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Further properties of null-additive fuzzy measure on metric spaces

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Abstract

We shall continue to discuss further properties of null-additive fuzzy measure on metric spaces following the previous results. Under the null-additivity condition, some properties of the inner/outer regularity and the regularity of fuzzy measure are shown. Also the strong regularity of fuzzy measure is discussed on complete separable metric spaces. As an application of strong regularity, we present a characterization of atom of null-additive fuzzy measure.

Keywords: Fuzzy measure; null-additivity; regularity;

1 Introduction

Recently, various regularities of set function are proposed and investigated by many authors ([2, 4, 5, 6, 7, 8, 9, 10]). As it is seen, the regularities play important role in nonadditive measure theory. In [4, 9] we discussed the regularity of null-additive fuzzy measure and proved Egoroff's theorem and Lusin's theorem for fuzzy measure on metric space.

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In this paper, we shall continue to investigate regularity of fuzzy measure on metric spaces following the results by [4, 9]. Under the null-additivity, weekly null-additivity and converse null-additivity condition, we shall discuss the relation among the inner regularity, the outer regularity and the regularity of fuzzy measure. Also we define the strong regularity of fuzzy measure and show our main result: the null-additive fuzzy measures possess strong regularity on complete separable metric spaces. Using strong regularity we shall show a version of Egoroff's theorem and Lusin's theorem for null-additive fuzzy measure on complete separable metric spaces, respectively. Lastly, as an application of strong regularity, we present a characterization of atom of null-additive fuzzy measure.

2 Preliminaries

Throughout this paper, we assume that $(X,\ d)$ is a metric space, and that $\mathcal{O}$, $\mathcal{C}$ and $\mathcal{K}$ are the classes of all open, closed and compact sets in (X, d), respectively. $\mathcal{B}$ denotes Borel σ-algebra on X, i.e., it is the smallest σ-algebra containing $\mathcal{O}$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{B}$.

A set function $\mu : \mathcal{B} \to [0, +\infty]$ is said to be *continuous from below*, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$; *continuous from above*, if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \searrow A$; *strongly order continuous*, if $\lim_{n\to+\infty}\mu(A_n) = 0$ whenever $A_n \searrow B$ and $\mu(B) = 0$; *null-additive*, if $\mu(E \cup F) = \mu(E)$ for any E whenever $\mu(F) = 0$; *weakly null-additive*, if $\mu(E \cup F) = 0$ whenever $\mu(E) = \mu(F) = 0$; *converse-null-additive*, if $\mu(E-F) = 0$ whenever $F \subset E$ and $\mu(F) = \mu(E) < +\infty$; *finite*, if $\mu(X) < \infty$.

Obviously, the null-additivity of μ implies weakly null-additivity.

Definition 2.1 A fuzzy measure on $(X, \mathcal{B})$ is an extended real valued set function $\mu : \mathcal{F} \to [0, +\infty]$ satisfying the following conditions:
(1) $\mu(\emptyset) = 0$;
(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$ (monotonicity).

We say that a fuzzy measure μ is continuous if it is continuous both from below and from above.

Note that in this paper we always assume that μ is a finite fuzzy measure.

3 Regularity of fuzzy measure

Definition 3.1 [10] A fuzzy measure μ is called *outer regular* (resp. *inner* *regular*), if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exists a set $G \in \mathcal{O}$ (resp. $F \in \mathcal{C}$) such that $A \subset G$, $\mu(G - A) < \epsilon$ (resp. $F \subset A$, $\mu(A - F) < \epsilon$). μ is called *regular*, if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exist a closed set $F \in \mathcal{C}$ and an open set $G \in \mathcal{O}$ such that $F \subset A \subset G$ and $\mu(G - F) < \epsilon$.

Obviously, if fuzzy measure μ is regular, then it is both outer regular and inner regular.

