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完備距離空間におけるシャウダーの不動点定理と無限区間ファジィ境界値問題

Schauder's Fixed Point Theorems in Complete Metric Spaces and Fuzzy Boundary Value Problems on an Infinite Interval

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Abstract

Aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder's fixed point theorem. Finally we discuss a sufficient conditions for the existence of solutions of fuzzy differential equations on an infinite interval with boundary conditions.

1 Complete Metric Space of Fuzzy Numbers

Denote $I = [0,1]$. The following definition means that a fuzzy number can be identified with a membership function.

Definition 1 *Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by*

$$\mathcal{F}_{\mathbf{b}}^{st} = \{\mu : \mathbf{R} \to I \textit{ satisfying } \text{(i)-(iv) } \textit{below}\}.$$

(i) *μ has a unique number $m \in \mathbf{R}$ such that $\mu(m) = 1$ (normality);*

(ii) *$supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ is bounded in $\mathbf{R}$ (bounded support);*

(iii) *μ is strictly fuzzy convex on $supp(\mu)$ as follows:*

  (a) *if $supp(\mu) \neq \{m\}$, then*
  $$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$
  for $\xi_1, \xi_2 \in supp(\mu)$ with $\xi_1 \neq \xi_2$ and $0 < \lambda < 1$;

  (b) *if $supp(\mu) = \{m\}$, then $\mu(m) = 1$ and $\mu(\xi) = 0$ for $\xi \neq m$;*

(iv) *μ is upper semi-continuous on $\mathbf{R}$ (upper semi-continuity).*

It follows that $\mathbf{R} \subset \mathcal{F}_{\mathbf{b}}^{st}$. Because m has a membership function as follows:

$$\mu(m) = 1\ ; \quad \mu(\xi) = 0\ (\xi \neq m) \tag{1.1}$$

Then μ satisfies the above (i)-(iv).

In usual case a fuzzy number x satisfies *fuzzy convex on* **R**, *i.e.*,

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min[\mu(\xi_1), \mu(\xi_2)] \tag{1.2}$$

for $0 \leq \lambda \leq 1$ and $\xi_1, \xi_2 \in \mathbf{R}$. Denote $\alpha-$cut sets by

$$L_\alpha(\mu) = \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}$$

for $\alpha \in I$. When the membership function is fuzzy convex, then we have the following remarks.

Remark 1 *The following statements* (1) - (4) *are equivalent each other, provided with* (i) *of Definition* 1.

(1) (1.2) *holds;*

(2) $L_\alpha(\mu)$ *is convex with respect to* $\alpha \in I$;

(3) μ *is non-decreasing in* $\xi \in (-\infty, m)$, *non-increasing in* $\xi \in [m, +\infty)$, *respectively;*

(4) $L_\alpha(\mu) \subset L_\beta(\mu)$ *for* $\alpha > \beta$.

Remark 2 *The above condition* (iiia) *is stronger than* (1.2). *From* (iiia) *it follows that* $\mu(\xi)$ *is strictly monotonously increasing in* $\xi \in [\min supp(\mu), m]$. *Suppose that* $\mu(\xi_1) \geq \mu(\xi_2)$ *for* $\xi_1 < \xi_2 \leq m$. *From Remark* 1(3), *it follows that* $\mu(\xi_1) = \mu_1(\xi_2)$ *for some* $\xi_1 < \xi_2$, *so we get* $\mu(\xi) = \mu(\xi_1) = \mu(\xi_2)$ *for* $\xi \in [\xi_1, \xi_2]$. *This contradicts with Definition* 1 (iiia). *Thus* μ *is strictly monotonously increasing. In the similar way* μ *is strictly monotonously decreasing in* $\xi \in [m, \max supp(\mu)]$. *This condition plays an important role in Theorem* 1.

We introduce the following parametric representation of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$ as

$$\begin{aligned} x_1(\alpha) &= \min L_\alpha(\mu), \\ x_2(\alpha) &= \max L_\alpha(\mu) \end{aligned}$$

for $0 < \alpha \leq 1$ and

$$\begin{aligned} x_1(0) &= \min supp(\mu), \\ x_2(0) &= \max supp(\mu). \end{aligned}$$

In the following example we illustrate typical types of fuzzy numbers.

