数理解析研究所講究録 1457 巻 2005 年 44-53

Optimization of Recursive Utility in the Incomplete Market Driven by Jump Diffusion Processes*

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2004.11.24

Abstract

This paper examines the maximization problem of the recursive utility in an incomplete market driven by jump-diffusion processes (compound-Poisson processes with the finite jump distribution). We state *A Priori Estimates* of the spread between the solutions of two Backward Stochastic Differential Equations (BSDEs) with jumps, from which we prove the existence and uniqueness of the solution which is originally shown by Tang and Li [16]. By using BSDE techniques, we show the first order condition for optimality and derive a state price density process explicitly. The optimal wealth and utility processes are characterized as the solution of forward-backward system which is applicable to the similar problems.

1 Introduction

In this paper, we consider the continuous-time portfolio-consumption problem of an agent with recursive utility who operates in an incomplete market driven by jump-diffusion processes (compound-Poisson processes with the finite jump distribution).

A conventional asset pricing model in financial economics is structured under the assumption that agents' preferences have a time-additive von Neumann-Morgenstern representation. However such a model has been exposed to many critics mainly for two reasons that are possibly related each other. First, it has failed to perform well when applied to real market. *The equity premium puzzle* by Mehra and Prescott [13] is one of the most famous counter examples. Second, the above specification of the utility confounds risk aversion and intertemporal substitutability, while these two aspects of preference are conceptually different. There have been a number of studies to solve this paradox.

Motivated by these drawback of such a conventional model, Epstein and Zin [5] studied recursive intertemporal utility functions that generalize conventional, time-additive, expected utility in a discrete-time setting. These utility functions permit a degree of separation to be achieved between substitution and risk aversion. In their following paper [6], the empirical results show that the recursive utility improves the goodness-of-fit of the model.

Duffie and Epstein [2] developed the recursive utility in a continuous-time setting as Stochastic Differential Utility (SDU). Also, they investigated fundamental properties of SDU and induced its Bellman equation. Moreover, in their subsequent paper [3], the state price density process of the normalized SDU was formulated explicitly by using the utility gradient approach of Duffie and Skiadas [4] in the case of the information generated by Brownian motion. Recently, there was an argument on the CCAPM under SDU in the economy with a jump-diffusion process [9]. However as the state price density process in his study is a quotation of Duffie and Skiadas [4], his result may be inappropriate and may not be applicable to a jump-diffusion process.

El Karoui, Peng and Quenez [7] showed that the maximization problem of SDU can be solved by using Backward Stochastic Differential Equation (BSDE) which was advocated by Pardoux and Peng [14]. By

*研究集会 不確実性科学と意思決定の数理と応用 講究録.

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using this result, they solved the optimization problem of SDU when the agent faces a nonlinear wealth process under the influence of taxation system or uncompetitive price formation [8].

Lazrak and Zapatero [10] characterized a set of consumption processes which are optimal for a given SDU when beliefs are unknown by using the forward-backward system of El Karoui *et al.* This problem is known as the inverse problem of Mas-Colell [12], and also known that the parameters of a utility function are not able to be determined uniquely only from the given demand curve. Lazrak and Zapatero presented the martingale conditions and advocated the verification method which uses a short rate.

In an attempt to develop these studies further, this study will apply a recursive utility to an incomplete market driven by jump-diffusion processes. As it has been noted in many papers, a recursive utility process can be seen as the solution of a BSDE. Firstly, we state *A Priori Estimates* of the spread between the solutions of two BSDEs with jumps, from which we prove the existence and uniqueness of the solution which was originally shown by Tang and Li [16]. (This result was extended to non-Lipschitzian case by Rong [15].) Additionally, we show the comparison theorem for BSDEs with jumps which gives that the positive constraint on the wealth process is equivalent to the positively of the terminal wealth only. Consequently, for a given consumption process and positive terminal wealth, the utility and the wealth processes are both solutions of BSDEs with jumps. To extend the result of El Karoui *et al.*, we study the property of differentiability of the solutions of BSDEs with jumps. Using the differentiability, we derive a first order condition which gives a necessary and sufficient condition of optimality. This characterization can be written in terms of not only the optimal utility and wealth processes but also their deflators. From first order condition, the optimal wealth and utility and their associated deflators are characterized as the solution of a forward-backward system.

2 Backward Stochastic Differential Equations with Jumps

This section focuses on introduction of some important extensions for BSDEs with jumps, which was originally shown by El Karoui, Peng and Quenez [7] in a Brownian information setting.

For the sake of analysis, some notation must be defined. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an $\mathbb{R}^m$-valued Brownian motion W and a random measure $N(dt, dy)$ which has a predicable measure $\nu(dy)dt$. $\{\mathcal{F}_t; t \in [0,T]\}$ denotes the filtration generated by a Brownian motion W and a random measure $N(dt, dy)$. $\tilde{N}(dt, dy)$ is the martingale measure such that $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$. For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidian norm such that $|x| = \sqrt{\mathrm{trace}(x^*x)}$, $\langle x, y\rangle$ denotes an inner product such that $\langle x, y \rangle = \mathrm{trace}(x^*y)$, and $*$ denotes a transpose.

