数理解析研究所講究録 1456 巻 2005 年 195-205

Stability of Gröbner bases and ACGB -revised-

佐藤洋祐

YOSUKE SATO

東京理科大学数理情報科学科

DEPARTMENT OF MATHEMATICAL INFORMATION SCIENCE, TOKYO UNIVERSITY OF SCIENCE *

Abstract

We prove that a Gröbner basis of an ideal I in a polynomial ring $(K[\bar{A}])[\bar{X}]$ over the coefficient ring $K[\bar{A}]$ with a field K also becomes a Gröbner basis in a polynomial ring $R[\bar{X}]$ over the commutative von Neumann regular ring $R = K[\bar{A}]/I \cap K[\bar{A}]$ under the assumption that the elimination ideal $I \cap K[\bar{A}]$ is a zero-dimensional proper radical ideal in $K[\bar{A}]$. This result gives us an alternative natural proof for the former results concerning stability of the Gröbner basis property under specializations obtained by T. Becker first and further generalized by M. Kalkbrener. We also give a modified ACGB algorithm to compute parametric Gröbner bases, which starts from Gröbner bases in a polynomial ring $K[\bar{A}, \bar{X}]$ over a field K. Our algorithm is a similar but different approach from the algorithms obtained by A. Montes, which are essentially based on computation of Gröbner bases in a polynomial ring $K(\bar{A})[\bar{X}]$ over the quotient field $K(\bar{A})$.

1 Introduction

Stability of Gröbner bases is an important notion in computer algebra. There have been published many papers by many authors. In , [1] the following result is shown

Theorem 1.1 (T. Becker) *Let I be an ideal of a polynomial ring $K[\bar{A}, \bar{X}]$ over a field K with variables $\bar{A}$ and $\bar{X}$ such that $I \cap K[\bar{A}]$ is a zero-dimensional radical ideal in $K[\bar{A}]$. Let $G = \{g_1(\bar{A}, \bar{X}), \ldots, g_l(\bar{A}, \bar{X})\}$ be a Gröbner basis of I w.r.t. a term order $\geq$ of $T(\bar{A}, \bar{X})$ such that each variable X_i is greater than any term in $T(\bar{A})$ and the restriction of $\geq$ on $T(\bar{A})$ is a lexicographical term order. Let $\bar{a}$ be an m-tuple of elements of the algebraic closure $\overline{K}$ of K which is a zero of the ideal $I \cap K[\bar{A}]$. Then, G is stable for $\bar{a}$, that is G becomes a Gröbner basis with the specialization by $\bar{a}$, i.e. $\{g_1(\bar{a}, \bar{X}), \ldots, g_l(\bar{a}, \bar{X})\}$ becomes a Gröbner basis in $\overline{K}[\bar{X}]$ w.r.t. the term order that is a restriction of $\geq$ on $T(\bar{X})$.*

In [2], the above result is further generalized by the following result.

Theorem 1.2 (M. Kalkbrener) *Let J be an ideal of a polynomial ring $K[\bar{A}]$ and $\bar{a} = a_1, a_2, \ldots, a_m$ be a zero of J in some algebraic extension field K' of K, then we have the following properties.*

1. *The maximal ideal $\langle A_1 - a_1, A_2 - a_2, \ldots, A_m - a_m\rangle$ is the associated prime of some isolated primary component of J in $K'[\bar{A}]$ if and only if G is strongly stable for $\bar{a}$ for any Gröbner basis G in $(K[\bar{A}])[X]$ such that $\langle G\rangle \cap K[\bar{A}] = J$.*

*ysato@rs.kagu.tus.ac.jp

2. *In case the number of the variables $\bar{X}$ is more than 1, the maximal ideal $\langle A_1-a_1, A_2-a_2, \ldots, A_m-a_m\rangle$ is some isolated primary component of J in $K'[\bar{A}]$ if and only if G is strongly stable for $\bar{a}$ for any Gröbner basis G in $(K[\bar{A}])[\bar{X}]$ such that $\langle G\rangle \cap K[\bar{A}] = J$.*

(See Definition 4.2 for the definition of strong stability.)

This result actually extends Theorem 1.1 because of the following facts.

Proposition 1.3 *Let G be a Gröbner basis of an ideal I of a polynomial ring $K[\bar{A}, \bar{X}]$ w.r.t. a term order $\geq$ such that each variable X_i is greater than any term in $T(\bar{A})$, then G is also a Gröbner basis of I in $(K[\bar{A}])[\bar{X}]$ w.r.t. the term order that is a restriction of $\geq$ on $T(\bar{X})$.*

Proposition 1.4 *Let J be a 0-dimensional ideal of a polynomial ring $K[\bar{A}]$, K' be an algebraic extension field of K and $\bar{a} = a_1, a_2, \ldots, a_m \in K'^m$ be a zero of J', then we have the following properties.*

1. *The maximal ideal $\langle A_1-a_1, A_2-a_2, \ldots, A_m-a_m\rangle$ is the associated prime of some primary component of J' in $K'[\bar{A}]$, where J' is the extension of J in $K'[\bar{A}]$.*
2. *In case J is a radical ideal, the maximal ideal $\langle A_1 - a_1, A_2 - a_2, \ldots, A_m - a_m\rangle$ is some primary component of J' in $K'[\bar{A}]$, where J' is the extension of J in $K'[\bar{A}]$.*

The result also shows that the condition $\langle G\rangle \cap K[\bar{A}]$ being 0-dimensional(and radical) is indispensable as far as concerning strong stability, i.e. we have the following.

Lemma 1.5 *Let J be an ideal of $K[\bar{A}]$ such that J is not a 0-dimensional ideal, then we have the following properties.*

1. *There exists an ideal I of $K[\bar{A}, X]$ such that $I \cap K[\bar{A}] = J$ and any Gröbner basis G of I in $(K[\bar{A}])[X]$ is not strongly stable for some zero of J.*
2. *In case the number of $\bar{X}$ is more than 1, with an additional assumption that J is not a radical ideal, there exists an ideal I of $K[\bar{A}, \bar{X}]$ such that $I \cap K[\bar{A}] = J$ and any Gröbner basis G of I in $(K[\bar{A}])[\bar{X}]$ is not strongly stable for some zero of J.*

In [9, 10], we showed that alternative of comprehensive Gröbner bases can be defined in terms of Gröbner bases in polynomial rings over commutative von Neumann regular rings, and we called them ACGB (Alternative Comprehensive Gröbner Bases). In [6], we further optimized ACGB to get the following result.

