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A Pincer Randomization Method for Valuing American Options

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1 Introduction

Variety has come to the options market nowadays since Black & Scholes (1973) and Merton (1973) published the seminal paper. In particular, the valuation of American options (i.e., options which can be exercised before the pre-specified date) written on dividend-paying assets is an important issue in the market due to that they have a much broader range of applications. Many academics and practitioners have attempted to resolve the value of American option analytically since McKean (1965) and Merton (1973) formulated the option value as a free boundary problem. However, there have been no closed-form formula and analytical solutions. The difficulties in such pricing options originate from the possibility of early exercise and the early exercise boundary not known priori must be determined as a part of the solution. Researchers have also made further efforts toward developments of numerical approximation methods for pricing American options.

A brand-new approximation is the randomization method proposed by Carr (1998), which is based on an American option with a random maturity. The random maturity follows the n-stage Erlangian distribution with mean equal to the pre-specified maturity. Although the idea is easy to understand, the probability density function (pdf) of Erlangian distribution is not suitable for obtaining a simple formula for the n-th approximation. Actually, Carr's formula for the n-th approximation of the American put value is given by a recursion of complex triple sums. To improve this shortcoming, an alternative randomization method has been recently developed by Kimura (2004), which used an order statistic for the random maturity. The order statistic also plays a key role in our new randomization method in this thesis, and hence the details of his method will be specified later. Kimura's approximation not only has a much simpler expression than Carr's one, but also its numerical results have almost the same accuracy as Carr's. However, computational results sometimes behave unstably under a certain condition. Improving this inadequacy is a principal goal of our randomization method, which we call a *pincer randomization*. The primal focus of this thesis is on the American put option because the call case can be analyzed by put-call symmetry relations.

The rest of this thesis is organized as follows: In Section 2, we provide some preliminaries for the analysis. Section 3 provides an idea of the pincer randomization method. To examine the accuracy of our method, numerical comparisons with other approximations are shown in Section 4. Finally, we give a conclusion and some comments on future research in Section 5.

2 Preliminalies

2.1 Basic Framework

Assume that the stock price is a risk-neutralized process governed by the stochastic differential equation

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t, \tag{2.1}$$

where $W \equiv \{W_t : t \in [0,T]\}$ is a standard Brownian motion process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen so that the stock has mean rate of return r. Here, r is the risk-free rate of interest, δ is the dividend rate, and σ is the volatility coefficient of the asset price.

We define the value of American put option with maturity date T and exercise price K, which is expressed as $C(S_t,t)$ through this article, satisfies the Black and Scholes (1973) partial differential equation (PDE)

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(S,t)}{\partial S^2} + (r-\delta)S\frac{\partial P(S,t)}{\partial S} + \frac{\partial P(S,t)}{\partial t} - rP(S,t) = 0 \tag{2.2}$$

subject to the boundary conditions

$$\lim_{S\downarrow 0} P(S,t) = 0, \tag{2.3}$$

$$\lim_{S\uparrow B_t} P(S,t) = K - B_t, \tag{2.4}$$

$$\lim_{S\uparrow B_t} \frac{\partial P(S,t)}{\partial S} = -1, \tag{2.5}$$

and the terminal condition

$$P(S_T, T) = (K - S_T)^+. \tag{2.6}$$

Equation (2.4) is usually called the "value matching" condition and Equation (2.5) is the "smooth pasting" condition. These conditions guarantee that premature exercise strategy on the early exercise boundary B_t will be optimal.

2.2 Randomization Methods

2.2.1 Carr's randomization

Carr's randomization method consists of the following three steps:

1. Randomize the maturity T by an exponentially distributed random variable $\tilde{T}$ with mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$ in order to value the so-called Canadian option.

2. Extend the result to the case that $\tilde{T}$ is distributed as the n-stage Erlangian distribution with the same mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$.

3. Take the limit of the randomized option value by letting $n \to \infty$ to obtain the underlying American option value.