Let $\mathbb{L}^2(\mathbb{R}^n)$ be the space of all measurable random variables $X : \Omega \to \mathbb{R}^n$ satisfying $\|X\| < \infty$ where $\|X\| := (E[|X|^2])^{\frac{1}{2}}$, and $\mathbb{H}^2_T(\mathbb{R}^n)$ be the space of all predictable process $\varphi : \Omega \times [0,T] \to \mathbb{R}^n$ satisfying $\|\varphi\| < \infty$ where $\|\varphi\| := (E[\int_0^T |\varphi_t|^2 dt])^{\frac{1}{2}}$. Denote also that $\mathbb{L}_{\nu(\cdot)}(\mathbb{R}^n)$ be the space of all measurable $\mathbb{R}^n$-valued functions $\varphi(y)$ such that $|\varphi|_{\nu(\cdot)} < \infty$, where $|\varphi|_{\nu(\cdot)} := (\int_y |\varphi(y)|\nu(dy))$, and $\mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}^n)$ be the space of all predictable function process $\varphi : \Omega \times [0,T] \to \mathbb{R}^n$ satisfying $\|\varphi\|_{\nu(\cdot)} < \infty$ where $\|\varphi\|_{\nu(\cdot)} := (E[\int_0^T \int_y |\varphi_t(y)|^2 N(dt, dy)])^{\frac{1}{2}}$. For $\beta > 0$ and $\varphi \in \mathbb{H}^2_T(\mathbb{R}^n)$, $\|\varphi\|^2_\beta$ denotes $E[\int_0^T e^{\beta t}|\varphi_t|^2 dt]$. $\mathbb{H}^2_{T,\beta}(\mathbb{R}^n)$ denotes the space $\mathbb{H}^2_T(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_\beta$. $\|\varphi\|^2_{\nu(\cdot),\beta}$ also denotes $E[\int_0^T e^{\beta t} \int_y |\varphi_t(y)|^2 N(dt, dy)]$ where $\varphi \in \mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}^n)$. $\mathbb{H}^2_{T,\nu(\cdot),\beta}(\mathbb{R}^n)$ denotes the space $\mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{\nu(\cdot),\beta}$.

Consider the BSDE,

$$-dY_t = f(t, Y_t, Z_t, \Lambda_t(\cdot))dt - Z_t^* dW_t - \int_{\mathbb{R}\setminus 0} \Lambda_t(y)\tilde{N}(dt, dy), \quad Y_T = \xi, \tag{1}$$

where

$$f(t, Y_t, Z_t, \Lambda_t(\cdot)) = \int_{\mathbb{R}\setminus 0} g(t, Y_t, Z_t, \Lambda_t(y))\nu(dy).$$

We now introduce *standard parameters* of the BSDE.

Definition 2.1 *(f, ξ) are said to be standard parameters for the BSDE where $\xi \in \mathbb{L}^2(\mathbb{R}^n)$, $f \in \mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}^n)$, and f is uniformly Lipschitz; i.e., there exists L such that,*

$$|f(\omega, t, y_1, z_1, \lambda_1(\cdot)) - f(\omega, t, y_2, z_2, \lambda_2(\cdot))| \le L\left(|y_1 - y_2| + |z_1 - z_2| + |\lambda_1 - \lambda_2|_{\nu(\cdot)}\right),$$

for $\forall (y_1, z_1, \lambda_1(y)), \forall (y_2, z_2, \lambda_2(y))$.

Tang and Li [16] showed that the BSDE with the standard parameters has a unique solution. This paper shall attempt to show an existence and uniqueness of the solution with a different approach. For this purpose, let us start with next proposition, so called *A Priori Estimates*.

Proposition 2.2 *(A Priori Estimates) Let* $((f^i,\xi^i);i=1,2)$ *be two standard parameters of the BSDE and* $((Y^i,Z^i,\Lambda^i);i=1,2)$ *be solutions. Put* $\delta Y_t = Y_t^1 - Y_t^2$ *and* $\delta_2 f_t = f^1(t,Y_t^2,Z_t^2,\Lambda_t^2(\cdot)) - f^2(t,Y_t^2,Z_t^2,\Lambda_t^2(\cdot))$. *For any* (κ,λ,μ) *such that* $\kappa^2 > L$, $\lambda^2 > L$, $\beta \geq L(2+\kappa^2+\lambda^2)+\mu^2$, *it follows that,*

$$\|\delta Y\|_\beta^2 \leq T\left\{E[e^{\beta T}|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2\right\},$$

$$\|\delta Z\|_\beta^2 \leq \frac{\kappa^2}{\kappa^2 - L}\left\{E[e^{\beta T}|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2\right\},$$

$$\|\delta \Lambda\|_\beta^2 \leq \frac{\lambda^2}{\lambda^2 - L}\left\{E[e^{\beta T}|\delta Y_T|^2] + \frac{1}{\mu^2}\|\delta_2 f\|_\beta^2\right\}.$$

Using this property, we can give a proof of the existence of a unique solution of BSDE.