Theorem 1.6 *Let $F = \{f_1(\bar{A}, \bar{X}), \ldots, f_s(\bar{A}, \bar{X})\}$ be a set of polynomials in $K[\bar{A}, \bar{X}]$, let I be a zero-dimensional proper radical ideal in $K[\bar{A}]$. Then the quotient ring $K[\bar{A}]/I$ becomes a commutative von Neumann regular ring.*
Let $G = \{g_1(\bar{A}, \bar{X}), \ldots, g_l(\bar{A}, \bar{X})\}$ be a Gröbner basis of $\langle F\rangle$ in the polynomial ring $(K[\bar{A}]/I)[\bar{X}]$ over $K[\bar{A}]/I$. Then, $\{g_1(\bar{a}, \bar{X}), \ldots, g_l(\bar{a}, \bar{X})\}$ becomes a Gröbner basis of the ideal $\langle f_1(\bar{a}, \bar{X}), \ldots, f_s(\bar{a}, \bar{X})\rangle$ for any m-tuple of elements $\bar{a}$ which lies on the variety $\mathbf{V}(I)$ in an algebraic extension field of K.

In this paper, we prove that G in Theorem 1.1 actually becomes a Gröbner basis in the polynomial rings $(K[\bar{A}]/I \cap K[\bar{A}])[\bar{X}]$ over the commutative von Neumann regular ring $K[\bar{A}]/I \cap K[\bar{A}]$. ¿From this result together with Theorem 1.6, the extended version of Theorem 1.1 by Theorem 1.2 directly follows.

Our proof is not only easy but also natural since the notion of Gröbner bases in polynomial rings over commutative von Neumann regular rings and the notion of comprehensive Gröbner bases are essentially same as is shown in [13]. We also give a redescription of Lemma 1.2 in terms of ACGB.

We also give a new method to comute a parametric Gröbner basis of an ideal I in $K[\bar{A}, \bar{X}]$ even when $I \cap K[\bar{A}]$ is not a zero-dimensional ideal, which is an improvement of our algorithm introduced in [9, 10]. Our new algorithm starts from a usual Gröbner basis of an ideal I in the polynomial ring $K[\bar{A}, \bar{X}]$ over the field K. Our method is a similar but different approach from the algorithms presented in [4] that compute parametric Gröbner bases with careful consideration of parameters, which are essentially based on computation of Gröbner bases in a polynomial ring $K(\bar{A})[\bar{X}]$ over the quotient field $K(\bar{A})$.

We assume the reader is familiar with a theory of Gröbner bases in polynomial rings over commutative von Neumann regular rings, which was introduced in [11]. Though we give a minimum review in section 2, we strongly recommend reading [5] or [11] for the reader who are not familiar with the theory. In section 2, we also prove some properties which will be used for proving our main results. In section 3, we give a brief review of ACGB. Though the contents is self contained, we also refer the reader to [6, 10] for more detailed description. Our main results are proved in section 4. In section 5, we discuss a new approach to computation of parametric Gröbner bases.

2 von Neumann regular rings and Gröbner bases

A commutative ring R with identity 1 is called a *von Neumann regular ring* if it has the following property:

$$\forall a \in R\, \exists b \in R \quad a^2 b = a.$$

For such a b, $a^* = ab$ and $a^{-1} = ab^2$ are uniquely determined and satisfy $aa^* = a$, $aa^{-1} = a^*$, and $(a^*)^2 = a^*$.

Notice that every direct product of fields is a von Neumann regular ring. Conversely, any von Neumann regular ring is shown to be isomorphic to a subring of a direct product of fields as follows.

Definition 2.1 *Let R be a von Neumann regular ring. If we define $\neg a = 1 - a$, $a \wedge b = ab$ and $a \vee b = \neg(\neg a \wedge \neg b)$ for each $a, b \in R$ such that $a^2 = a, b^2 = b$, then $(\{x \in R \mid x^2 = x\}, \neg, \wedge, \vee)$ becomes a boolean algebra, which is denoted by $B(R)$.*

Considering $B(R)$ as a boolean ring, Stone representation theorem gives the following isomorphism Φ from $B(R)$ to a subring of $\prod_{I \in St(B(R))} B(R)/I$ by $\Phi(x) = \prod_{I \in St(B(R))} [x]_I$, where $St(B(R))$ is the set of all maximal ideals of $B(R)$. This representation of $B(R)$ is extended to a representation of R as follows.

Theorem 2.1 (Saracino-Weispfenning) *For a maximal ideal I of $B(R)$, if we put $I_R = \{xy \mid x \in R, y \in I\}$, then I_R is a maximal ideal of R. If we define a map Φ from R into $\prod_{I \in St(B(R))} R/I_R$ by $\Phi(x) = \prod_{I \in St(B(R))} [x]_{I_R}$, then Φ is a ring embedding.*

A maximal ideal coincides with a prime ideal in a boolean ring. In the rest of the paper $St(B(R))$ is denoted by $Spec(B(R))$. We use p for an element of $Spec(B(R))$ as in the papers [11, 13]. We also use the same notations R_p to denote the field R/p_R and x_p to denote the element $[x]_{p_R}$ in R_p.

In the following unless mentioned, Greek letters α, β, γ are used for terms, Roman letters a, b, c for elements of R, and f, g, h for polynomials over R. Throughout this section, we work in a polynomial ring over R which is a von Neumann regular ring unless mentioned. We also assume that some term order is given. The leading term of f is denoted by $lt(f)$ and its coefficient by $lc(f)$. By $li(f)$ we denote $lc(f)^*$. The leading monomial of f, i.e., $lc(f)lt(f)$ is denoted by $lm(f)$. The set of all terms consisting of variables $\bar{X}$ is denoted by $T(\bar{X})$.

Definition 2.2 *For a polynomial $f = a\alpha + g$ with $lm(f) = a\alpha$, a* ***monomial reduction*** $\rightarrow_f$ *is defined as follows:*

$$b\alpha\beta + h \rightarrow_f b\alpha\beta + h - ba^{-1}\beta(a\alpha + g)$$

where $ab \neq 0$ and $b\alpha\beta$ need not be the leading monomial of $b\alpha\beta + h$.

A monomial reduction $\rightarrow_F$ by a set F of polynomials is also naturally defined. When F is a finite set, $\rightarrow_F$ has a termination property. Using this monomial reduction, Gröbner bases are defined as follows.

Definition 2.3 *Let I be an ideal of a polynomial ring over R. A finite subset G of I is called a* ***Gröbner basis*** *of I, if it satisfies the following property:*

$$\forall f \quad f \in I \Longleftrightarrow f \xrightarrow{*}_G 0.$$

We simply say G is a Gröbner basis if G is a Gröbner basis of the ideal $\langle G \rangle$ generated by itself.