Figure 1 illustrates the n-stage Erlangian distribution converges to Dirac's delta function concentrated at the mean $\lambda^{-1} = T$.

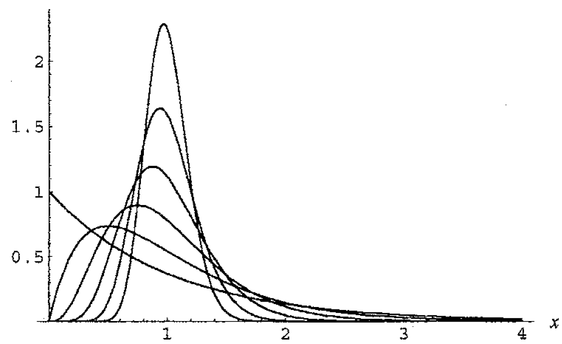


Figure 1: n-stage Erlangian probability density functions ($n = 1, 2, 4, 8, 16, 32$)

Let $g_n^*(T) = \mathbb{E}[g(\tilde{T})]$ for a continuous function g. Then, we have

$$g_n^*(T) = \int_0^\infty g(t)\, \frac{(nt/T)^{n-1}}{(n-1)!}\, \frac{n}{T}\, e^{-nt/T} dt, \tag{2.7}$$

for which we obtain

$$\lim_{n\to\infty} g_n^*(T) = g(T) \tag{2.8}$$

that is the mathematical essence of Carr's randomization method.

2.2.2 Kimura's randomization

Instead of the n-stage Erlangian distribution, Kimura (2004) used an order static for the random maturity. In much the same way as in Carr's randomization, his method consists of the following three steps:

1. Randomize the maturity T by an exponentially distributed random variable $\tilde{T}$ with mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$ in order to value the Canadian option.
2. Extend the result to the case that $\tilde{T}$ is distributed as an order statistic whth the same mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$.
3. Take the limit of the randomized option value by letting $n, m \to \infty$ to obtain the underlying American option value.

Let $X_1, \ldots, X_{n+m}$ be independent and exponentially distributed random variables with parameter α (> 0), and let $X_{(i)}$ denote the i-th smallest of these random variables ($i = 1, \ldots, n+m$). The pdf of $X_{(n+1)}$ is given by

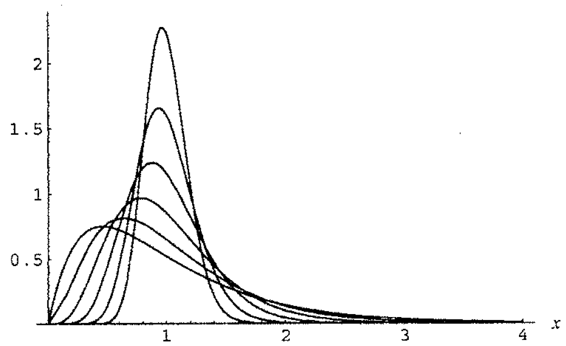
$$f(t) = \frac{(n+m)!}{n!(m-1)!}(1-e^{-\alpha t})^n \alpha e^{-m\alpha t}, \qquad t \geq 0. \tag{2.9}$$

The mean and variance of $X_{(n+1)}$ are given by

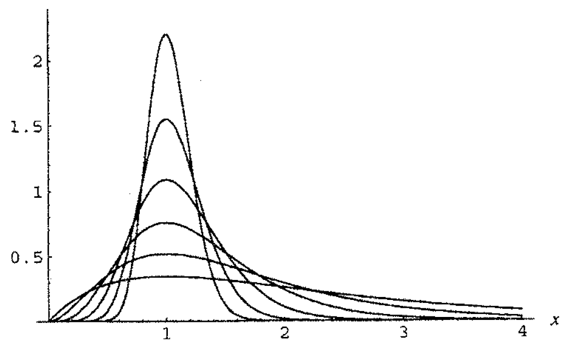
$$\mathbb{E}[X_{(n+1)}] = \frac{1}{\alpha}\sum_{i=0}^{n}\frac{1}{m+i}, \qquad \mathbb{V}[X_{(n+1)}] = \frac{1}{\alpha^2}\sum_{i=0}^{n}\frac{1}{(m+i)^2}. \tag{2.10}$$