Theorem 2.3 *Given standard parameters* (f,ξ), *there exists a unique solutions* $(Y,Z,\Lambda) \in \mathbb{H}_T^2(\mathbb{R}^n) \times \mathbb{H}_T^2(\mathbb{R}^{m\times n}) \times \mathbb{H}_{T,\nu(\cdot)}^2(\mathbb{R})$ *which solves* (1).

3 The utility and wealth processes

This section focuses on defining the utility and the wealth processes which formulate simultaneous Forward Backward Stochastic Differential Equations (FBSDEs). Our main problem is formulated as optimization of the solution of the FBSDEs.

3.1 The utility process

Duffie and Epstein [2] introduced a recursive utility in the continuous case under uncertainty. This permits risk aversion to be disentangled from the degree of intertemporal substitutability.

Let us consider an agent with the recursive utility who operates within an incomplete market driven by jump-diffusion process.

Following the notation given by Duffie and Epstein, a certainty equivalent m is defined on a probability measure p (representing the distribution of utility),

$$m(\tilde{Y}) := h^{-1}\left(\mathbb{E}[h(\tilde{Y})]\right),$$

where h denotes the von Neumann-Morgenstern index, which is continuous, strictly increasing and satisfy a growth condition; i.e., there exists some constants k such that, $h(x) \leq k(1+|x|)$.

With this certainty equivalent, the recursive utility is defined as,

$$Y_t := \lim_{s\to 0} m(Y_{t+s}|\mathcal{F}_t). \tag{2}$$

Also the aggregator f is defined as,

$$f(c_t, Y_t) := \lim_{s\to 0}\frac{\partial}{\partial t} m(Y_{t+s}|\mathcal{F}_t).$$

In these settings, it is natural to assume that the utility process Y_t follows that,

$$dY_t = \mu_Y dt + \sigma_Y dW_t + \int_{\mathbb{R}\backslash 0} J_Y(y)\tilde{N}(dt,dy). \tag{3}$$

By applying Feynman-Kac to $h(Y_t) = \lim_{s\to 0} E[h(Y_{t+s})|\mathcal{F}_t]$, we obtain the drift term of the utility process (3),

$$\mu_Y = -\left(f(c_t,Y_t) + \frac{h''(Y_t)}{h'(Y_t)}\sigma_Y^2 + \int_{\mathbb{R}\backslash 0}\frac{h(Y_{t-}+J_Y) - h(Y_{t-}) - J_Y h'(Y_t)}{h'(Y_t)}\nu(dy)\right).$$

Therefore the recursive utility can be expressed as follows.

$$Y_t = E\left[Y_T + \int_t^T \left\{ f(c_s, Y_s) + \frac{h''(Y_s)}{h'(Y_s)}\sigma_Y^2 + \int_{\mathbb{R}\backslash 0} \frac{h(Y_{s-} + J_Y) - h(Y_{s-}) - J_Y h'(Y_t)}{h'(Y_s)}\nu(dy)\right\} ds\right].$$

In the remaining of this paper, the utility process shall be expressed in more general form as,

$$-dY_t = f(t, c_t, Y_t, Z_t, \Lambda_t(\cdot))dt - Z_t^* dW_t - \int_{\mathbb{R}\backslash 0} \Lambda_t(y)\tilde{N}(dt, dy), \quad Y_T = Y, \tag{4}$$

where the driver of this BSDE is,

$$f(t, c_t, Y_t, Z_t, \Lambda_t(\cdot)) = \int_{\mathbb{R}\backslash 0} g(t, c_t, Y_t, Z_t, \Lambda_t(y))\nu(dy).$$

We make some assumptions in order to ensure that the BSDE has a unique solution,

Assumption 1 *f is supposed to be uniformly Lipschitz with respect to Y, Z and $\Lambda(y)$.*

Assumption 2 *There exists some constants k_1, k_2 such that $|f(t, c, 0, 0, 0)| \leq k_1 + k_2\frac{c^p}{p}$ a.s. with $0 < p < 1$ $\forall c \in \mathbb{R}^+$.*

These assumptions ensure that, for each $c \in \mathbb{H}^2_{T,\nu(\cdot)}$, and each terminal reward $Y \in \mathbb{L}^2$, BSDE (4) has a unique solution $(Y, Z, \Lambda) \in \mathbb{H}^2_T(\mathbb{R}^1) \times \mathbb{H}^2_T(\mathbb{R}^{m\times 1}) \times \mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R})$.

2 additional assumptions may be necessary in the later section.

Assumption 3 *f is strictly concave with respect to c, Y, Z and $\Lambda(y)$ and f is a strictly nondecreasing function with respect to c.*

By using the comparison theorem which is shown in next section, assumption 3 ensures the usual properties of utility functions, that is, monotonicity with respect to the terminal value and to the consumption, and concavity with respect to the consumption.