Notice that a Gröbner basis G of I is clearly a basis of I.
It is not difficult to show the following property.

Lemma 2.2 *A finite subset G of an ideal I is a Gröbner basis of I if and only if* $\langle\{lm(f)|f \in I\}\rangle = \langle\{lm(g)|g \in G\}\rangle$

Proof. Assume that G is a Gröbner basis of I. Let f be a non-zero polynomial in I. Since $f \xrightarrow{*}_G 0$, there must exist polynomials $g_1, \ldots, g_s \in G$ such that $lt(g_i)|lt(f)$ for each $i = 1, \ldots, s$ and $((li(g_1) \vee \cdots \vee li(g_s))li(f) = li(f)$. Define $c_1, \ldots, c_s \in R$ inductively as follows. $c_i = b_i li(g_i)$ for each $i = 1, \ldots, s$, where $b_i = 1-(c_1+\cdots+c_{i-1})$ for each $i = 2, \ldots, s$. (We put $b_1 = 1$ for convenience.) Then we have $c_ic_j = 0$ for each distinct i and j and $c_1 + \cdots + c_s = li(g_1) \vee \cdots \vee li(g_s)$. Since $lc(f) = li(f)lc(f)$, we have $lc(f) = (c_1 + \cdots + c_s)lc(f)$. Hence, $lm(f) = (c_1 + \cdots + c_s)lc(f)lt(f) = c_1 lc(f)lt(f) + \cdots + c_s lc(f)lt(f) = b_1 li(g_1)lc(f)lt(f) + \cdots + b_s li(g_s)lc(f)lt(f) = b_1 li(g_1)lc(f)lt(g_1)(lt(f)/lt(g_1)) + \cdots + b_s li(g_s)lc(f)lt(g_s)(lt(f)/lt(g_s)) = b_1 lc(g_1)^{-1}lc(g_1)lc(f)lt(g_1)(lt(f)/lt(g_1)) + \cdots + b_s lc(g_s)^{-1}lc(g_s)lc(f)lt(g_s)(lt(f)/lt(g_s))$
$= b_1 lc(g_1)^{-1}lc(f)(lt(f)/lt(g_1))lm(g_1) + \cdots + b_s lc(g_s)^{-1}lc(f)(lt(f)/lt(g_s))lm(g_s)$.
It follows that $\langle\{lm(f)|f \in I\}\rangle \subseteq \langle\{lm(g)|g \in G\}\rangle$.
$\langle\{lm(f)|f \in I\}\rangle \supseteq \langle\{lm(g)|g \in G\}\rangle$ is trivial.

Assume conversely that $\langle\{lm(f)|f \in I\}\rangle = \langle\{lm(g)|g \in G\}\rangle$.
To get a contradiction suppose there exists a non-zero polynomial f in I which is irreducible by $\rightarrow_G$. This means that $lc(f)lc(g) = 0$ for any g in G satisfying $lt(g)|lt(f)$. By our assumption, there exists $g_1, \ldots, g_s \in G$ and monomials $a_1\alpha_1, \ldots, a_s\alpha_s$ such that $lm(f) = a_1 lm(g_1)\alpha_1 + \cdots + a_s lm(g_s)\alpha_s$. Multiplying $lc(f)$ from both sides, we get a contradiction $lc(f)lm(f) = 0$. ∎

Definition 2.4 *For a polynomial f, $li(f)f$ is called the* ***boolean closure*** *of f and denoted by $bc(f)$. A polynomial f such that $f = bc(f)$ is said to be* ***boolean closed****. Notice that $bc(f)$ is boolean closed.*

Lemma 2.3 *Let G be a Gröbner basis of an ideal I, then $G' = \{bc(g)|g \in G\}$ also becomes a Gröbner basis of I.*

Proof. By the definition of boolean closure, G' is clearly a subset of I. Since $lm(g) = lm(bc(g))$ for each polynomial g, $\langle\{lm(g)|g \in G\}\rangle = \langle\{lm(g)|g \in G'\}\rangle$. So, G' is a Gröbner basis of I by Lemma 2.2. ∎

The following result of [3] will be used for proving our main results.

Lemma 2.4 *Let R be a commutative ring with identity, which need not to be a von Neumann regular ring. Let I be an ideal in a polynomial ring $R[\bar{X}]$ and $G = \{g_1, \ldots, g_m\}$ be a finite subset of I. Then the following properties are equivalent:*

1. $\langle\{lm(f)|f \in I\}\rangle = \langle\{lm(g)|g \in G\}\rangle$
2. *For any polynomial $f \in I$, f has a Gröbner representation w.r.t. G, that is there exist polynomials $p_1 \ldots, p_m$ such that $f = \sum_{i=1}^{m} p_i g_i$ and $lt(f) \geq lt(p_i)lt(g_i)$ for each $i = 1, \ldots, m$.*

We conclude this section with the following fact.

Lemma 2.5 *For a polynomial f in a polynomial ring $R[\bar{X}]$ and $p \in Spec(B(R))$, f_p denotes the polynomial in $R_p[\bar{X}]$ given from f by replacing each coefficient a with a_p. For a set F of polynomials in $R[\bar{X}]$, F_p denotes the set $\{f_p|f \in F\} - \{0\}$. Let G be a Gröbner basis of an ideal I in a polynomial ring $R[\bar{X}]$. Then G_p becomes a Gröbner basis of the ideal I_p in the polynomial ring $R_p[\bar{X}]$ for each $p \in Spec(B(R))$.*

Proof. Notice first that for each element e in R_p there exists an element a in R such that $a_p = e$. Hence, for each polynomial h in $R_p[\bar{X}]$ there exists a polynomial f in $R[\bar{X}]$ such that $f_p = h$, from which it follows that I_p is an ideal in $R_p[\bar{X}]$. In case each element of G is boolean closed, this lemma is already shown in [11]. (Where the converse also holds.) If G is not a set of boolean closed polynomials, let $G' = \{bc(g)|g \in G\}$. Then G' is also a Gröbner basis of I by Lemma 2.3. Therefore, G'_p is also a Gröbner basis of I_p. We claim that G'_p is a subset of G_p. Let g be a polynomial in G. Notice the following two properties:

If $li(g)_p = 0$, then $bc(g)_p = 0$. If $li(g)_p = 1$, then $bc(g)_p = g_p$.