In addition, the modal value of $X_{(n+1)}$ is given by

$$\mathbb{M}[X_{(n+1)}] \equiv \operatorname*{argmax}_t f(t) = \frac{1}{\alpha}\log\frac{n+m}{m}. \tag{2.11}$$



(a) mean matching: $\mathbb{E}[X_{(n+1)}] = 1$



(b) mode matching: $\mathbb{M}[X_{(n+1)}] = 1$

Figure 2: Probability density functions of the order statistic ($n = m = 1, 2, 4, 8, 16, 32$)

Figure 2(a) and 2(b) show the convergence of the pdf as n ($= m$) $\to \infty$. There is not all that much difference between these figures and they converge to Dirac's delta function concentrated at the mean $\mathbb{E}[X_{(n+1)}] = 1$. By setting either $\mathbb{E}[X_{(n+1)}] = T$ or $\mathbb{M}[X_{(n+1)}] = T$, $X_{(n+1)}$ can be another candidate for the random maturity $\tilde{T}$, because $\lim_{n,m\to\infty}\mathbb{V}[X_{(n+1)}] = 0$.

Kimura (2004) adopted the mode-matching $\mathbb{M}[X_{(n+1)}] = T$ in his randomization for computational convenience, because there is no significant difference between the two matchings. For the mode-matching, α can be determined by

$$\alpha = \frac{1}{T}\log\frac{n+m}{m}. \tag{2.12}$$

For a continuous function $g(t)$ ($t \geq 0$), let $g^*_{n,m} \equiv g^*_{n,m}(T) = \mathbb{E}[g(\tilde{T})]$, then

$$g^*_{n,m}(T) = \frac{(n+m)!}{n!(m-1)!}\int_0^\infty g(t)(1-e^{-\alpha t})^n \alpha e^{-m\alpha t}dt, \tag{2.13}$$

for which we have

$$\lim_{n,m\to\infty} g^*_{n,m}(T) = g(T). \tag{2.14}$$

Proposition 1 (Kimura) The sequence $(g^*_{n,m})_{n,m\geq 1}$ satisfies the recursion

$$\begin{cases} g^*_{0,m} = \displaystyle\int_0^\infty m\alpha e^{-m\alpha t} g(t)dt \\ g^*_{n,m} = \dfrac{n+m}{n} g^*_{n-1,m} - \dfrac{m}{n} g^*_{n-1,m+1}, \quad n \geq 1. \end{cases} \tag{2.15}$$

Let $L^*(m\alpha)$ denote a root of the equation for the early exercise boundary of Canadian options. The N-th randomized approximation $g^*_{N,N} \equiv \beta_N \approx B_t$ $(N \geq 1)$ can be obtained by the algorithm named OS-Random.

The algorithm also can be applied to computing the option value $P(t,S)$ by assuming that we have a functional program for computing $P^*(m\alpha, S)$ for a set of the parameters $\{t, S, K, T, r, \delta, \sigma\}$.

2.3 Canadian Options

The randomization methods are based on the value of Canadian option whose maturity is exponentially distributed to introduce not only Carr's randomization but also the alternative one proposed by Kimura (2004).

Proposition 2 (Kimura) The value of the European-style Canadian put option is given by

$$p^*(\lambda, S) = \begin{cases} \xi(S) + \dfrac{\lambda}{\lambda+r}K - \dfrac{\lambda}{\lambda+\delta}S, & S < K \\ \eta(S), & S \geq K, \end{cases} \tag{2.16}$$

where

$$\begin{aligned} \xi(S) &= \frac{1}{\theta_+ - \theta_-}\frac{\lambda}{\lambda+\delta}\left(1 - \frac{r-\delta}{\lambda+r}\theta_-\right)K\left(\frac{S}{K}\right)^{\theta_+}, \quad S < K \\ \eta(S) &= \frac{1}{\theta_+ - \theta_-}\frac{\lambda}{\lambda+\delta}\left(1 - \frac{r-\delta}{\lambda+r}\theta_+\right)K\left(\frac{S}{K}\right)^{\theta_-}, \quad S \geq K, \end{aligned} \tag{2.17}$$