In general, the terminal value Y is a utility of the terminal wealth X_T, that is, $Y = u(X_T)$, where u satisfies the below assumption.

Assumption 4 *u is strictly concave and strictly nondecreasing real function defined on $\mathbb{R} \times \Omega$ which is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$ measurable, and satisfies $|u(x)| \leq k_1 + k_2\frac{x^p}{p}$ a.s. with $0 < p < 1$ $\forall x \in \mathbb{R}^+$.*

This assumption ensures that the variable $h(X_T) \in \mathbb{L}^{2/p} \subset \mathbb{L}^2$ for each $X_T \in \mathbb{L}^2$, and that the recursive utility associated with this terminal value is increasing and concave with respect to terminal wealth.

3.2 The wealth process

This subsection focuses on a specification of the dynamics of the wealth process. There exists n+1 assets to which the agent can invest some of his wealth in the market. One is a riskless asset whose price process P_t^0 is,

$$dP_t^0 = P_t^0 r_t dt. \tag{5}$$

where r_t is a short rate. In addition to the bond, n risky assets (the stocks) are continuously traded. The price process of ith stock P_t^i is,

$$\frac{dP_t^i}{P_t^i} = a_t^i dt + \sum_{j=1}^m \sigma_t^{i,j} dW_t^j + \int_{\mathbb{R}\backslash 0} d_t^i(y) N(dt, dy) \tag{6}$$

The agent invests to these assets with portfolio $\pi_t = (\pi_t^1, \pi_t^2, \cdots, \pi_t^n)^*$ and $\pi_t^0 = X_t - \sum_{i=1}^n \pi_t^i$. In these settings, the wealth process of the agent is,

$$\begin{aligned} dX_t &= (X_t - \sum_{i=1}^n \pi_t^i)\frac{dP_t^0}{P_t^0} + \sum_{i=1}^n \pi_t^i \frac{dP_t^i}{P_t^i} - c_t dt \\ &= \left(X_t r_t + \pi_t^* \sigma_t \theta_t + \int_{\mathbb{R}\backslash 0} \pi_t^* d_t(y)\nu(dy) - c_t\right) dt + \pi_t^* \sigma_t dW_t + \int_{\mathbb{R}\backslash 0} \pi_t^* d_t(y)\tilde{N}(dt, dy). \end{aligned} \tag{7}$$

where θ_t is the risk premium vector, such that $a_t - r_t\mathbf{1} = \sigma_t\theta_t$.

A general setting for the wealth process can be given by

$$-dX_t = a(t, c_t, X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot))dt - \pi_t^*\sigma_t dW_t - \int_{\mathbb{R}\backslash 0} \pi_t^* d_t(y)\tilde{N}(dt, dy), \quad X_0 = x, \tag{8}$$

where the driver of this forward SDE is,

$$a(t, c_t, X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot)) = \int_{\mathbb{R}\backslash 0} b(t, c_t, X_t, \pi_t^*\sigma_t, \pi_t^* d_t(y))\nu(dy).$$

We suppose similar assumptions on the driver a of the wealth process.

Assumption 5 *a is supposed to be uniformly Lipschitz with respect to X, $\pi^*\sigma$ and $\pi^* d(y)$.*

Assumption 6 *There exists a positive constant k such that, $|a(t, c, 0, 0, 0)| \leq kc$ a.s. for $\forall c \in \mathbb{R}^+$.*

Assumption 7 *a is convex with respect to c, x, $\pi^*\sigma$ and $\pi^* d(y)$, and a is nondecreasing with respect to c.*

Assumption 8 *$a(t, c, 0, 0, 0) \geq 0$ a.s. for $\forall c \in \mathbb{R}^+$.*

Let $(X_t^{x,c,\pi}; 0 \leq t \leq T)$ be the wealth process associated with initial wealth x and strategy $(c, \pi) \in \mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}) \times \mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}^n)$. One may notice that given an initial wealth and portfolio, there exists a pathwise unique wealth process which solves forward equation (8) since a is Lipschitz. (Growth condition $\int_{\mathbb{R}\backslash 0} |\pi_t^* d_t(y)|^2 \nu(dy) \leq k(1 + |X_t|^2)$ for $\exists k$, $\forall X_t$ is also satisfied since $d_t(y) \in \mathbb{L}^2_{\nu(\cdot)}(\mathbb{R}^n)$.)