Since $li(g)_p$ is 0 or 1 for each p, we have $bc(g)_p = 0$ or $bc(g)_p = g_p$, from which our claim follows. Since G_p is clearly a subset of I_p, G_p is a Gröbner basis of I_p in $R_p[\bar{X}]$. ∎

3 ACGB

A polynomial ring $K[\bar{A}]$ over a field K with variables $\bar{A} = A_1, \ldots, A_m$ is not a von Neumann regular ring. But considering a polynomial in $K[\bar{A}]$ as a function from $\overline{K}^m$ to $\overline{K}$, $K[\bar{A}]$ can be considered as a subring of a von Neumann regular ring $\overline{K}^{\overline{K}^m}$. This idea leads us to define an *ACGB*(Alternative Comprehensive Gröbner Basis) as follows.

Definition 3.1 *Let F be a finite set of polynomials in a polynomial ring $K[\bar{A}, \bar{X}]$ over a field K with variables $\bar{A} = A_1, \ldots, A_m$ and $\bar{X} = X_1, \ldots, X_n$. Let G be a Gröbner basis of $\langle F\rangle$ in the polynomial ring $\overline{K}^{\overline{K}^m}[\bar{X}]$. G is called an **ACGB** of F with parameters $\bar{A}$.*

Theorem 3.1 *Let $G = \{g_1, \ldots, g_l\}$ be an ACGB of $F = \{f_1(\bar{A}, \bar{X}), \ldots, f_s(\bar{A}, \bar{X})\}$ with parameters $\bar{A}$. Then, for each m-tuple $\bar{a} = a_1, \ldots, a_m$ of elements in $\overline{K}$, $G_{\bar{a}}$ becomes a Gröbner basis of the ideal $\langle\{f_1(\bar{a}, \bar{X}), \ldots, f_s(\bar{a}, \bar{X})\}\rangle$ in $\overline{K}[\bar{X}]$. Where $G_{\bar{a}}$ denotes the set $\{g_{1\bar{a}}, \ldots, g_{l\bar{a}}\}$ of polynomials $g_{1\bar{a}}, \ldots, g_{l\bar{a}}$ in $\overline{K}[\bar{X}]$ given from $g_1, \ldots, g_l$ by replacing each coefficient c with $c(\bar{a})$.*
(Remember that c is an element of $\overline{K}^{\overline{K}^m}$).

Proof. Let $R = \overline{K}^{\overline{K}^m}$. Notice that for any element c of R, $c^2 = c$ if and only if $c(\bar{a}) = 0$ or $c(\bar{a}) = 1$ for each element $\bar{a}$ of $\overline{K}^m$. Hence, the boolean ring $B(R)$ consists of all c of R such that $c(\bar{a}) = 0$ or $c(\bar{a}) = 1$ for each element $\bar{a}$ of $\overline{K}^m$. ($B(R)$ is not a subring of R, the addition of $c(\bar{a})$ and $c'(\bar{a})$ for c and c' in $B(R)$ is defined as the addition of the finite field $\mathbf{Z}_2$.) Clearly the set $\{c \in B(R) | c(\bar{a}) = 0\}$ forms a prime ideal in $B(R)$ for any element $\bar{a}$ of $\overline{K}^m$. Let $\bar{a}$ be an element of $\overline{K}^m$ and p be the prime ideal $\{c \in B(R) | c(\bar{a}) = 0\}$. Notice also that the maximal ideal $p_R = \{xy | x \in R, y \in p\}$ in R has the following form: $p_R = \{c \in R | c(\bar{a}) = 0\}$. Remember that R_p is the quotient field R/p_R. Since $c - c' \in p_R$ if and only if $c(\bar{a}) = c'(\bar{a})$ for any c and c' in R, the mapping θ from R/p_R to $\overline{K}$ defined by $\theta([c]_{p_R}) = c(\bar{a})$ is an isomorphism. If we identify R/p_R with $\overline{K}$ by this isomorphism, $[c]_{p_R}$ is equal to $c(\bar{a})$. Remember that $[c]_{p_R}$ is denoted by c_p. So the theorem follows from Lemma 2.5. ∎

In ACGB, we implicitly assume that a specialization can take any value from $\overline{K}^m$. If we give a restriction on specializations, we can generalize ACGB as follows.

Definition 3.2 *Let S be a subset of $\overline{K}^m$. Let F be a finite set of polynomials in a polynomial ring $K[\bar{A}, \bar{X}]$. Let G be a Gröbner basis of $\langle F \rangle$ in the polynomial ring $\overline{K}^S[\bar{X}]$. G is called an* ***ACGB*** *on S of F with parameters $\bar{A}$. We simply simply call G an ACGB on S of F when $\bar{A}$ is clear from contexts. We also simply call G an ACGB on S when G is an ACGB on S of G.*

We have the following theorem by an exactly same proof of Theorem 3.1.

Theorem 3.2 *Let S be a subset of $\overline{K}^m$ and $G = \{g_1, \dots, g_l\}$ be an ACGB on S of $F = \{f_1(\bar{A}, \bar{X}), \dots, f_s(\bar{A}, \bar{X})\}$. Then, for each m-tuple $\bar{a}$ in S, $G_{\bar{a}}$ becomes a Gröbner basis of the ideal $\langle F(\bar{a}) \rangle$ in $\overline{K}[\bar{X}]$.*

Notice that we can not generally construct ACGB's on S for arbitrary set S. Even when we can construct ACGB's on S such as the case $S = \overline{K}^m$, we usually can not represent them in a form of a set of polynomials in $K[\bar{A}, \bar{X}]$. In the rest of this section, we show that we can always construct ACGB's on S in a form of a set of polynomials in $K[\bar{A}, \bar{X}]$ when S is given in a form of a variety of zero-dimensional ideal.

Let V be the variety of an ideal I. Let $K[V]$ denote a subring of K^V which consists of all elements that can be represented as polynomial functions. Notice that $K[V]$ is isomorphic to the quotient ring $K[\bar{A}]/\mathbf{I}(V)$, where $\mathbf{I}(V)$ denotes the ideal $\{f \in K[\bar{A}] | f(\bar{a}) = 0 \text{ for every } \bar{a} \in V\}$. In general, $K[\bar{A}]/\mathbf{I}(V)$ is not a von Neumann regular ring. However, in case $\mathbf{I}(V)$ is zero-dimensional, it becomes a von Neumann regular ring. Since $\mathbf{I}(V)$ is a radical ideal, it can be represented as an intersection of distinct prime ideals $P_1 \cap \cdots \cap P_k$. If $\mathbf{I}(V)$ is zero-dimensional, each P_i is also zero-dimensional, so it is maximal. Therefore, $K[\bar{A}]/\mathbf{I}(V)$ is isomorphic to the direct product $K[\bar{A}]/P_1 \times \cdots \times K[\bar{A}]/P_k$ of fields by the Chinese remainder theorem. So, $K[\bar{A}]/\mathbf{I}(V)$ becomes a von Neumann regular ring. These observations lead us to have the following theorem.