and the parameters $\theta_\pm$ are two roots of the following quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - (\lambda + r) = 0, \tag{2.18}$$

i.e.,

$$\theta_\pm = \frac{1}{\sigma^2}\left\{-(r-\delta-\tfrac{1}{2}\sigma^2) \pm \sqrt{(r-\delta-\tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+r)}\right\}. \tag{2.19}$$

Proposition 3 (Kimura) For $L^* \leq K$, the value of the American-style Canadian put option is given by

$$P^*(\lambda, S) = \begin{cases} K - S, & S \leq L^* \\ p^*(\lambda, S) + e^*(\lambda, S), & S > L^*, \end{cases} \tag{2.20}$$

where

$$e^*(\lambda, S) = -\frac{1}{\theta_-}\left\{\theta_+\xi(L^*) + \frac{\delta}{\lambda+\delta}\right\}\left(\frac{S}{L^*}\right)^{\theta_-}, \quad S > L^*. \tag{2.21}$$

3 A Pincer Randomization Method

Kimura's randomization method is not only much simpler than Carr's one, but also as accurate as Carr's one; however, the method shows unstable behaviors near the expiry under certain conditions. The reasons for the unstability are considered as

(i) the algorithm is sensitive to the precision of the root L^*.

(ii) the (n, m)-th approximation $g^*_{n,m}$ cannot appropriately satisfies the value matching condition in the recursive procedure.

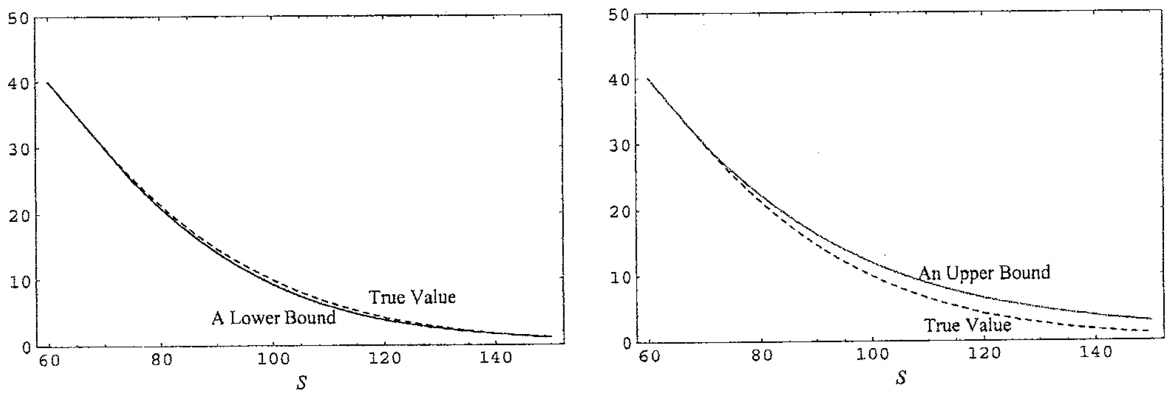
In this section, we propose a new randomization scheme named a *pincer randomization* (PR) method to overcome those difficulties. The PR method is based on a pair of lower and upper bounds for a true value (say TRUE), and then TRUE is sandwiched in between the bounds. This methods reflect some fundamental properties of the option Greek *Theta* and the order statistic. It is generally known that Theta indicates the ratio of the change in an American put option price to the decrease in time to expiration, so that the shorter the remaining time to expiration, the option value is cheaper.

Remark 1 Note that the relation of a pair of lower and upper bounds inverts if and only if the Theta is negative under deep-in-the-money.