4 Formulation of the maximization problem

Let us consider an agent, endowed with initial wealth $x > 0$, who invests at each time t with portfolio π_t and consumes with consumption process c_t. Recall that the wealth process of the agent is,

$$-dX_t = a(t, c_t, X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot))dt - \pi_t^*\sigma_t dW_t - \int_{\mathbb{R}\backslash 0} \pi_t^* d_t(y)\tilde{N}(dt, dy), \quad X_0 = x. \tag{9}$$

The agent may choose a portfolio-consumption strategy which maximizes his utility of consumption and terminal wealth. This optimization problem is expressed as,

$$\sup_{(c,\pi)\in\mathcal{A}(x)} Y_0^{(x,c,\pi)}, \tag{10}$$

where $Y^{(x,c,\pi)}$ is the recursive utility which follows,

$$-dY_t = f(t, c_t, Y_t, Z_t, \Lambda_t(\cdot))dt - Z_t^* dW_t - \int_{\mathbb{R}\backslash 0} \Lambda_t(y)\tilde{N}(dt, dy), \quad Y_T = u(X_T). \tag{11}$$

Assumption 9 *The function g and b are to be Lipschitz with respect to $\Lambda_t(y)$ and $\pi^* d(y)$ for $\forall y$ respectively; i.e. there exists L such that,*

$$g(t, y, z, \Lambda_t^1(y)) - g(t, y, z, \Lambda_t^2(y)) \leq L(\Lambda_t^1(y) - \Lambda_t^2(y)) \tag{12}$$

Proposition 4.1 *(Adjoint process) Let $(\beta, \gamma, \eta(y))$ be a bounded $(\mathbb{R}, \mathbb{R}^n, \mathbb{R})$-valued predictable process for $\forall y \in \mathbb{R}\backslash 0$, $\varphi \in \mathbb{H}^2$, $\xi \in \mathbb{L}$ and $\Lambda \in \mathbb{L}^2_{\nu(\cdot)}$. Then the linear BSDE,*

$$-dY_t = \left[\varphi_t + Y_{t-}\beta_t + Z_t^*\gamma_t + \int_{\mathbb{R}\backslash 0} \Lambda_t(y)\eta_t(y)\nu(dy)\right] dt - Z_t^* dW_t - \int_{\mathbb{R}\backslash 0} \Lambda_t(y)\tilde{N}(dt, dy), \quad Y_T = \xi,$$

has a unique solution $(Y, Z, \Lambda(y))$ *and* Y_t *is given by the closed form,*

$$Y_t = E\left[\Gamma_T \xi + \int_t^T \Gamma_{s-}\varphi_s ds \Big| \mathcal{F}_t\right],$$

where Γ_t *is the adjoint process defined by the linear forward SDE,*

$$\frac{d\Gamma_t}{\Gamma_{t-}} = \beta_t dt + \gamma_t^* dW_t + \int_{\mathbb{R}\backslash 0} \eta_t(y)\tilde{N}(dt, dy), \quad \Gamma_t = 1. \tag{13}$$

By using this proposition, we can prove next useful theorem.

Theorem 4.2 *(Comparison Theorem) Suppose that the assumptions hold, let two BSDEs be,*

$$-dY_t^i = f^i(t, c_t, Y_t^i, Z_t^i, \Lambda_t^i(\cdot))dt - Z_t^{i*}dW_t - \int_{\mathbb{R}\backslash 0} \Lambda_t^i(y)\tilde{N}(dt, dy), \quad Y_T^i = \xi^i, \tag{14}$$

where $i = 1, 2$ *and let us suppose that* $\xi^1 \geq \xi^2$ *and* $f^1(t, c, Y, Z, \Lambda(\cdot)) \geq f^2(t, c, Y, Z, \Lambda(\cdot))$. *Then, we have* $Y_t^1 \geq Y_t^2$, *a.s.* $\forall t \in [0, T]$.

Remark 4.1 Assumption 9 is a sufficient condition for the comparison theorem 4.2. If the process $\int_{\mathbb{R}\backslash 0} \Delta_\Lambda g^1(t)\delta\Lambda_t(y)\nu(dy)$ is Lipschitz to $\delta\Lambda_t(y)$ in the space of $\mathbb{L}_{\nu(\cdot)}$, the comparison theorem hold.

Let us define the *maximal reward* by

$$V(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} Y_0^{(x,c,\pi)}. \tag{15}$$

Consider two optimization problems, by using the comparison theorem, we state the following proposition.

Proposition 4.3 *Let* (a^1, f^1, u^1),(a^2, f^2, u^2) *be two standard parameters satisfying the above assumptions with,*

$$\begin{aligned} u^1(x) &\leq u^2(x), \\ f^1(t, c, y, z, \lambda(\cdot)) &\leq f^2(t, c, y, z, \lambda(\cdot)), \\ a^1(t, c, x, \pi^*\sigma, \pi^* d(\cdot)) &\geq a^2(t, c, x, \pi^*\sigma, \pi^* d(\cdot)). \end{aligned}$$

Let $V^1(x)$ *(respectively* $V^2(x)$*) be maximal reward associated with* (a^1, f^1, u^1) *(respectively* (a^2, f^2, u^2)*). Then,*

$$V^1(x) \leq V^2(x). \tag{16}$$

By Assumption 8 and the comparison theorem, the positive constraint on the wealth process $X_t^{x,c,\pi} \geq 0$ is equivalent to the positive constraint on the terminal wealth $X_T^{x,c,\pi} \geq 0$. Using this property, it becomes possible to take the terminal wealth as control variable instead of portfolio process. Let $\mathcal{D}$ denotes a consumption space, the subset of predictable measurable positive process c_t which belongs to $\mathbb{H}_T^2(\mathbb{R})$. And let $\mathcal{L}$ denotes a terminal wealth space, the set of square-integrable $\mathcal{F}_T$ measurable positive random variable ξ.