Theorem 3.3 *Let $F = \{f_1(\bar{A}, \bar{X}), \dots, f_s(\bar{A}, \bar{X})\}$ be a finite set of polynomials in a polynomial ring $K[\bar{A}, \bar{X}]$ with variables $\bar{A} = A_1, \dots, A_m$ and $\bar{X}$. Let I be a zero-dimensional proper radical ideal in $K[\bar{A}]$. Then the quotient ring $K[\bar{A}]/I$ becomes a von Neumann regular ring. Let G be a Gröbner basis of $\langle F \rangle$ in the polynomial ring $(K[\bar{A}]/I)[\bar{X}]$ over $K[\bar{A}]/I$. Each coefficient of a polynomial $h(\bar{X})$ in $(K[\bar{A}]/I)[\bar{X}]$ is a member of $K[\bar{A}]/I$, so it can be represented by a polynomial in $K[\bar{A}]$. Hence, $h(\bar{X})$ can also be represented as a polynomial in $K[\bar{A}, \bar{X}]$. Therefore, G can be represented by a set of polynomials $\{g_1(\bar{A}, \bar{X}), \dots, g_l(\bar{A}, \bar{X})\}$ in $K[\bar{A}, \bar{X}]$. Then, for any m-tuple $\bar{a}$ of elements in the algebraic closure $\overline{K}$ of K which is a zero of I, $\{g_1(\bar{a}, \bar{X}), \dots, g_l(\bar{a}, \bar{X})\}$ becomes a Gröbner basis of the ideal $\langle f_1(\bar{a}, \bar{X}), \dots, f_s(\bar{a}, \bar{X}) \rangle$ in $\overline{K}[\bar{X}]$.*

Proof. If K is an algebraically closed field, let V be the variety of I. Since $\mathbf{I}(V) = I$, $K[V]$ is isomorphic to $K[\bar{A}]/I$. Therefore G is actually an ACGB on V of F with parameters $\bar{A}$, from which the theorem directly follows from Theorem 3.2. In case K is not an algebraically closed field, we need to optimize the above proof. Represent $I = P_1 \cap \cdots \cap P_k$ as an intersection of distinct prime(maximal) ideals in $K[\bar{A}]$. For each $i = 1, \ldots, k$, let $\bar{a}_i \in \overline{K}^m$ be a zero of P_i. If we put $K_i = \{f(\bar{a}_i) | f(\bar{A}) \in K[\bar{A}]\}$ for each i, K_i becomes a field which is isomorphic to $K[\bar{A}]/P_i$. Define a map Φ from $K[\bar{A}]/I$ to $K_1 \times \cdots \times K_k$ by $\Phi(f(\bar{A})) = (f(\bar{a}_1), \ldots, f(\bar{a}_k))$. Then Φ is an isomorphism. Hence, $B(K[\bar{A}]/I)$ can be considered as a boolean algebra which consists of all subsets of $\{1, \ldots, k\}$. A prime ideal of $B(K[\bar{A}]/I)$ has a form $\{s \subseteq \{1, \ldots, k\} | i \notin s\}$ for some i. If we denote $\{s \subseteq \{1, \ldots, k\} | i \notin s\}$ by p_i, $(K[\bar{A}]/I)_{p_i}$ can be identified with K_i and $f(\bar{A})_{p_i}$ is equal to $f(\bar{a}_i)$ for each $f(\bar{A}) \in K[\bar{A}]/I$. By Lemma 2.5, $\{g_1(\bar{a}_i, \bar{X}), \ldots, g_l(\bar{a}_i, \bar{X})\}$ becomes a Gröbner basis of the ideal $\langle f_1(\bar{a}_i, \bar{X}), \ldots, f_s(\bar{a}_i, \bar{X})\rangle$ in $K_i[\bar{X}]$. Since the Gröbner basis property is conservative under a field extension, it is also a Gröbner basis in $\overline{K}[\bar{X}]$. ∎

4 Stability of Gröbner bases

In this section we prove our main results. Let us begin with the standard definition of Gröbner bases.

Definition 4.1 *Let I be an ideal of a polynomial ring $K[\bar{A}, \bar{X}]$ over a field K with variables $\bar{A}$ and $\bar{X}$. Let G be a finite subset of I. Consider $K[\bar{A}, \bar{X}]$ as a polynomial ring $(K[\bar{A}])[\bar{X}]$ over the coefficient ring $K[\bar{A}]$. If we have $\langle\{lm(f) | f \in I\}\rangle = \langle\{lm(g) | g \in G\}\rangle$ in $(K[\bar{A}])[\bar{X}]$ with a term order $\geq$ of $T(\bar{X})$, G is called a Gröbner basis of I in $(K[\bar{A}])[\bar{X}]$ w.r.t. $\geq$.*

The next fact is a special instance of a well-known result shown in [8, 14].

Proposition 4.1 *Let I be an ideal of a polynomial ring $K[\bar{A}, \bar{X}]$ over a field K with variables $\bar{A}$ and $\bar{X}$ such that $I \cap K[\bar{A}]$ is a zero-dimensional proper radical ideal in $K[\bar{A}]$. Let $G = \{g_1(\bar{A}, \bar{X}), \ldots, g_l(\bar{A}, \bar{X})\}$ be a Gröbner basis of I in $(K[\bar{A}])[\bar{X}]$ w.r.t. a term order $\geq$ of $T(\bar{X})$. If we consider G as a set of polynomials in a polynomial ring $(K[\bar{A}]/I \cap K[\bar{A}])[\bar{X}]$ over the von Neumann regular ring $K[\bar{A}]/I \cap K[\bar{A}]$, then G also becomes a Gröbner basis in this polynomial ring w.r.t. $\geq$.*

Together with Theorem 3.3, we directly have the following.