Proposition 3.1 [4] *If μ is weekly null-additive and continuous, then it is regular. Furthermore, if μ is null-additive, then for any $A \in \mathcal{B}$,*

$$\begin{aligned} \mu(A) &= \sup\{\ \mu(F) \mid F \subset A,\ F \in \mathcal{C}\ \} \\ &= \inf\{\ \mu(G) \mid G \supset A,\ G \in \mathcal{O}\ \} \end{aligned}$$

In the following we present some properties of the inner regularity and outer regularity of fuzzy measure, their proofs can be easily obtained:

Proposition 3.2 *If μ is weekly null-additive and strongly order continuous, then both outer regularity and inner regularity imply regularity.*

Proposition 3.3 *Let μ be null-additive fuzzy measure.*

(1) *If μ is continuous from below, then inner regularity implies*

$$\mu(A) = \sup\{\ \mu(F) \mid F \subset A,\ F \in \mathcal{C}\ \}$$

for all $A \in \mathcal{B}$;

(2) *If μ is continuous from above, then outer regularity implies*

$$\mu(A) = \inf\{\ \mu(G) \mid A \subset G,\ G \in \mathcal{O}\ \}$$

for all $A \in \mathcal{B}$.

Proposition 3.4 *Let μ be converse-null-additive fuzzy measure.*

(1) *If μ is continuous from below and strongly order continuous, and for any $A \in \mathcal{B}$,*

$$\mu(A) = \sup\{\ \mu(F) \mid F \subset A,\ F \in \mathcal{C}\ \},$$

then μ is inner regular.

(2) *If μ is continuous from above, and for any $A \in \mathcal{B}$,*

$$\mu(A) = \inf\{\ \mu(G) \mid A \subset G,\ G \in \mathcal{O}\ \},$$

then μ is outer regular.

Definition 3.2 μ is called *strongly regular*, if for each $A \in \mathcal{B}$ and each $\epsilon > 0$, there exist a compact set $K \in \mathcal{K}$ and an open set $G \in \mathcal{O}$ such that $K \subset A \subset G$ and $\mu(G - K) < \epsilon$.

The strongly regularity implies regularity, and hence inner regularity and outer regularity.

Proposition 3.5 *Let μ be null-additive and continuous from below. If μ is strongly regular, then for any $A \in \mathcal{B}$,*

$$\mu(A) = \sup\{\ \mu(K) \mid K \subset A,\ K \in \mathcal{K}\ \}.$$

Proposition 3.6 *Let μ be null-additive and order continuous. If for any $A \in \mathcal{B}$,*

$$\mu(A) = \sup\{\ \mu(K) \mid K \subset A,\ K \in \mathcal{K}\ \},$$

then μ is strongly regular.

In the rest of the paper, we assume that $(X,\ d)$ is complete and separable metric space, and that μ is finite continuous fuzzy measure.

In the following we show the main result in this paper.

Theorem 3.1 *If μ is null-additive, then μ is strongly regular.*

To prove the theorem, we first present two lemmas.

Lemma 3.1 *Let μ be a finite continuous fuzzy measure. Then for any $\epsilon > 0$ and any double sequence $\{A_n^{(k)} \mid n \geq 1, k \geq 1\} \subset \mathcal{B}$ satisfying $A_n^{(k)} \searrow \emptyset$ $(k \to \infty)$, $n = 1, 2, \ldots$, there exists a subsequence $\{A_n^{(k_n)}\}$ of $\{A_n^{(k)} \mid n \geq 1, k \geq 1\}$ such that*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n^{(k_n)}\right) < \epsilon \quad (k_1 < k_2 < \ldots)$$

Proof. Since for any fixed $n = 1, 2, \ldots$, $A_n^{(k)} \searrow \emptyset$ as $k \to \infty$, for given $\epsilon > 0$, using the continuity from above of fuzzy measures, we have $\lim_{k\to+\infty} \mu(A_1^{(k)}) = 0$, therefore there exists k_1 such that $\mu(A_1^{(k_1)}) < \frac{\epsilon}{2}$; For this k_1, $(A_1^{(k_1)} \cup A_2^{(k)}) \searrow A_1^{(k_1)}$, as $k \to \infty$. Therefore it follows, from the continuity from above of μ, that

$$\lim_{k\to+\infty} \mu(A_1^{(k_1)} \cup A_2^{(k)}) = \mu(A_1^{(k_1)}).$$

Thus there exists k_2 $(> k_1)$, such that

$$\mu(A_1^{(k_1)} \cup A_2^{(k_2)}) < \frac{\epsilon}{2}.$$

Generally, there exist $k_1, k_2, \ldots, k_m$, such that

$$\mu(A_1^{(k_1)} \cup A_2^{(k_2)} \cup \ldots A_m^{(k_m)}) < \frac{\epsilon}{2}.$$