Example 1 *Consider the following* $L - R$ *fuzzy number* $x \in \mathcal{F}_{\mathbf{b}}^{st}$ *with a membership function as follows:*

$$\mu(\xi) = \begin{cases} L(\frac{|m-\xi|}{\ell})_+ & (\xi \leq m) \\ R(\frac{|\xi-m|}{r})_+ & (\xi > m) \end{cases}$$

Here it is said that $m \in \mathbf{R}$ *is a center and* $\ell > 0, r > 0$ *are spreads.* L, R *are* $I-$*valued functions. Let* $L(\xi)_+ = \max(L(|\xi|), 0)$ *etc. We identify* μ *with* $x = (x_1, x_2)$. *As long as there exist* L^{-1} *and* R^{-1}, *we have* $x_1(\alpha) = m - L^{-1}(\alpha)\ell$ *and* $x_2(\alpha) = m + R^{-1}(\alpha)r$.

Let $L(\xi) = -c_1\xi + 1$, *where* $c_1 > 0$ *and* $|x_1 - m| \leq \ell$. *We illustrate the following cases* (i)-(iv).

(i) *Let* $R(\xi) = -c_2\xi + 1$, *where* $c_2 > 0$. *Then* $c_2\ell(x_2 - m) = c_1 r(m - x_1)$.

(ii) *Let* $R(\xi) = -c_2\sqrt{\xi} + 1$, *where* $c_2 > 0$. *Then* $c_2\ell(x_2 - m)^2 = c_1 r^2(m - x_1)$.

(iii) *Let* $R(\xi) = -c_2\xi^2 + 1$, *where* $c_2 > 0$. *Then* $c_2^2\ell^2(x_2 - m) = c_1^2 r(x_1 - m)^2$.

(iv) *Let c be a real number such that $0 < c < 1$. Denote*

$$L(\xi) = \begin{cases} 1 & (\xi = 0) \\ -c\xi + c & (0 < \xi \leq 1) \end{cases}$$

and let $R(\xi) = L(\xi)$. Then we have $\ell(x_2 - m) = r(m - x_1)$ for $|x_1 - m| \leq \ell$. The representation of $x = (x_1, x_2)$ is as follows:

$$\begin{aligned} x_1(\alpha) &= m - (1 - \frac{\alpha}{c})\ell, \\ x_2(\alpha) &= m + (1 - \frac{\alpha}{c})r \quad (0 \leq \alpha < c) \\ x_1(\alpha) &= x_2(\alpha) = m \quad (c \leq \alpha \leq 1) \end{aligned}$$

The membership function is given by as follows:

$$\mu(\xi) = \begin{cases} 0 & (\xi < x_1(0), \xi > x_2(0)) \\ x_1^{-1}(\xi) & (x_1(0) \leq \xi < m) \\ 1 & (\xi = m) \\ x_2^{-1}(\xi) & (m < \xi \leq x_2(0)) \end{cases}$$

Denote by $C(I)$ the set of all the continuous functions on I to $\mathbf{R}$. The following theorem shows a membership function is characterized by x_1, x_2.

Theorem 1 *Denote the left-, right-end points of the α–cut set of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$ by $x_1(\alpha), x_2(\alpha)$, respectively. Here $x_1, x_2 : I \to \mathbf{R}$. The following properties* (i)-(iii) *hold.*

(i) $x_1, x_2 \in C(I)$;

(ii) $\max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1)$;

(iii) *x_1, x_2 are non-decreasing, non-increasing on I, respectively, as follows :*

(a) *there exists a positive number $c \leq 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c)$ and that $x_1(\alpha) = m = x_2(\alpha)$ for $\alpha \in [c, 1]$;*

(b) *$x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$;*

Conversely, under the above conditions (i) -(iii), *if we denote*

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \tag{1.3}$$

for $\xi \in \mathbf{R}$, then $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$.

