数理解析研究所講究録 1458 巻 2005 年 80-88

ON THE HYPERREFLEXIVITY OF SUBSPACES OF TOEPLITZ OPERATORS

MAREK PTAK

ABSTRACT. The review of recent reflexivity and hyperreflexivity results for subspaces of Toeplitz operators will be presented. We will consider Toeplitz operators on the unit disc, the bidisc, the unit ball and generalized Toeplitz operators. The problems in this area are given.

1. INTRODUCTION

The notion of reflexivity and hyperreflexivity was at first stated for algebras of operators on Hilbert spaces. Both notions have their roots in invariant subspace problem. An algebra of operators is reflexive if there are so many invariant subspaces for all operators from the algebra that the set of these subspaces determine the algebra itself. An algebra of operators is hyperreflexive if the usual distance from any operator to the given algebra is controlled by distance given by invariant subspaces. Later this notion was extended to subspaces of operators, which seems to be more suitable setting, [16, 4]. As we will see later the most convenient setting for the above notions is duality between the trace class operators and the algebra of all bounded operators. In Section 2 we put the definitions in this setting which is equivalent to classical one, see [7].

Toeplitz operators are naturally given thus investigating their properties is one of main point in operator theory. Our aim is to present reflexivity and hyperreflexivity results on Toeplitz operators on various spaces. In Section 3 we consider Toeplitz operators on the classical Hardy space. Section 4 is devoted to Toeplitz operators on the Hardy space on the bidisc and unit ball. The generalized Toeplitz operators are considered in Section 5.

2. DEFINITIONS

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For a set of operators $\mathcal{S} \subset B(\mathcal{H})$ we denote by $\mathcal{W}(\mathcal{S})$ the smallest algebra containing the set $\mathcal{S}$, the identity operator I and closed in weak operator topology. By $\tau c(\mathcal{H})$ we denote the trace class operators and by F_k the set of operators of rank at most k. Duality between trace class

2000 *Mathematics Subject Classification.* Primary: 47L80; Secondary: 47L05, 47B35.

Key words and phrases. Reflexive subspaces, hyperreflexive subspace, hyperreflexive constant, k–hyperreflexive subspaces, Toeplitz operator.

operators $\tau c(\mathcal{H})$ and the algebra $B(\mathcal{H})$ is given by

$$\langle A, t\rangle = \operatorname{tr}(A\,t) \ \text{ for } \ A \in B(\mathcal{H}), t \in \tau c(\mathcal{H}).$$

An important role in reflexivity and hyperreflexivity is played by rank one operators; for $x, y \in \mathcal{H}$ we define $(x \otimes y)z = (z, y)x$ for $z \in \mathcal{H}$. The action of a rank one operator $x \otimes y$ on any operator $A \in B(\mathcal{H})$ can be expressed as

$$\langle A, x \otimes y\rangle = \operatorname{tr}(A(x \otimes y)) = (Ax, y).$$

Let $\mathcal{M} \subset B(\mathcal{H})$ be a subspace, then by ${}_\perp\mathcal{M} \subset \tau c(\mathcal{H})$ we denote the preannihilator of $\mathcal{M}$ and by $\text{ball}_\perp\mathcal{M}$ the unit ball in ${}_\perp\mathcal{M}$. The reflexivity and transitivity can be described by rank one operators. The w^*–closed subspace $\mathcal{M}$ is called *reflexive* if rank one operators in the preannihilator spans the preannihilator, i.e. $\text{span}({}_\perp\mathcal{M} \cap F_1) = {}_\perp\mathcal{M}$. In other words any operator out of the subspace can be separated from the subspace by rank–one operator. The subspace $\mathcal{M}$ is *transitive* if there is no rank one operator (except the zero operator) in its preannihilator, i.e. ${}_\perp\mathcal{M} \cap F_1 = \{0\}$. In other words there is no operator which can be separated from the subspace by rank–one operator. The antonyms of this two notions become clear. We will call the subspace k–reflexive if $\text{span}({}_\perp\mathcal{M}\cap F_k) = {}_\perp\mathcal{M}$. In other words any operator out of the subspace can be separated from the subspace by operator of rank k. This condition is equivalent to require that the k ampliation $\mathcal{M}^{(k)} = \{T \oplus T \oplus \cdots \oplus T : k - \text{times}, T \in \mathcal{M}\}$ of the subspace $\mathcal{M}$ is reflexive, see [2].