3.1 Lower and upper bounds for the option value

Assume that the maturity T is a random variable $\tilde{T}$ distributed as the order statistic $X_{(n+1)}$ with mean $\mathbb{E}[\tilde{T}] = T$, as in the OS-Random algorithm. From Figure 2(a) and the Theta property of American put options that the mean-matching approximation for the option value always underestimates the true value when n, m is not large enough, giving a lower bound. Note that mean-matching approximation for the early exercise boundary provides the upper bound. Figure 3(a) shows that the lower bound is a tight one over the true value derived by the CRR binomial method with $n = 1000$.

In the same manner as the mean-matching case, Figure 2(b) and the Theta property shows that the mode-matching approximation always overestimates the true value when n, m is small, *i.e.*, it is an upper bound. For the early exercise boundary, the mode-matching approximation provides the lower bound. Figure 3(b) shows that the upper bound is less tight than the lower bound, where TRUE values are also computed by the CRR binomial method with $n = 1000$.



(a) A lower bound for the TRUE value

(b) An upper bound for the TRUE value

Figure 3: Lower and upper bounds ($T = 1.0$, $S = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\delta = 0$)

3.2 Interpolating lower and upper bounds

Figure 4 illustrates a relationship between the lower and upper bounds for the early exercise boundary. This figure shows that the the TRUE value is appropriately sandwiched in between the bounds, and that the upper bound derived by the mean matching is a good approximation for the TRUE value. For the

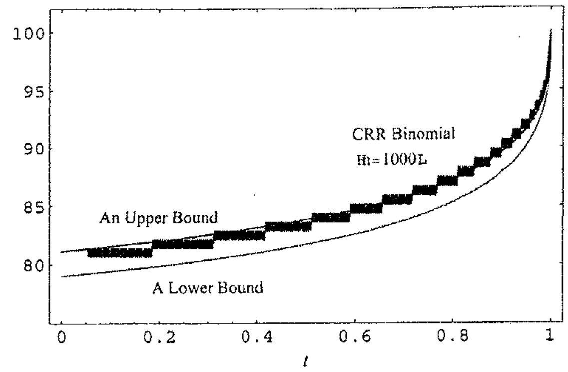


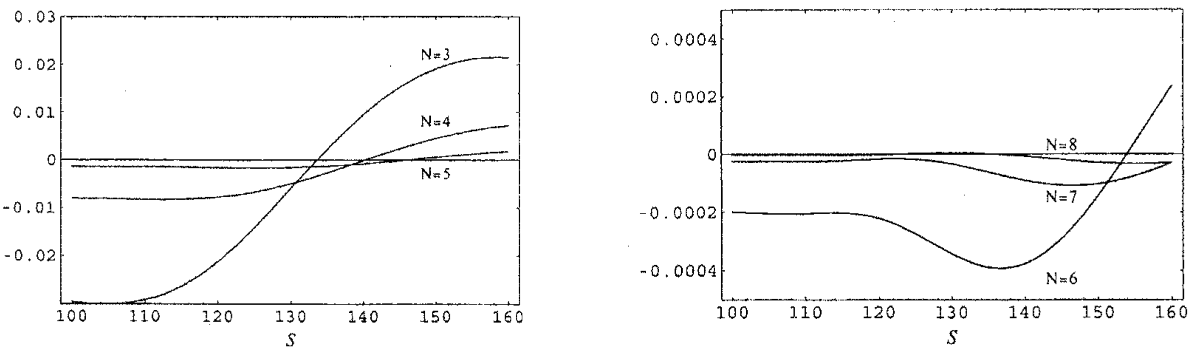
Figure 4: Lower and upper bounds for the early exercise boundary ($K = 100$, $r = 0.05$, $\sigma = 0.3$, $\delta = 0$)

option value, the TRUE value is appropriately sandwiched in between the bounds, and the lower bound is a good approximation for the TRUE one. From Figure 3, the mean matching provides more accurate approximations for the option value. From these observations, we employ the two methods below for valuing American put options.

