending to 0 such that

$$\sum (1 - \delta_n) |\alpha_n| \leq \left\| \sum \alpha_n x_n \right\| \leq \sum |\alpha_n|$$

for every $\{\alpha_n\} \in \ell_1$. In this case we will denote $x_n \sim_{(asy)} \ell_1$.

The abstract measure topology (τ_μ) is defined by considering the class of convergent sequences. (Cf. [8].) Namely, if $\{x_n\}$ is a sequence in a Banach space X , we say that $\{x_n\}$ tends to 0 in the abstract measure topology $(\tau_\mu - \lim_n x_n = 0)$ if $\{x_n\}$ is norm bounded and every subsequence $\{x_{n_k}\}$ contains a subsequence $\{x_{n_{k_\ell}}\}$ such that $x_{n_{k_\ell}} \sim_{(asy)} \ell_1$ or $x_{n_{k_\ell}} \rightarrow 0$. A sequence $\{x_n\}$ tends to x in τ_μ if

$(\tau_\mu - \lim_n(x_n - x) = 0)$ and a subset $A \subset X$ is τ_μ -closed if it is τ_μ -sequentially closed.

Pfizer[13] proved the following theorem.

Theorem A.([13]) *Let X be an L -embedded Banach space ($X^{**} = X \oplus_1 X_s$). Let P be the natural projection on X^{**} with range X , and consider $C \subset X$ which is closed, bounded and convex. Then the following two assertions are equivalent:*

- (i) $P(\text{cl}_{\sigma(X^{**}, X^*)} C) = C$.
- (ii) C is closed for the abstract measure topology.

By Theorem A, Japón Pineda[14] proved the following theorems.

Theorem B.([14]) *Let X be an L -embedded Banach space. If C is a convex, bounded, closed for the abstract measure topology, subset of X which is diametral, then C is weakly compact.*

Theorem C.([14]) *Let X be the dual of an M -embedded space E . Then the abstract measure topology τ_μ is finer than the $\sigma(X, E)$ topology on bounded subsets of X .*

Theorem D.([14]) *Let X be the dual of an M -embedded space E . Then the following are equivalent:*

- (i) X has the $\sigma(X, X^*)$ -FPP.
- (ii) X has the $\sigma(X, E)$ -FPP.

By introducing the concept of ψ^* -embedded Banach space E , we obtained a generalization of Theorem C.

Theorem 1. *Let X be the dual of an ψ^* -embedded Banach space E with $\psi > \psi_\infty$. If C is a $\sigma(X, E)$ -compact subset of X which is diametral, then C is weakly compact.*

By the Theorem 1, we obtain a generalization of Theorem D.

Theorem 2. *Let X be the dual of an ψ^* -embedded Banach space E with $\psi > \psi_\infty$. The following are equivalent:*

- (i) X has the $\sigma(X, X^*)$ -FPP.
- (ii) X has the $\sigma(X, E)$ -FPP.

Banach space X has Schur property if every weakly convergent sequence of X converges strongly.

Using a Dominguez's generalization of the Garcia-Falset coefficient $R(X)$, $M(X)$ ([6]) is defined by

$$M(X) = \sup \left\{ \frac{1+a}{R(a, X)} : a \geq 0 \right\},$$

where

$$R(a, X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\| \right\},$$

where the supremum is taken over all $a > 0$, x with $\|x\| \leq a$ and weakly null sequences $\{x_n\}$ of the unit ball of X such that its double limit of $\{\|x_n - x_m\|\}_{n,m}$ exists and $\limlim_{n, m} \|x_n - x_m\| \leq 1$.

The following theorem is known.

Theorem E (cf. [1]). *Let X be a Banach space. If $M(X) > 1$, then X has weak fixed point property.*

The following two lemma have important roles for proving Theorem 5.

Lemma 1. Let $\{x_n^{(k)}\}, \{y_n^{(k)}\}$ of a Banach space X be nonzero double sequences with $\lim_{n \rightarrow \infty} \|x_n^{(k)}\| > 0$, $\lim_{n \rightarrow \infty} \|y_n^{(k)}\| > 0$ for each k . The following are equivalent.