Let $A \in B(\mathcal{H})$. Then by duality the usual distance $\text{dist}(A, \mathcal{M}) = \inf\{\|A - T\| : T \in \mathcal{M}\}$ can be calculated by trace class operators, i.e.

$$\text{dist}(A, \mathcal{M}) = \sup\{|\langle A, t\rangle| : t \in \tau c, \ t \in \text{ball}_\perp\mathcal{M}\}.$$

On the other hand, we can introduce the distance α which uses only rank one operators, see [7]. Namely

$$\alpha(A, \mathcal{M}) = \sup\{|(Ax, y)| = |\langle A, x \otimes y\rangle| : x \otimes y \in \text{ball}_\perp\mathcal{M}\}. \tag{1}$$

We have the following inequality

$$\alpha(A, \mathcal{M}) \leqslant \text{dist}(A, \mathcal{M}).$$

A subspace $\mathcal{M}$ is called *hyperreflexive* if there is a constant κ such that

$$\text{dist}(A, \mathcal{M}) \leqslant \kappa\, \alpha(A, \mathcal{M}) \ \text{ for all } \ A \in B(\mathcal{H}). \tag{2}$$

The smallest constant κ fulfilling (2) is called *constant of hyperreflexivity* and it is denoted by $\kappa_\mathcal{M}$.

Hence the hyperreflexivity of a subspace $\mathcal{M}$ means that the distance from an operator to a subspace can be controlled by the distance calculated using rank one operators.

Taking the operators of rank k instead of rank–one we define

$$\alpha_k(A, \mathcal{M}) = \sup\{|\langle A, t\rangle| : t \in \text{ball}_\perp\mathcal{M} \cap F_k\}. \tag{3}$$

Similarly a subspace $\mathcal{M}$ is called *k-hyperreflexive* if there is a constant κ such that

$$\text{dist}(A, \mathcal{M}) \leqslant \kappa\, \alpha_k(A, \mathcal{M}) \quad \text{for all } A \in B(\mathcal{H}). \tag{4}$$

As it was observed in [14] it is not equivalent to hyperreflexivity of $\mathcal{M}^{(k)}$.

The reason why we call the subspace hyperreflexive can be seen by the following.

Proposition 2.1. *Let $\mathcal{M} \subset B(\mathcal{H})$ be a norm-closed subspace. If $\mathcal{M}$ is hyperreflexive then $\mathcal{M}$ is reflexive.*

Proof. Assume that $A \in \text{ref}\,\mathcal{M}$ Using (2) it is clear that $\alpha(A, \mathcal{M}) = 0$. Thus by hyperreflexivity $\text{dist}(A, \mathcal{M}) = 0$ and by norm–closeness $A \in \mathcal{M}$. □

The reverse implication is not true, see [15], i.e. there is a subspace and even an algebra which is reflexive, but not hyperreflexive.

3. Toeplitz operators on the disc

Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. Set $L^2(\mathbb{T}) = L^2(\mathbb{T}, m)$ and $L^\infty(\mathbb{T}) = L^\infty(\mathbb{T}, m)$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Let $H^2(\mathbb{D})$ be the Hardy space corresponding to $L^2(\mathbb{T})$ and let $P_{H^2(\mathbb{D})}$ be the projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. By $H^\infty(\mathbb{D})$ we denote the Hardy space corresponding to $L^\infty(\mathbb{T})$, i.e. the space of these functions from $L^\infty(\mathbb{T})$ which have an analytic extension to the whole unit disc $\mathbb{D}$.

For a given function $\varphi \in L^\infty(\mathbb{T})$ we can define a Toeplitz operator with the symbol φ as

$$T_\varphi f = P_{H^2(\mathbb{D})}(\varphi f) \quad \text{for } f \in H^2(\mathbb{D}).$$

The unilateral shift can be seen as the operator $S \in B(H^2(\mathbb{D}))$, $(Sf)z = z\, f(z)$ for $f \in H^2(\mathbb{D})$, i.e. $S = T_z$. By $\mathcal{T}(\mathbb{D})$ we denote the space of all Toeplitz operators and by $\mathcal{A}(\mathbb{D})$ the space of Toeplitz operators with symbols from $H^\infty(\mathbb{D})$. Note also that $\mathcal{T}(\mathbb{D}) = \mathcal{W}(S)$. Moreover, ([12, Corollary to Problem 194]),

$$\mathcal{T}(\mathbb{D}) = \{A \in B(H^2(\mathbb{D})) : A = T_z^* A T_z\} \tag{5}$$

and by [12, Problem 116],

$$\mathcal{A}(\mathbb{D}) = \{A \in B(H^2(\mathbb{D})) : AT_z = T_z A\}. \tag{6}$$

First reflexivity result concerning Toeplitz operators was shown by Sarason.