- Arithmetic Average:
$$P_A(t, S_t) = \frac{L(t, S_t) + U(t, S_t)}{2} \tag{3.1}$$
where $L(t, S_t)$ and $U(t, S_t)$ are the lower and upper bound for the option value, respectively.

- Geometric Average:
$$P_G(t, S_t) = \sqrt{L(t, S_t) \times U(t, S_t)} \tag{3.2}$$

As described above, the upper bound of the early exercise boundary and the lower bound of the option value are good approximations for the TRUE values. Hence, we also add the lower-bound approximation for the option value in comparisons.



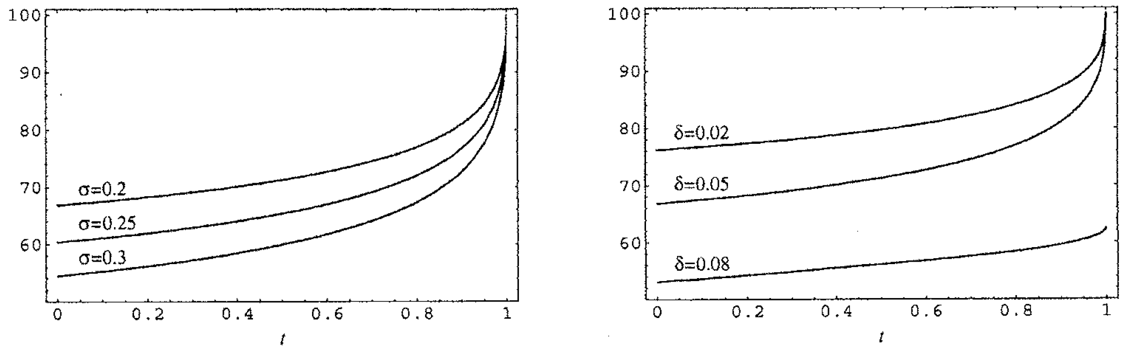
(a) $N = 3, 4, 5$ (b) $N = 6, 7, 8$

Figure 5: Relative percentage errors of the approximations for the vanilla European put value $p(0, S)$ ($T = 1.0$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\delta = 0.05$)

To determine the level N of the approximation, we make a comparison between $p(t, S)$ and its PR approximation. Obviously, the exact values of $p(t, S)$ can be computed by the Black-Scholes formula (2.2). Figure 5 illustrates the relative percentage errors of approximations for $p(0, S)$ as functions of S. The approximations become better as N increases and we have sufficient accuracy for $N \geq 6$. Hence, we will employ $N = 8$ in our numerical experiments.