Hence we obtain a sequence $\{k_n\}_{n=1}^{\infty}$ of numbers and a sequence $\{A_n^{(k_n)}\}_{n=1}^{\infty}$ of sets. By using the monotonicity and the continuity from below of μ, we have

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n^{(k_n)}\right) \leq \frac{\epsilon}{2} < \epsilon.$$

Lemma 3.2 *If μ be continuous fuzzy measure, then for each $\epsilon > 0$, there exists a compact set $K_\epsilon \in \mathcal{K}$ such that $\mu(X - K_\epsilon) < \epsilon$.*

Proof. Since $(X,\ d)$ is separable, there exists a countable dense subsets $\{x_1, x_2, \ldots, x_n, \ldots\}$. For any for any $n, k \geq 1$, we put

$$\overline{S_k}(x_n) = \left\{x \ :\ x \in X,\ d(x, x_n) \leq \frac{1}{k}\right\},$$

then, for fixed $k = 1, 2, \cdots$, as $m \to +\infty$

$$\bigcup_{n=1}^{m} \overline{S_k}(x_n) \nearrow \bigcup_{n=1}^{\infty} \overline{S_k}(x_n) = X.$$

Thus, as $m \to +\infty$

$$X - \bigcup_{n=1}^{m} \overline{S_k}(x_n) \searrow \emptyset,$$

for fixed $k = 1, 2, \cdots$. Applying Lemma 1 to the double sequence $\{X - \bigcup_{n=1}^{m} \overline{S_k}(x_n) \mid m \geq 1, k \geq 1\}$, then there exists a subsequence $\{m_k\}_k$ of the positive integers such that

$$\mu\left(\bigcup_{k=1}^{+\infty}\left(X - \bigcup_{n=1}^{m_k} \overline{S_k}(x_n)\right)\right) < \epsilon$$

Put

$$K_\epsilon = \bigcap_{k=1}^{+\infty} \bigcup_{n=1}^{m_k} \overline{S_k}(x_n).$$

Thus, the closed set K_ϵ is totally bounded. From the completeness of X, we know that K_ϵ is compact in X and satisfies

$$\mu(X - K_\epsilon) = \mu\left(\bigcup_{k=1}^{+\infty}\left(X - \bigcup_{n=1}^{m_k} \overline{S_k}(x_n)\right)\right) < \epsilon.$$

The lemma is now proved.

Proof of Theorem 3.1. Let $A \in \mathcal{B}$ and given $\epsilon > 0$. From Proposition 3.1 we know that μ is regular. Therefore, there exist a sequence $\{F^{(k)}\}_{k=1}^{\infty}$ of closed sets and a sequence $\{G^{(k)}\}_{k=1}^{\infty}$ of open sets such that for every $k = 1, 2, \ldots$, $F^{(k)} \subset A \subset G^{(k)}$,

$$\mu(G^{(k)} - F^{(k)}) < \frac{1}{k}.$$

Without loss of generality, we can assume that the sequence $\{F^{(k)}\}_{k=1}^{\infty}$ is increasing in k and the sequence $\{G^{(k)}\}_{k=1}^{\infty}$ is decreasing in k. Thus, $\{G^{(k)} - F^{(k)}\}_{k=1}^{\infty}$ is a decreasing sequence of sets with respect to k, and as $k \to \infty$

$$G^{(k)} - F^{(k)} \searrow \bigcap_{k=1}^{\infty}(G^{(k)} - F^{(k)}).$$

Denote $D_1 = \bigcap_{k=1}^{\infty}(G^{(k)} - F^{(k)})$, and noting that $\mu(D_1) \leq \mu(G^{(k)} - F^{(k)}) < \frac{1}{k}$, $k = 1, 2, \ldots$, then $\mu(D_1) = 0$.