Theorem 3.1. [22] *If S is the unilateral shift then $\mathcal{W}(S)$ is reflexive.*

Proof. Note that vectors $k_\lambda = (1 - \bar{\lambda} z)^{-1}$, $\lambda \in \mathbb{D}$, are eigenvectors of the backward shift (i.e. $S^* k_\lambda = \bar{\lambda}\, k_\lambda$). Moreover, the set $\{k_\lambda : \lambda \in \mathbb{D}\}$ is linearly dense in the Hardy space $H^2(\mathbb{D})$.

Now when we take an operator $A \notin \mathcal{W}(S)$, then A do not commute with S by (6), thus A^* do not commute with S^*. Hence there is k_λ such that $A^* S^* k_\lambda =$

$\bar{\lambda}A^*k_\lambda \neq S^*A^*k_\lambda$ and $(\bar{\lambda} - S^*)A^*k_\lambda \neq 0$. Hence there is vector y such that $((\bar{\lambda} - S^*)A^*k_\lambda, y) \neq 0$. Thus

$$\langle A, (\lambda - S)y \otimes k_\lambda \rangle = (A(\lambda - S)y, k_\lambda) = (y, (\bar{\lambda} - S^*)A^*k_\lambda) \neq 0.$$

On the other hand if $B \in \mathcal{W}(S)$ then $(\bar{\lambda} - S^*)B^*k_\lambda = 0$ since B commutes with S and B^* commutes with S^*. Therefore $\langle A, (\lambda - S)y \otimes k_\lambda \rangle = 0$. □

In general, the reflexivity and the hyperreflexivity are not hereditary, but, in this case using property $\mathbb{A}$ and $\mathbb{A}_1(1)$ (see [7]) it is not hard to show that

Proposition 3.2. [22] *Let $\mathcal{M} \subset \mathcal{A}(\mathbb{D})$ be a w^*-closed subspace. Then $\mathcal{M}$ is reflexive.*

On the other hand

Proposition 3.3. [3] *The space of all Toeplitz operators $\mathcal{T}(\mathbb{D})$ is transitive.*

Proof. Let $f, g \in H^2(\mathbb{D})$ and $f \otimes g \in {}_\perp\mathcal{T}(\mathbb{D})$. Then, for all $\varphi \in L^\infty(\mathbb{T})$, we have

$$0 = \langle T_\varphi, f \otimes g \rangle = \langle T_\varphi f, g \rangle = \int \varphi f \overline{g}\, dm.$$

Since the equality holds for all functions $\varphi \in L^\infty(\mathbb{T})$ thus $f\,\overline{g} = 0$ as a function in $L^1(\mathbb{T})$. Since $f, g \in H^2(\mathbb{D})$ thus both functions f, g can not be equal to 0 on the set of positive measure on $\mathbb{T}$. Hence $f = 0$ or $g = 0$ thus ${}_\perp\mathcal{T}(\mathbb{D}) \cap F_1 = \{0\}$. □

If we consider not rank-one but rank–two operators, the space of Toeplitz operators is "close" to reflexivity.

Proposition 3.4. [3] *The space of all Toeplitz operators $\mathcal{T}(\mathbb{D})$ is 2–reflexive.*

Proof. By characterization (5) of Toeplitz operators, if $A \notin \mathcal{T}(\mathbb{D})$ there is rank–two operator $T_z x \otimes T_z y - x \otimes y$ which is not zero on A, but zero on $\mathcal{T}(\mathbb{D})$. □

The following dichotomy between transitivity and reflexivity of Toeplitz operators holds.