Theorem 4.2 *Let I be an ideal of a polynomial ring $K[\bar{A}, \bar{X}]$ over a field K such that $I \cap K[\bar{A}]$ is a zero-dimensional radical ideal in $K[\bar{A}]$.*
Let $G = \{g_1(\bar{A}, \bar{X}), \ldots, g_l(\bar{A}, \bar{X})\}$ be a Gröbner basis of I in $(K[\bar{A}])[\bar{X}]$ w.r.t. a term order $\geq$ of $T(\bar{X})$. Let $\bar{a}$ be an m-tuple of elements of the algebraic closure $\overline{K}$ of K which is a zero of the ideal $I \cap K[\bar{A}]$. Then, G becomes a Gröbner basis with the specialization by $\bar{a}$, that is $\{g_1(\bar{a}, \bar{X}), \ldots, g_l(\bar{a}, \bar{X})\}$ becomes a Gröbner basis of the ideal $\langle g_1(\bar{a}, \bar{X}), \ldots, g_l(\bar{a}, \bar{X})\rangle$ in $\overline{K}[\bar{X}]$ w.r.t. $\geq$.

Proof. When $I \cap K[\bar{A}]$ is not a proper ideal, the result is trivial, otherwise apply Proposition 4.1 and Theorem 3.3. ∎

We can get a slightly stronger result as follows.

Definition 4.2 *Let $G = \{g_1(\bar{A}, \bar{X}), \ldots, g_l(\bar{A}, \bar{X})\}$ be a finite set of polynomials of $K[\bar{A}, \bar{X}]$ with a field K. Let $\geq$ be a term order of $T(\bar{X})$ and $\bar{a}$ be an m-tuple of elements of some extension field K' of K. G is said to be* ***strongly stable*** *for $\bar{a}$ w.r.t. $\geq$ if $\{g_{n_1}(\bar{a}, \bar{X}), \ldots, g_{n_k}(\bar{a}, \bar{X})\}$ becomes a Gröbner basis of the* *ideal $\langle g_1(\bar{a},\bar{X}),\ldots,g_l(\bar{a},\bar{X})\rangle$ in $K'[\bar{X}]$ w.r.t. $\geq$, where $\{g_{n_1},\ldots,g_{n_k}\}$ is the set of all polynomials g of G such that $lc(g)(\bar{a}) \neq 0$.(We consider g as a polynomial in $(K[\bar{A}])[\bar{X}]$, so $lc(g)$ is a polynomial in $K[\bar{A}]$.) G is also simply said to be* **stable** *for $\bar{a}$ w.r.t. $\geq$ if $\{g_1(\bar{a},\bar{X}),\ldots,g_l(\bar{a},\bar{X})\}$ becomes a Gröbner basis of the ideal $\langle g_1(\bar{a},\bar{X}),\ldots,g_l(\bar{a},\bar{X})\rangle$ in $K'[\bar{X}]$ w.r.t. $\geq$.*

The strong stability condition is actually much stronger than the stability condition.

Example 1.
Let $G = \{A*X_1+X_2-1, X_2^2-1\}$ with a lexicographic term order $X_1 > X_2$. Then G is not strongly stable for 0 since $\{X_2^2-1\}$ is not a Gröbner basis of $\langle X_2-1, X_2^2-1\rangle$, but G is stable for 0 i.e. $\{X_2-1, X_2^2-1\}$ is a Gröbner basis of $\langle X_2-1, X_2^2-1\rangle$

Corollary 4.3 *Let I be an ideal of a polynomial ring $K[\bar{A},\bar{X}]$ over a field K such that $I \cap K[\bar{A}]$ is a zero-dimensional radical ideal in $K[\bar{A}]$.*
Let $G = \{g_1(\bar{A},\bar{X}),\ldots,g_l(\bar{A},\bar{X})\}$ be a Gröbner basis of I in $(K[\bar{A}])[\bar{X}]$ w.r.t. a term order $\geq$ of $T(\bar{X})$. Let $\bar{a}$ be an m-tuple of elements of the algebraic closure $\overline{K}$ of K which is a zero of the ideal $I \cap K[\bar{A}]$. Then G is strongly stable for $\bar{a}$ w.r.t. $\geq$.

Proof. Notice that $\{g_{n_1}(\bar{a},\bar{X}),\ldots,g_{n_k}(\bar{a},\bar{X})\}$ and $\{g_1(\bar{a},\bar{X}),\ldots,g_l(\bar{a},\bar{X})\}$ in Definition 8 correspond to G'_p and G_p in the proof of Lemma 2.5. So we can replace $\{g_1(\bar{a},\bar{X}),\ldots,g_l(\bar{a},\bar{X})\}$ by $\{g_{n_1}(\bar{a},\bar{X}),\ldots, g_{n_k}(\bar{a},\bar{X})\}$ in Theorem 3.3, from which our corollary follows.∎
By Proposition 1, the following facts are direct consequences from Proposition 4.1 and Corollary 4.3.

Theorem 4.4 *Let I be an ideal of a polynomial ring $K[\bar{A},\bar{X}]$ over a field K with variables $\bar{A}$ and $\bar{X}$ such that $I \cap K[\bar{A}]$ is a zero-dimensional proper radical ideal in $K[\bar{A}]$. Let $G = \{g_1(\bar{A},\bar{X}),\ldots,g_l(\bar{A},\bar{X})\}$ be a Gröbner basis of I w.r.t. a term order $\geq$ such that each variable X_i is greater than any term in $T(\bar{A})$. If we consider G as a set of polynomials in the polynomial ring $(K[\bar{A}]/I \cap K[\bar{A}])[\bar{X}]$ over the von Neumann regular ring $K[\bar{A}]/I \cap K[\bar{A}]$, then G also becomes a Gröbner basis of the ideal $\langle G\rangle$ in this polynomial ring w.r.t. the term order that is a restriction of $\geq$ on $T(\bar{X})$.*

Corollary 4.5 *Let I be an ideal of a polynomial ring $K[\bar{A},\bar{X}]$ over a field K such that $I \cap K[\bar{A}]$ is a zero-dimensional radical ideal in $K[\bar{A}]$.*
Let $G = \{g_1(\bar{A},\bar{X}),\ldots,g_l(\bar{A},\bar{X})\}$ be a Gröbner basis of I w.r.t. a term order $\geq$ such that each variable X_i is greater than any term in $T(\bar{A})$. Let $\bar{a}$ be an m-tuple of elements of the algebraic closure $\overline{K}$ of K which is a zero of the ideal $I \cap K[\bar{A}]$. Then, G is strongly stable for $\bar{a}$ w.r.t. the term order that is a restriction of $\geq$ on $T(\bar{X})$.

We conclude this section by the following fact which is a redescription of Lemma 1 in terms of ACGB.