Definition 4.4 *A pair* $(\xi, c) \in \mathcal{L} \times \mathcal{D}$ *is a "consumption plan".* $(X_t^{(\xi,c)}, \pi_t^{(\xi,c)})$ *denotes the wealth and portfolio associated with given consumption plan* (ξ, c)*, which is the solution of the BSDE,*

$$-dX_t^{(\xi,c)} = a(t, c_t, X_t^{(\xi,c)}, \pi_t^{(\xi,c)*}\sigma_t, \pi_t^{(\xi,c)*}d_t(\cdot))dt - \pi_t^{(\xi,c)*}\sigma_t dW_t - \int_{\mathbb{R}\backslash 0} \pi_t^{(\xi,c)*}d_t(y)\tilde{N}(dt, dy), \quad X_T^{(\xi,c)} = \xi.$$

And $(Y_t^{(\xi,c)}, Z_t^{(\xi,c)}, \Lambda_t^{(\xi,c)}(y))$ *denotes the utility of consumption plan* (ξ, c)*, which is the solution of the BSDE,*

$$-dY_t^{(\xi,c)} = f(t, c_t, Y_t^{(\xi,c)}, Z_t^{(\xi,c)}, \Lambda_t^{(\xi,c)}(\cdot))dt - Z_t^{(\xi,c)*}dW_t - \int_{\mathbb{R}\backslash 0} \Lambda_t^{(\xi,c)}(y)\tilde{N}(dt, dy), \quad Y_T^{(\xi,c)} = u(\xi).$$

In the context of consumption plans, the initial wealth x is considered as a constraint. In case of a is nonlinear, the set of consumption plan such that $X_0^{(\xi,c)} = x$ is not convex. To obtain a convex set, the milder constraint $X_0^{(\xi,c)} \leq x$ is required.

Definition 4.5 *A consumption plan $(\xi, c) \in \mathcal{L} \times \mathcal{D}$ is called admissible for initial wealth x if and only if $X_0^{(\xi,c)} \leq x$. Let $\mathcal{A}(x)$ be the set of consumption plans admissible for the initial wealth x.*

It follows that the optimization problem can be written in the following *backward formulation*:

$$V(x) = \sup_{(\xi,c)\in\mathcal{A}(x)} Y_0^{(\xi,c)}. \tag{17}$$

Consequently, the agent optimize a consumption plan (ξ, c) belonging to $\mathcal{A}(x)$ so that it maximize the recursive utility function given by $Y_0^{(\xi,c))}$.

5 Maximum principle

Following the utility gradient approach of Duffie and Skiadas [4], we consider another optimization problem without constraint, which is equivalent to the original problem. Because of the assumption that f and h are concave and a is convex, the function defined on $\mathcal{L} \times \mathcal{D}$ by,

$$\begin{aligned}(\xi, c) &\mapsto x - X_0^{(\xi,c)},\\ (\xi, c) &\mapsto Y_0^{(\xi,c)},\end{aligned}$$

are concave. Using the result of convex analysis, we find that there exists a constant $v > 0$ such that,

$$\sup_{(\xi,c)\in\mathcal{L}\times\mathcal{D}} \left(Y_0^{(\xi,c)} + v\left(x - X_0^{(\xi,c)}\right)\right) \tag{18}$$

Furthermore, if the supreme is achieved in (17) by (ξ^0, c^0), then it is achieved by (ξ^0, c^0) in (18) with $X_0^{(\xi,c)} = x$.

This section shall attempt to derive a necessary and sufficient condition of the optimization problem of (18) under additional assumption. Let us start with the introduction of differentiability assumption.

Assumption 10 *a and f are supposed to be continuously differentiable with respect to $(c, X, \pi^*\sigma)$ and (c, Y, Z), and b and g are continuously differentiable with respect to $\pi^*\sigma$ and Λ, h is continuously differentiable. Also, h', f_c, a_c are supposed to be bounded.*

Let (ξ^0, c^0) be an optimal consumption plan for (18), and $(Y_t^0, Z_t^0, \Lambda_t^0(y))$ and (X_t^0, π_t^0) be the utility and wealth/portfolio processes associated (ξ^0, c^0). Let (ξ, c) be a consumption plan such that $\xi - \xi^0$, $c - c^0$ are uniformly bounded. Then $(\xi^0 + \alpha(\xi - \xi^0), c^0 + \alpha(c - c^0))$ is consumption plan for $\forall\alpha, 0 \leq \alpha \leq 1$. Let $(Y_t^\alpha, Z_t^\alpha, \Lambda_t^\alpha(y))$ and $(X_t^\alpha, \pi_t^\alpha)$ be the utility and wealth/portfolio processes associated $(\xi^0 + \alpha(\xi - \xi^0), c^0 + \alpha(c - c^0))$.