Remark 3 *From the above Condition* (i) *a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $\mathbf{R}^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$.*

In what follows we denote $\mu = (x_1, x_2)$ for $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$. The parametric representation of μ is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be an $\mathbf{R}$–valued function. The corresponding binary operation of two fuzzy numbers $x, y \in \mathcal{F}_{\mathbf{b}}^{st}$ to $g(x, y) : \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} \to \mathcal{F}_{\mathbf{b}}^{st}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x,y)}$ of g is as follows:

$$\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))$$

Here $\xi, \xi_1, \xi_2 \in \mathbf{R}$ and μ_x, μ_y are membership functions of x, y, respectively. From the extension principle, it follows that, in case where $g(x,y) = x+y$,

$$\begin{aligned}&\mu_{x+y}(\xi)\\&= \max_{\xi=\xi_1+\xi_2} \min_{i=1,2}(\mu_i(\xi_i))\\&= \max\{\alpha \in I : \xi = \xi_1 + \xi_2,\ \xi_i \in L_\alpha(\mu_i), i = 1,2\}\\&= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}.\end{aligned}$$

Thus we get $x + y = (x_1 + y_1, x_2 + y_2)$. In the similar way $x - y = (x_1 - y_2, x_2 - y_1)$.

Denote a metric by

$$d_\infty(x,y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}$.

Theorem 2 *$\mathcal{F}_{\mathbf{b}}^{st}$ is a complete metric space in $C(I)^2$.*

2 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions μ_x, μ_y of $x, y \in \mathcal{F}_{\mathbf{b}}^{st}$ and $\lambda \in \mathbf{R}$, the following addition and a scalar product are given as follows :

$$\begin{aligned}\mu_{x+y}(\xi) &= \sup\{\alpha \in [0,1] :\\ &\qquad \xi = \xi_1 + \xi_2,\ \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\};\\ \mu_{\lambda x}(\xi) &= \begin{cases} \mu_x(\xi/\lambda) & (\lambda \neq 0)\\ 0 & (\lambda = 0,\ \xi \neq 0)\\ \sup_{\eta \in \mathbf{R}} \mu_x(\eta) & (\lambda = 0,\ \xi = 0)\end{cases}\end{aligned}$$

In [5] they introduced the following equivalence relation $(x,y) \sim (u,v)$ for $(x,y), (u,v) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$, *i.e.*,

$$(x,y) \sim (u,v) \Longleftrightarrow x + v = u + y. \tag{2.4}$$

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (2.4) means that the following equations hold.

$$x_i + v_i = u_i + y_i \quad (i = 1,2)$$

Denote an equivalence class by $[x,y] = \{(u,v) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st} : (u,v) \sim (x,y)\}$ for $x, y \in \mathcal{F}_{\mathbf{b}}^{st}$ and the set of equivalence classes by

$$\mathcal{F}_{\mathbf{b}}^{st}/\sim = \{[x,y] : x, y \in \mathcal{F}_{\mathbf{b}}^{st}\}$$

such that one of the following cases (i) and (ii) hold:

(i) if $(x,y) \sim (u,v)$, then $[x,y] = [u,v]$;

(ii) if $(x,y) \not\sim (u,v)$, then $[x,y] \cap [u,v] = \emptyset$.

Then $\mathcal{F}_{\mathbf{b}}^{st}/\sim$ is a linear space with the following addition and scalar product

$$[x,y] + [u,v] = [x+u, y+v] \tag{2.5}$$

$$\lambda[x,y] = \begin{cases} [(\lambda x, \lambda y)] & (\lambda \geq 0)\\ [((-\lambda)y, (-\lambda)x)] & (\lambda < 0)\end{cases} \tag{2.6}$$

for $\lambda \in \mathbf{R}$ and $[x, y], [u, v] \in \mathcal{F}_{\mathrm{b}}^{st}/\sim$. They denote a norm in $\mathcal{F}_{\mathrm{b}}^{st}/\sim$ by

$$\| [x,y] \| = \sup_{\alpha \in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here d_H is the Hausdorff metric is as follows:

$$\begin{aligned} & d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)) \\ & = \max(\sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|, \\ & \sup_{\eta \in L_\alpha(\mu_x)} \inf_{\xi \in L_\alpha(\mu_y)} |\xi - \eta|) \end{aligned}$$

It can be easily seen that $\| [x,y] \| = d_\infty(x,y)$.