Proposition 4.6 *Let J be an ideal of $K[\bar{A}]$ such that J is not a 0-dimensional ideal, then we have the following properties.*

1. *There exits an ideal I of $K[\bar{A},X]$ such that $I \cap K[\bar{A}] = J$ and any Gröbner basis G of I in $(K[\bar{A}])[X]$ is not an ACGB on $V(J)$.*
2. *In case the number of $\bar{X}$ is more than 1, with an additional assumption that J is not a radical ideal, there exits an ideal I of $K[\bar{A},\bar{X}]$ such that $I \cap K[\bar{A}] = J$ and any Gröbner basis G of I in $(K[\bar{A}])[\bar{X}]$ is not an ACGB on $V(J)$.*

5 Computation of parametric Gröbner bases

Let $F = \{f_1(\bar{A},\bar{X}),\dots,f_s(\bar{A},\bar{X})\}$ be a finite set of polynomials in $K[\bar{A},\bar{X}]$. Compute a Gröbner basis $G = \{g_1(\bar{A},\bar{X}),\dots,g_l(\bar{A},\bar{X}),h_1(\bar{A}),\dots,h_k(\bar{A})\}$ of the ideal $\langle F\rangle$ w.r.t. a term order $\geq$ such that any variable X_i is greater than any term in $T(\bar{A})$, where $\{h_1(\bar{A}),\dots,h_k(\bar{A})\}$ is a set of all polynomials in G that do not include any variable X_i, it might be an empty set. In case $\langle h_1(\bar{A}),\dots,h_k(\bar{A})\rangle$ is a zero dimensional radical ideal in $K[\bar{A}]$, G becomes a parametric Gröbner basis of F that is $\{g_1(\bar{a},\bar{X}),\dots,g_l(\bar{a},\bar{X}),h_1(\bar{a}),\dots,h_k(\bar{a})\}$ becomes a Gröbner basis of the ideal $\langle f_1(\bar{a},\bar{X}),\dots,f_s(\bar{a},\bar{X})\rangle$ in $\overline{K}[\bar{X}]$ for any m-tuple $\bar{a}$ of elements of $\overline{K}$, as is shown in Corollary 4.5. When $\langle h_1(\bar{A}),\dots,h_k(\bar{A})\rangle$ is not a zero dimensional radical ideal, G is not a parametric Gröbner basis of F in general. If we want to construct a parametric Gröbner basis of F using G, the most interesting question is whether we can construct a computable condition of $\bar{a}$ which is necessary and sufficient for G to be stable for $\bar{a}$ w.r.t. $\geq$.
With a minor change, that is replacing 'stable' by 'strongly stable', we can construct such a condition using the following fact which is also presented in [2](Theorem 3.1).

Theorem 5.1 *Let I be an ideal of a polynomial ring $K[\bar{A},\bar{X}]$. Let $G = \{g_1(\bar{A},\bar{X}),\dots,g_l(\bar{A},\bar{X})\}$ be a Gröbner basis of I in $(K[\bar{A}])[\bar{X}]$ w.r.t. a term order $\geq$ of $T(\bar{X})$. Let $\bar{a}$ be an m-tuple of elements of the algebraic closure*
$\overline{K}$ of K which is a zero of the ideal $I\cap K[\bar{A}]$. Let $\{g_{n_1},\dots,g_{n_k}\}$ be the set of all polynomials g of G such that $lc(g)(\bar{a})\neq 0$. Then G is strongly stable for $\bar{a}$ w.r.t. $\geq$, that is $\{g_{n_1}(\bar{a},\bar{X}),\dots,g_{n_k}(\bar{a},\bar{X})\}$ becomes a Gröbner basis of $\langle g_1(\bar{a},\bar{X}),\dots,g_l(\bar{a},\bar{X})\rangle$, if and only if $g(\bar{a},\bar{X})$ is reducible to 0 modulo
$\{g_{n_1}(\bar{a},\bar{X}),\dots,g_{n_k}(\bar{a},\bar{X})\}$ for every g in G.

Algorithm 1.
Let I and G be as in the above theorem. We can compute an algebraically constructible set S such that $\bar{a}\in S$ if and only if G is strongly stable for $\bar{a}$ w.r.t. $\geq$. We call $\rho = (\{p_1,\dots,p_r\},\{q_1,\dots,q_s\})$ be a *binary partition* of G, if $\{p_1,\dots,p_r\}\cap\{q_1,\dots,q_s\}=\emptyset$ and $\{p_1,\dots,p_r\}\cup\{q_1,\dots,q_s\}=G$. (When $r=0(s=0)$, we abuse the notation $\{p_1,\dots,p_r\}(\{q_1,\dots,q_s\})$ to denote an empty set.) For such a binary partition ρ, we put a case distinction $C_\rho = \{lc(p_1)\neq 0,\dots,lc(p_r)\neq 0,\ lc(q_1)=0,\dots,lc(q_s)=0\}$. Compute a normal form q_i' of q_i modulo $\{p_1,\dots,p_r\}$ in $K(\bar{A})[\bar{X}]$ for each $i=1,\dots,s$. Let $\{h_1,\dots,h_t\}$ be the set of all polynomials of $K[\bar{A}]$ that is a numerator of some coefficient of some q_i'. For each $\bar{a}$ that satisfies all conditions of the case distinction C_ρ, we can see that G is strongly stable for $\bar{a}$ w.r.t. $\geq$ if and only if $h_i(\bar{a})=0$ for every $i=1,\dots,t$, by the theorem. Let $C_\rho' = \{h_1=0,\dots,h_t=0\}\cup C_\rho$ and $S_\rho = \{\bar{a}\in\overline{K}^m \mid \bar{a}$ satisfies all conditions of $C_\rho'\}$ and put an algebraically constructible set $S=\cup_\rho S_\rho$. Then S has the desired property.

The above algorithm is simple and fast when we do not have so many binary partitions. When l is not small, however, if G does not include any polynomial consisting of only variables $\bar{A}$, we have to take care of 2^l many case distinctions, which of course is not a light job. This difficulty is overcome by using ACGB. The following algorithm produces the above algebraically constructible set S with a minimum cost. Using the output of the algorithm we can also get an ACGB of I, which can be considered as a parametric Gröbner basis of I.