On the other hand, from Lemma 3.2 there exists a sequence $\{K^{(k)}\}_{k=1}^{\infty}$ of compact subsets in X such that for every $k = 1, 2, \ldots$

$$\mu(X - K^{(k)}) < \frac{1}{k},$$

and we can assume that $\{K^{(k)}\}_{k=1}^{\infty}$ is decreasing in k. Therefore, as $k \to \infty$

$$X - K^{(k)} \searrow \bigcap_{k=1}^{\infty}(X - K^{(k)}).$$

Denote $D_1 = \bigcap_{k=1}^{\infty}(X - K^{(k)})$, then $\mu(D_1) = 0$. Thus, we have

$$(X - K^{(k)}) \cup (G^{(k)} - F^{(k)}) \searrow D_1 \cup D_2$$

as $k \to \infty$. Noting that $\mu(D_1 \cup D_2) = 0$, by the continuity of μ, then

$$\lim_{k \to +\infty} \mu\left((X - K^{(k)}) \cup (G^{(k)} - F^{(k)})\right) = 0.$$

Therefore there exists k_0 such that

$$\mu\left((X - K^{(k_0)}) \cup (G^{(k_0)} - F^{(k_0)})\right) < \epsilon.$$

Denoting $K_\epsilon = K^{(k_0)} \cap F^{(k_0)}$ and $G_\epsilon = G^{(k_0)}$, then K_ϵ is a compact set and G_ϵ is an open set, and $K_\epsilon \subset A \subset G_\epsilon$. Since $G_\epsilon - K_\epsilon \subset (X - K^{(k_0)}) \cup (G^{(k_0)} - F^{(k_0)})$, we obtain

$$\mu(G_\epsilon - K_\epsilon) \leq \mu(X - K^{(k_0)}) \cup (G^{(k_0)} - F^{(k_0)}) < \epsilon.$$

This shows that μ is strongly regular. $\square$

Corollary 3.1 *If μ is null-additive, then for any $A \in \mathcal{B}$ the following statements hold:*

(1) *For each $\epsilon > 0$, there exist a compact set $K_\epsilon \in \mathcal{K}$ such that $K_\epsilon \subset A$ and $\mu(A - K_\epsilon) < \epsilon$;*

(2) $\mu(A) = \sup\{ \mu(K) \mid K \subset A,\ K \in \mathcal{K} \}$.

By using the strongly regular of fuzzy measure, similar to the proof of Theorem 3 and 4 in [4], we can prove the following theorems. They are a version of Egoroff's theorem and Lusin's theorem on complete separable metric space, respectively.

Theorem 3.2 (Egoroff's theorem) *Let μ be null-additive continuous fuzzy measure. If $\{f_n\}$ converges to f almost everywhere on X, then for any $\epsilon > 0$, there exists a compact subset $K_\epsilon \in \mathcal{K}$ such that $\mu(X - K_\epsilon) < \epsilon$ and $\{f_n\}_n$ converges to f uniformly on K_ϵ.*

Theorem 3.3 (Lusin's theorem) *Let μ be null-additive continuous fuzzy measure. If f is a real measurable function on X, then, for each $\epsilon > 0$, there exists a compact subset $K_\epsilon \in \mathcal{K}$ such that f is continuous on K_ϵ and $\mu(X - K_\epsilon) \le \epsilon$.*

4 Atoms of fuzzy measure

In this section, as an application of strongly regularity, we shall show a characterization of atom of null-additive fuzzy measure on complete separable metric space.

Definition 4.1 ([2]) A set $A \in \mathcal{B}$ with $\mu(A) > 0$ is call an atom if $B \subset A$ then

(i) $\mu(B) = 0$, or

(ii) $\mu(A) = \mu(B)$ and $\mu(A - B) = 0$.

Consider a nonnegative real-valued measurable function f on A. The *fuzzy integral* of f on A with respect to μ, denoted by $(S)\int_A f d\mu$, is defined by

$$(S)\int_A f d\mu = \sup_{0 \le \alpha < +\infty} [\alpha \wedge \mu(\{x : f(x) \ge \alpha\} \cap A)]$$

Theorem 4.1 *Let μ be null-additive and continuous. If A is an atom of μ, then there exists a point $a \in A$ such that the fuzzy integral satisfies*

$$(S)\int_A f d\mu = f(a) \wedge \mu(\{a\})$$

for any non-negative measurable function f on A.

Proof. It is similar to the proof of Theorem 8 in [2].

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