- (i) $\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \|x_n^{(k)} + y_n^{(k)}\| = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (\|x_n^{(k)}\| + \|y_n^{(k)}\|)$.
- (ii) $\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \left\| \frac{x_n^{(k)}}{\|x_n^{(k)}\|} + \frac{y_n^{(k)}}{\|y_n^{(k)}\|} \right\| = 2$.

Lemma 2. Let $\{x_n^{(k)}\}, \{y_n^{(k)}\}$ of a Banach space X be nonzero double sequences with $\lim_{n \rightarrow \infty} \|x_n^{(k)}\| > 0$, $\lim_{n \rightarrow \infty} \|y_n^{(k)}\| > 0$ for each k . The following are equivalent.

- (i) $\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \|x_n^{(k)} + y_n^{(k)}\| = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (\|x_n^{(k)}\| + \|y_n^{(k)}\|)$.
- (ii) For every $\alpha > 0$, the following holds:
 $\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \|x_n^{(k)} + \alpha y_n^{(k)}\| = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (\|x_n^{(k)}\| + \alpha \|y_n^{(k)}\|)$.
- (iii) For some $\alpha > 0$, the following holds:

$$\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \|x_n^{(k)} + \alpha y_n^{(k)}\| = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (\|x_n^{(k)}\| + \alpha \|y_n^{(k)}\|).$$

In [9], Property M was introduced to be a necessary and sufficient condition of that $K(X)$, the Banach space of all linear compact operator of a Banach space X , is M-ideal in $L(X)$, the Banach space of all continuous linear operator.

X has property M if $\liminf \|x_n - x\| = \liminf \|x_n - y\|$ for every weakly null sequence $\{x_n\}$ and $x, y \in X$ with $\|x\| = \|y\|$ ([9]). In Lemma 2.1 of [9], he showed that X has property M if and only if $\liminf \|x_n - x\| \leq \liminf \|x_n - y\|$ for every weakly null sequence $\{x_n\}$ and $x, y \in X$ with $\|x\| \leq \|y\|$.

We gives another characterization of Propety M by using a norm of ψ -direct sum $X \oplus_\psi X$.

Theorem 3. Let X be a Banach space. The following are equivalent.

- (1) X has Property M;
- (2) For every weakly null sequence $\{x_n\}$ of B_X , there exists $\psi \in \Psi$ such that $\liminf_n \|x_n - x\| = \|(\liminf_n \|x_n\|, \|x\|)\|_\psi$ for every $x \in B_X$.

By Theorem 3, we can prove the following propostion included in [5] without proof.

Propostion 4.([5]) Let X be a Banach space with Property M and $\{x_n\}$ a sequence converging weakly to x . Then

$$\liminf_n \|x_n\| \leq \liminf_m \liminf_n \|x_n - x_m\| + (\|x\| \vee \liminf_n \|x_n - x\| - \liminf_n \|x_n - x\|).$$

We recall that

$$R(1, X) = \sup\{\liminf \|x_n - x\| : x \in B_X, x_n \in B_x, x_n \text{ converges weakly } 0, \liminf_m \liminf_n \|x_n - x_m\| \leq 1\}$$

Theorem 4. Let X be a Banach space. If X has property M, then $R(1, X) \leq \frac{3}{2}$, i.e. $M(X) \geq \frac{4}{3} > 1$.

We shall prove the following theorem by Propostion 2 , Lemma 1 and Lemma 2.

Theorem 5. Let $\psi \neq \psi_1$. $M(X \oplus_\psi Y) > 1$ if and only if $M(X) > 1$ and $M(Y) > 1$

By Theorem 5 and Theorem F, we obtain weak fixed point property for $X \oplus_\psi Y$.

Theorem 6. Let $\psi \neq \psi_1$. If $M(X) > 1$ and $M(Y) > 1$, then $X \oplus_\psi Y$ has weak fixed point property.

Theorem 3 and Theorem 4 provide the following corollary.

Corollary 1. ([5]) *Let X and Y be Banach spaces and $\psi \in \Psi$. If X and Y have Property M and $\psi \neq \psi_1$, then $X \oplus_\psi Y$ has weak fixed point property for nonexpansive mappings.*

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