Algorithm 2.
Let I and G be as in the above theorem. For each $i=1,\dots,l$, compute a normal form g_i' of g_i modulo G

in the polynomial ring $T[\bar{X}]$ over the von Neumann regular ring T which is defined as the smallest von Neumann regular subring of $\overline{K}^{\overline{K}^m}$ that includes $K[\bar{A}]$. T is defined as a computable ring using so called terrace. (See [10] for more detail.) Notice that g_i' is not necessary to be 0 since we do not generally have the property $f \rightarrow_f 0$ in a polynomial ring over a von Neumann regular ring. Put $G' = \{g_1', \ldots, g_l'\}$. Let $\{\theta_1, \ldots, \theta_N\}$ be the set of all coefficients which appear in some polynomial g_i'. Let $\theta = \theta_1^* \vee \cdots \vee \theta_N^*$, where $\vee$ is a boolean sum, i.e. $x \vee y$ is defined by $x + y + xy$ for any pair of idempotent elements x and y. θ has the following property for any $\bar{a} \in \overline{K}^m$:

$$\theta(\bar{a}) = \begin{cases} 1 & \text{if} \quad \theta_i(\bar{a}) \neq 0 \quad \text{for some} \quad i \\ 0 & \quad \text{otherwise.} \end{cases}$$

Notice the fact that normal forms of monomial reductions by a set of boolean closed polynomials in $T[\bar{X}]$ are specialization invariant, that is, if $f(\bar{A}, \bar{X}) \xrightarrow{*}_H f'(\bar{A}, \bar{X})$ and $f'(\bar{A}, \bar{X})$ is irreducible by $\rightarrow_H$ then $f(\bar{a}, \bar{X}) \xrightarrow{*}_{H_{\bar{a}}} f'(\bar{a}, \bar{X})$ and $f'(\bar{a}, \bar{X})$ is irreducible by $\rightarrow_{H_{\bar{a}}}$ for any m-tuple $\bar{a}$ of elements of $\overline{K}$ and any set H of boolean closed polynomials in $T[\bar{X}]$. Since $\rightarrow_h$ and $\rightarrow_{bc(h)}$ do an exactly same monomial reduction, we have the following property for any m-tuple $\bar{a}$ of elements of $\overline{K}$ and any set H of polynomials in $T[\bar{X}]$:

If $f(\bar{A}, \bar{X}) \xrightarrow{*}_H f'(\bar{A}, \bar{X})$ and $f'(\bar{A}, \bar{X})$ is irreducible by $\rightarrow_H$
then $f(\bar{a}, \bar{X}) \xrightarrow{*}_{H'_{\bar{a}}} f'(\bar{a}, \bar{X})$ and $f'(\bar{a}, \bar{X})$ is irreducible by $\rightarrow_{H'_{\bar{a}}}$.
(Where $H' = \{bc(h) | \, h \in H\}$.)

Notice also that $bc(h)(\bar{a}) = 0$ if and only if $lc(h)(\bar{a}) = 0$. Hence, with the above property of θ, we can see that G is strongly stable for $\bar{a}$ if and only if $\theta(\bar{a}) = 0$.
Notice also that we can also express the set $S = \{\bar{a} \in \overline{K}^m | \theta(\bar{a}) = 0\}$ in a form of algebraically constructible set using the structure of terrace. For the construction of S, boolean simplification is also useful.
We can further obtain a parametric Gröbner basis of I in a form of ACGB as follows.
Let $G_1 = \{(1-\theta)g_1, \ldots, (1-\theta)g_l\}$ and compute a Gröbner basis G_2 of $\{\theta g_1, \ldots, \theta g_l\}$ in $T[\bar{X}]$. (Notice that G_2 is nothing but an ACGB on $\overline{K}^m \setminus S$ of G.) Then $G_1 \cup G_2$ forms a Gröbner basis of I in $T[\bar{X}]$, that is $G_1 \cup G_2$ is an ACGB of I.

6 Conclusions and Remarks

Theorem 1.2 is given in more general situations in terms of a homomorphism from an arbitrary commutative ring R to a field.(See theorem 3.2 and 3.3 in [2]). In our situation that is R is a polynomial ring $K[\bar{A}]$ over a field K, it is equivalent to Theorem 1.2 since a homomorphism is nothing but a specialization and its kernel is a maximal ideal.

The stability property defined in [2] corresponds to the strong stability property defined in this paper. We use this terminology since it reflects its meaning more precisely and there is a closed relationship between its notion and the notion of monomial reductions in polynomial rings over von Neumann regular rings as is shown in this paper.

Three theorems of section 3 are originally given for boolean closed Gröbner bases in [6, 9, 10]. In this paper, we optimize our proofs so that the theorems hold for arbitrary Gröbner bases.

References

[1] Becker,T. (1994). On Gröbner Bases under Specialization. Applicable Algebra in Engineering, Communication and Computing. 5, 1–8.

[2] Kalkbrener,M. (1997). On the Stability of Gröbner Bases Under Specializations. J. Symb. Comp. 24/1, 51–58.

[3] Möller,H.M. (1988). On the Construction of Gröbner Bases Using Syzygies. J. Symb. Comp. 6/2-3, 345–359.

[4] Montes,A.(2002). A New Algorithm for Discussing Gröbner Bases with Parameters J. Symb. Comp. 33/2, 183–208.

[5] Sato, Y. (1998). A new type of canonical Gröbner bases in polynomial rings over Von Neumann regular rings. International Symposium on Symbolic and Algebraic Computation (ISSAC 98), Proceedings, 317–321.

[6] Sato, Y., Suzuki, A. and Nabeshima, K. (2003). ACGB on Varieties, Proceedings of the Sixth International Workshop on Computer Algebra in Scientific Computing(CASC 2003), 313–318.

[7] Saracino, D., Weispfenning, V. (1975). On algebraic curves over commutative regular rings, Model Theory and Algebra, a memorial tribute to A. Robinson, Springer LNM 498, 307–387.

[8] Spear, D.A. (1977). A constructive approach to commutative ring theory, Proc. of the 1977 MACSYMA Users' Conference, NASA CP-2012, 369–376.

[9] Suzuki, A. and Sato, Y. (2002). An Alternative approach to Comprehensive Gröbner Bases. International Symposium on Symbolic and Algebraic Computation (ISSAC 2002), Proceedings, 255–261.

[10] Suzuki, A. and Sato, Y.(2003). An Alternative approach to Comprehensive Gröbner Bases. J. Symb. Comp. 36/3-4 649–667.

[11] Weispfenning, V. (1989). Gröbner bases in polynomial ideals over commutative regular rings, EUROCAL '87, J. H. Davenport Ed., Springer LNCS 378, 336–347.

[12] Weispfenning, V. (1992). Comprehensive Gröbner bases, J. Symb. Comp. 14/1, 1–29.

[13] Weispfenning, V. (2002). Comprehensive Gröbner bases and regular rings, Symposium in Honor of Bruno Buchberger's 60th Birthday (LMCS 2002) Proceedings, 256–265.

[14] Zacharias, G. (1978). Generalized Gröbner bases in commutative polynomial rings, Bachelor's thesis, M.I.T.