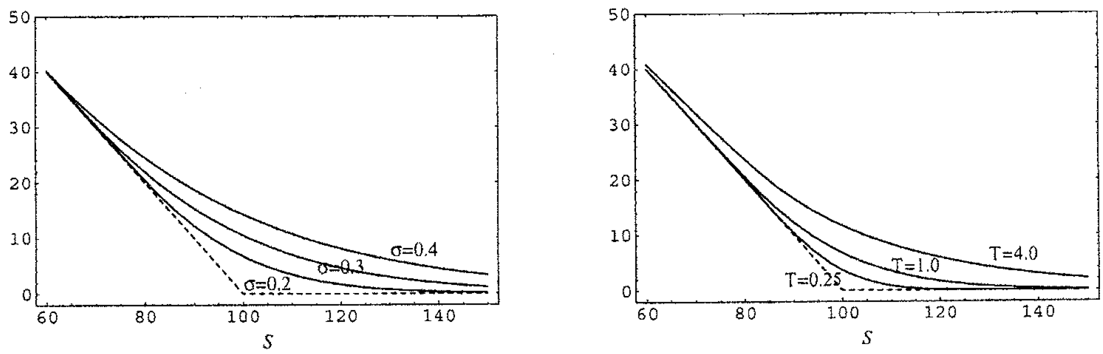
4 Computational Results

Figure 6(a) (6(b)) shows some relations between the early exercise boundary and the volatility (dividend rate). Also, Figure 7(a) (7(b)) shows some relations between the option values and the volatility (maturity). In order to check the performance of the PR method in detail, we compare them with other approximations for particular cases quoted from numerical experiments in AitSahlia and Carr (1997). Tables 1 and 2 summarize these results, in which we compute three approximation by the PR method with both the arithmetic and geometric averages and the lower-bound approximation named *LB-Rondom.* We employ the arithmetic average of the 1000- and 1001-steps binomial value as a bench mark of the TRUE value. For the methods of OS-Random, Carr, and Geske & Johnson, "N-pts" in these tables denote the number of steps of the N-point Richardson extrapolation. For the finite-difference results, the parameters N and M denote the numbers of time and state steps, respectively. See AitSahlia and Carr (1997) for details of their experiments.



(a) $\delta = 0.05$, $\sigma = 0.2, 0.25, 0.3$ (b) $\delta = 0.02, 0.05, 0.08$, $\sigma = 0.2$

Figure 6: Early exercise boundaries of put options ($K = 100$, $T = 1.0$, $r = 0.05$)



(a) $T = 1.0$, $\sigma = 0.2, 0.3, 0.4$ (b) $T = 0.25, 1.0, 4.0$, $\sigma = 0.2$

Figure 7: Values of put options ($K = 100$, $t = 0$, $r = 0.05$, $\delta = 0.02$)

The PR method performs very well and competes with OS-Random and Carr's randomization. In addition, the PR method succeeds in the way that modification of OS-Random that always underestimates the TRUE value, because the PR method provides not only much more accurate approximations for valuing put options but also better approximation than OS-Random. In addition, we see from these figures that the PR method is more accurate than LBA and LUBA, which are also the lower-bound and the lower-and-upper-bounds approximations, respectively.

Table 1 shows the impacts of the initial stock price S. The PR method with both of arithmetic and geometric average becomes accurate as S increases, because the early exercise premium relatively constitutes a smaller portion of the value for such cases. The fact is very well deserved from the viewpoints that the PR method can value European option values as accurate as the Black-Scholes formula and that we can decompose American option value into the early exercise premium plus European option value.

Table 2 demonstrates that remaining time impacts on the option values. The PR method with both arithmetic and geometric averages becomes accurate as the remaining time becomes long. For this tendency, we can give the same prospect from Table 1. In addition, from Tables 1 and 2, we can see that the PR method with arithmetic average is accurate enough and is greater than the one with geometric average. Clearly, this reflects the fact that $P_A(t, S_t) \geq P_G(t, S_t)$ for all (t, S_t).

From the observations in Figures 3 and 4, it was considered that the lower-bound approximations for the option values would perform well. However, we see from Tables 1 and 2 that the lower-bound approximations are less accurate than other approximations. We also see from other numerical experiments that the randomization method with mean matching performs well if and only if dividend is zero for which the root L^* can be computed via

$$L^* = K\left(\frac{r(\theta_+ - 1)}{\lambda}\right)^{\frac{1}{\theta_+}} \tag{4.1}$$

without using Newton's method. These observations would imply that the accuracy of the lower-bound (or mean-matching) approximation is highly sensitive to the computational accuracy of the root L^*.

5 Conclusion

The previous randomization methods have crucial problems such as (i) difficulty of implementation for Carr's one and (ii) unstable behavior near expiry for Kimura's one. To rectify these faults at the same time, we have employed an interpolation approximation using a pair of lower and upper bounds obtained by Kimura's randomization method. The idea is based on the Theta property of American put options. Our new method, the PR method, refines Kimura's one, removing another fault of underestimation.

The PR method generates accurate approximations when the initial stock price is in the out-of-money or the remaining time to maturity is long. It is straightforward to interpret these properties from the fact that American option value can decomposed into the early exercise premium and the associated European option value, the latter of which constitutes a greater portion of the whole value. However, the PR method still have a tendency of underestimation from the true value, which needs a further revision of the randomization.