Proposition 5.1 *The function $\alpha \mapsto (Y_t^\alpha, Z_t^\alpha, \Lambda_t^\alpha(y))$ is right-differentiable with derivatives given by $(\partial_\alpha Y_t^0, \partial_\alpha Z_t^0, \partial_\alpha \Lambda_t^0(y))$, the solution of the following BSDE,*

$$\begin{aligned}-d\partial_\alpha Y_t^0 =& \left[f_c^0(t)(c_t - c_t^0) + f_Y^0(t)\partial_\alpha Y_t^0 + f_Z^0(t)^*\partial_\alpha Z_t^0 + \int_{\mathbb{R}\backslash 0} g_\Lambda^0(t,y)\partial_\alpha \Lambda_t^0 \nu(dy)\right] dt\\ &- \partial_\alpha Z_t^{0*} dW_t - \int_{\mathbb{R}\backslash 0} \partial_\alpha \Lambda_t^0(y)\tilde{N}(dt, dy),\\ \partial_\alpha Y_T^0 =& u'(\xi^0)(\xi - \xi^0),\end{aligned}$$

where $f_c^0(t) = f_c(t, c_t^0, Y_t^0, Z_t^0, \Lambda_t^0(\cdot))$, $f_Y^0(t) = f_Y(t, c_t^0, Y_t^0, Z_t^0, \Lambda_t^0(\cdot))$, $f_Z^0(t) = f_Z(t, c_t^0, Y_t^0, Z_t^0, \Lambda_t^0(\cdot))$, $g_\Lambda^0(t,y) = g_\Lambda(t, c_t^0, Y_t^0, Z_t^0, \Lambda_t^0(y))$.

Applying this result to $(X_t^\alpha, \pi_t^\alpha)$, a pair $(\partial_\alpha X_t^0, \partial_\alpha \pi_t^0)$ is a solution of

$$
\begin{aligned}
-d\partial_\alpha X_t^0 = &\left[a_c^0(t)(c_t - c_t^0) + a_X^0(t)\partial_\alpha Y_t^0 + a_{\pi^*\sigma}^0(t)^*(\partial_\alpha \pi_t^{0*}\sigma_t) + \int_{\mathbb{R}\backslash 0} b_{\pi^*d}^0(t,y)\partial_\alpha \pi_t^{0*} d_t(y)\nu(dy)\right] dt \\
&- \partial_\alpha \pi_t^{0*}\sigma_t dW_t - \int_{\mathbb{R}\backslash 0} \partial_\alpha \pi_t^{0*} d_t(y)\tilde{N}(dt,dy), \\
\partial_\alpha X_T^0 = &\xi - \xi^0,
\end{aligned}
$$

where $a_c^0(t) = a_c(t, c_t^0, X_t^0, \pi_t^{0*}\sigma_t, \pi_t^{0*}d_t(\cdot))$, $a_X^0(t) = a_X(t, c_t^0, X_t^0, \pi_t^{0*}\sigma_t, \pi_t^{0*}d_t(\cdot))$, $a_{\pi^*\sigma}^0(t) = a_{\pi^*\sigma}(t,\ c_t^0,$ $X_t^0,\ \pi_t^{0*}\sigma_t,\ \pi_t^{0*}d_t(\cdot))$, $b_{\pi^*d}^0(t,y) = b_{\pi^*d}(t, c_t^0, X_t^0, \pi_t^{0*}\sigma_t, \pi_t^{0*}d_t(y))$.

Since these BSDEs are linear, using Proposition 4.1, there exist the adjoint processes associated with $\partial_\alpha Y^0$ and $\partial_\alpha X^0$. The adjoint process of $\partial_\alpha Y^0$ is,

$$
\frac{d\Gamma_t}{\Gamma_t} = f_Y^0(t)dt + f_Z^0(t)^* dW_t + \int_{\mathbb{R}\backslash 0} g_\Lambda^0(t)\tilde{N}(dt,dy).
$$

Also the adjoint process of $\partial_\alpha X^0$ is,

$$
\frac{dH_t}{H_t} = a_X^0(t)dt + a_{\pi^*\sigma}^0(t)^* dW_t + \int_{\mathbb{R}\backslash 0} b_{\pi^*d}^0(t)\tilde{N}(dt,dy).
$$

Before proceeding to next theorem, we introduce additional assumption so called the *Inada condition.*

Assumption 11 *(Inada condition) h and f satisfies $h'(0) = +\infty$ and $f_c(t, 0, Y, Z, \Lambda(\cdot)) = +\infty$.*

Theorem 5.2 *Suppose that the assumptions hold. Let $(Y_t^0, Z_t^0, \Lambda_t^0(y))$ and (X_t^0, π_t^0) be the utility and the wealth/portfolio process associated with (ξ^0, c^0) which is an optimal consumption plan. The maximum principle can be written as follows,*

$$
\Gamma_T u'(\xi^0) = \upsilon H_T, \tag{19}
$$

$$
\Gamma_t f_c^0(t) = \upsilon H_t a_c^0(t), \quad 0 \le t \le T, \tag{20}
$$

Theorem 5.3 *Suppose that the assumptions hold, Let (ξ^0, c^0) be a consumption plan. Let $(Y^0, Z^0, \Lambda^0(y))$ and (X^0, π^0) be the utility and the wealth processes associated with (ξ^0, c^0). Suppose that (19) and (20) are satisfied, then (ξ^0, c^0) is optimal.*

These results lead the state price process explicitly.