Note that $\| [x,y] \| = 0$ in $\mathcal{F}_{\mathrm{b}}^{st}/\sim$ if and only if $x = y$ in $\mathcal{F}_{\mathrm{b}}^{st}$.

3 Schauder's Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $\mathcal{F}_{\mathrm{b}}^{st}$ has an induced Banach space.

Theorem 3 *Let S be a bounded closed subset in $\mathcal{F}_{\mathrm{b}}^{st}$. Assume that S contains any segments of $x, y \in S$, i.e., $\lambda x + (1 - \lambda) y \in S$ for $\lambda \in I$. Let V be an into continuous mapping on S. Assume that the closure $cl(V(S))$ is compact in $\mathcal{F}_{\mathrm{b}}^{st}$. Then V has at least one fixed point x in S, i.e., $V(x) = x$.*

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

Theorem 4 *Let $\mathcal{F}$ be a complete metric space with a metric d. Assume that $\mathcal{F}$ is closed under addition and scalar product, and that $d(\lambda x, 0) = |\lambda| d(x, 0)$ for the scalar product λx and $\lambda \in \mathbf{R}, x \in \mathcal{F}$. Denote $X = \{[x, 0] : x, 0 \in \mathcal{F}\}$. Here $[x, y]$ for $x, y \in \mathcal{F}$ are equivalence classes of (2.4) and 0 is the origin. Then X is a Banach space concerning addition (2.5), scalar product (2.6) and norm $\| [x, 0] \| = d(x, 0)$ for $[x, 0] \in X$.*

Moreover let S be a bounded closed subset in $\mathcal{F}$. Assume that S contains any segments of $x, y \in S$ in the same meaning of Theorem 3. Let V be an into continuous mapping on S. Assume that the closure $cl(V(S))$ is compact in $\mathcal{F}$. Then V has at least one fixed point in S.

4 FBVP on Infinite Intervals

In this section we deal with the following FBVP on an infinite interval:

$$\frac{dx}{dt} = p(t)x + f(t,x), \quad x(\infty) = c \tag{4.7}$$

Here $p : \mathbf{R}_+ \to \mathcal{F}_{\mathrm{b}}^{st}$, $f : \mathbf{R}_+ \times \mathcal{F}_{\mathrm{b}}^{st} \to \mathcal{F}_{\mathrm{b}}^{st}$ are continuous functions. Let denote $\mathbf{R}_+ = [0, \infty)$ and $c \in \mathcal{F}_{\mathrm{b}}^{st}$. The following assumptions play important roles in considering the existence of solutions of (4.7).

Assupmtion.

(A1) Assume that

$$\int_0^\infty d(p(s), 0) ds = K < \infty.$$

(A2) There exist positive real numbers a, r, R and integrable function $m : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\begin{aligned} & d(f(t,x), 0) \le m(t) \text{ for } (t,x) \in \mathbf{R}_+ \times S_1; \\ & \int_0^\infty m(s) ds \le rR; \\ & [R + N_p(a + \| L \| R)]K < 1. \end{aligned}$$

Here

$$S_1 = \{x \in \mathcal{F}_b^{st} : d(x,0) \le \min(ar, r)\}$$

and N_p is independent on the function p.
$L : C_r^{\lim} \to \mathcal{F}_b^{st}$ is a linear operator as $L(x) = x(\infty)$ and

$$C_r^{\lim} = \{x \in C(\mathbf{R}_+ : \mathcal{F}_b^{st}) : \exists x(\infty), d(x,0) \le r\}.$$

(A3) There exists no solution of

$$\frac{dx}{dt} = p(t)x, L(x) = 0$$

except for the zero solution.

We expect the following existence theorem for solutions of FBVP on the infinite interval.

Under assumptions (A1) - (A3) we expect that there exists at least one solution of (4.7) in $C_r^{\lim}$ for any $c \in S_1$ by applying the Schauder's fixed point theorem in $C_r^{\lim}$.

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