Mathematically, the essential of randomization can be interpreted as an inversion of *Laplace* or *Fourier transforms*. This interpretation enables us to apply the randomization methods including the PR method to valuing other options, *e.g.*, *exotic* or *path-dependent* options such as Asian, lookback, barrier options and so on. This is a future theme of extensive research. Another extension of the randomization method is the case that the stock return jumps accidentally, that is, the stock price process follows not the Brownian motion but a jump-diffusion process such as Lévy processes. This remains as future work, too.

References

[1] AitSahlia, F., and Carr, P., 1997, "American Options: A Comparison of Numerical Methods," *Numerical Methods in Finance*, L.C.G. Rogers and D. Talay (eds.), Cambridge University Press, 67–87.

[2] Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–659.

[3] Carr, P., 1997, "Randomization and the American Put," *Review of Financial Studies*, 11, 597–626.

[4] Cox, J.C., S.A. Ross and M. Rubinstein, 1979, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229–264.

[5] Kimura, T., 2004, "Alternative Randomization for Valuing American Options," The 2004 Daiwa International Workshop on Financial Engineering, Kyoto, Japan.

[6] McKean, H.P., Jr., 1965, "Appendix: A Free Boundary Problem for the Heating Function Arising from a Problem in Mathematical Economics," *Industrial Management Review*, 6, 32–39.

[7] Merton, R., 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4, 141–183.

Table 1: A comarison of approximations for $P(0,S)$ ($T=3$, $K=100$, $r=0.06$, $\sigma=0.4$, $\delta=0.02$)

Method	$S=80$	$S=90$	$S=100$	$S=110$	$S=120$
Binomial	29.2601	24.8023	21.1294	18.0849	15.5428
PR method (Arithmetic Ave.)	28.8392	24.4533	20.8489	17.8681	15.3892
PR method (Geometric Ave.)	28.8373	24.4501	20.8445	17.8624	15.3823
LB-Random	28.5135	24.0614	20.4092	17.3971	14.9005
OS-Random (8-pts)	28.7998	24.4246	20.7891	17.7971	15.2995
Carr (3-pts)	29.2323	24.7692	21.0835	18.0369	15.4873
Geske and Johnson	31.0305	26.1543	22.1114	18.7646	15.9911
Quadratic	29.4377	25.0614	21.4484	18.4418	15.9239
LBA	29.2105	24.7669	21.1039	18.0635	15.5252
LUBA	29.2540	24.7989	21.1306	18.0860	15.5437
Bunch and Johnson	29.9382	25.1566	21.3092	18.1558	15.5755
Huang et al.	29.7147	25.0136	21.2121	18.1173	15.5729
Finite Difference ($N=200$, $M=300$)	29.0584	24.4744	20.6330	14.5535	14.5535

Table 2: A comparison of approximations for $P(0,100)$ ($K=100, r=0.06, \sigma=0.4, \delta=0.02$)

Method	$T=0.5$	$T=1.0$	$T=1.5$	$T=2.0$	$T=2.5$
Binomial	10.2741	13.8774	16.3682	18.2840	19.8349
PR method (Arithmetic Ave.)	10.3057	13.8392	16.2596	18.1109	19.6045
PR method (Geometric Ave.)	10.3016	13.8345	16.2547	18.1061	19.5998
LB-Random	10.0157	13.4765	15.8586	17.6884	19.1702
OS-Random (8-pts)	10.1802	13.7083	16.1399	18.0090	19.5321
Carr (3-pts)	10.2759	13.8670	16.3469	18.2533	19.7960
Geske and Johnson	10.3159	14.0553	16.7200	18.8388	20.5970
Quadratic	10.2728	13.9142	16.4627	18.4476	20.0743
LBA	10.2697	13.8679	16.3545	18.4476	19.8134
LUBA	10.2750	13.8796	16.3712	18.2869	19.8371
Bunch and Johnson	10.2679	13.8904	16.4070	18.3487	19.9434
Huang et al.	10.2813	13.8756	16.3657	18.2948	19.8742
Finite Difference ($N=200$, $M=300$)	10.2614	13.8578	16.3158	18.1500	19.5442