Corollary 5.4 *(State price density process) State price density process ϕ_t in this economy can be expressed as,*

$$
\begin{aligned}
\phi_t = f_c^0(t)\exp\Bigg(&\int_0^t \left\{ f_Y^0(s) - \frac{1}{2}|f_Z^0(s)|^2 - \int_{\mathbb{R}\backslash 0} g_\Lambda^0(s)\nu(dy)\right\} ds \\
&+ \int_0^t f_Z^0(t)^* dW_s + \int_0^t \int_{\mathbb{R}\backslash 0} \log(1 + g_\Lambda^0(t))N(dt,dy)\Bigg).
\end{aligned} \tag{21}
$$

6 Forward-Backward system

This section focuses on the *forward-backward system* which is derived from the maximal principle. Using theorem 5.2, the optimal terminal wealth satisfies,

$$
\xi^0 = u'^{-1}\left(\frac{\upsilon H_T}{\Gamma_T}\right),
$$

However in this section, the arguments shall be made under an additional assumption.

Assumption 12 *The driver of the wealth can be expressed as,* $a(t, c_t, X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot)) = a(t, X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot)) + c$, *for* $\forall (t, c, X, \pi)$.

In this case, the optimal consumption c_t^0 is,

$$c_t^0 = I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot))$$

where the function I is inverse function defined that

$$f_c(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), Y_t, Z_t, \Lambda_t(\cdot)) = vH_t\Gamma_t^{-1}.$$

From theorem 5.2, we see that the following theorem hold.

Theorem 6.1 *Let* $(Y_t, Z_t, \Lambda_t(y))$ *and* (X_t, π_t) *be the utility and the wealth/portfolio process. And let* Γ_t, H_t *be the associated adjoint processes. They are the optimal utility and wealth processes and their deflators if and only if they are the unique solution of the forward-backward system,*
Backward components

$$\begin{aligned}
-dX_t =& a(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot))dt - \pi_t^*\sigma_t dW_t - \int_{\mathbb{R}\backslash 0} \pi_t^* d_t(y)\tilde{N}(dt, dy), \\
X_T =& u'^{-1}(vH_T\Gamma_T^{-1}), \\
-dY_t =& f(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), Y_t, Z_t, \Lambda_t(\cdot))dt - Z_t^* dW_t - \int_{\mathbb{R}\backslash 0} \Lambda_t(y)\tilde{N}(dt, dy), \\
Y_T =& u(u'^{-1}(vH_T\Gamma_T^{-1})).
\end{aligned}$$

Forward components

$$\begin{aligned}
\frac{d\Gamma_t}{\Gamma_t} =& f_Y(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), Y_t, Z_t, \Lambda_t(\cdot))dt + f_Z(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), Y_t, Z_t, \Lambda_t(\cdot))^* dW_t \\
&+ \int_{\mathbb{R}\backslash 0} g_\Lambda(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), Y_t, Z_t, \Lambda_t(y))\tilde{N}(dt, dy), \Gamma_0 = 1, \\
\frac{dH_t}{H_t} =& a_X(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot))dt \\
&+ a_{\pi^*\sigma}(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^*\sigma_t, \pi_t^* d_t(\cdot))^* dW_t \\
&+ \int_{\mathbb{R}\backslash 0} b_{\pi^* d}(t, I(t, vH_t\Gamma_t^{-1}, Y_t, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^*\sigma_t, \pi_t^* d_t(y))\tilde{N}(dt, dy), H_0 = 1.
\end{aligned}$$

7 Summary and conclusions

Discussed in this study is the maximization problem of the recursive utility in an incomplete market driven by jump diffusion processes. To fulfill this purpose, we have developed *A Priori Estimates* of the spread between the solutions of two BSDEs with jumps, from which the existence of unique solution is derived. After defining the utility and the wealth processes, the comparison theorem for BSDEs with jump is proved. To extend the result of El Karoui *et al.*, we have studied differentiability of the solutions of BSDEs with jumps. Also, using the property of differentiability, a first order condition is derived, which gives a necessary and sufficient condition for optimality. Thus, through the above arguments, a characterization of the optimal wealth and utility and their associated deflators can be derived as a unique solution of a forward-backward system. These results can be easily extended to plural dimensional jumps.

All proofs in this paper are omitted due to space limitations. They are available to interested readers on request.

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