Theorem 3.5. [3] *Let $\mathcal{B} \subset \mathcal{T}(\mathbb{D})$ be a w^*–closed subspace. Then the following are equivalent:*

(1) *$\mathcal{B}$ is reflexive,*
(2) *$\mathcal{B}$ is not transitive,*
(3) *there is $f \in L^1(\mathbb{T})$ such that $\log|f| \in L^1(\mathbb{T})$ and $\int fg\, dm = 0$ for all g such that $T_g \in \mathcal{B}$.*

The condition (1) and (2) give the dichotomy while the condition (3) gives full characterization of a reflexive subspace of Toeplitz operators. There is also an extension of Theorem 3.2.

Theorem 3.6. [3] *Let $\mathcal{B}$ be w^*–closed subspace such that $\mathcal{A}(\mathbb{D}) \subset \mathcal{B} \subsetneq \mathcal{T}(\mathbb{D})$. Then $\mathcal{B}$ is reflexive.*

As a consequence of the above there are two following results.

Example 3.7. The subspace

$$S^{*n}\mathcal{A}(\mathbb{D}) = \operatorname{span}\{S^{*n}, S^{*n-1}, \dots, S^*, I, S, S^2, \dots\}$$

is reflexive.

Example 3.8. Let $E \subset \mathbb{T}$, such that $0 < m(E) < 1$ and χ_E denote the characteristic function of the set E. Then $T_{\chi_E}\mathcal{A}(\mathbb{D})$ is not reflexive.

The natural direction of study is extending reflexivity results to hyperreflexivity. Theorem 3.1 was improved to hyperreflexivity by Davidson and next the hyperreflexive constant was sharpened in [13].

Theorem 3.9. [9, 13] *Let S be the unilateral shift. Then $\mathcal{W}(S)$ is hyperreflexive and $\kappa_{\mathcal{W}(S)} < 13$.*

The proof of this theorem is not so elementary as in reflexivity case. The Nehari's theorem is strongly used. It says that

$$\|H_\varphi\| = \operatorname{dist}(\varphi, H^\infty(\mathbb{D})) \quad \text{for} \quad \varphi \in L^\infty(\mathbb{T}),$$

where $H_\varphi \in B(H^2(\mathbb{D}), L^2(\mathbb{T}) \ominus H^2(\mathbb{D}))$ is a Hankel operator defined as $H_\varphi f = P_{L^2(\mathbb{T})\ominus H^2(\mathbb{D})}(\varphi f)$ for $f \in H^2(\mathbb{D})$.

Now we can ask whether Proposition 3.4 can be strengthen. In [1, Proposition 5.2] Arveson constructed the projection $\pi : B(H^2(\mathbb{D})) \to \mathcal{T}(\mathbb{D})$, which has the property that for any $A \in B(H^2(\mathbb{D}))$ the operator $\pi(A)$ belongs to the w^*-closed convex hull of the set $\{T^*_{z^n} A T_{z^n} : n \in \mathbb{N}\}$. Using this projection it can be shown that

Theorem 3.10. [14] *The space of all Toeplitz operators $\mathcal{T}(\mathbb{D})$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}(\mathbb{D}) \leq 2$.*

Proof. Let $A \in B(H^2)$, then

$$\begin{aligned} d(A,\mathcal{T}) &\leq \|A - \pi(A)\| \leq \sup_{n\in\mathbb{N}} \|A - T^*_{z^n} A T_{z^n}\| \\ &= \sup_{n\in\mathbb{N}} \sup\{|\langle (A - T^*_{z^n} A T_{z^n})f, g\rangle| : f,\, g \in H^2,\, \|f\| = \|g\| = 1\} \\ &= \sup_{n\in\mathbb{N}} \sup\{|\langle Af, g\rangle - \langle Az^n f, z^n g\rangle| : f,\, g \in H^2,\, \|f\| = \|g\| = 1\} \\ &= \sup_{n\in\mathbb{N}} \sup\{|tr(A(f \otimes g - z^n f \otimes z^n g))| : f,\, g \in H^2,\, \|f\| = \|g\| = 1\}. \end{aligned}$$

Note that $\operatorname{rank}(f \otimes g - z^n f \otimes z^n g) \leq 2$ and $\|f \otimes g - z^n f \otimes z^n g\|_1 \leq 2$. Therefore the last expression is less than or equal to $2\,\alpha_2(A,\mathcal{T})$. Hence $\mathcal{T}$ is 2-hyperreflexive with constant $\kappa_2(\mathcal{T}) \leq 2$. ☐

To extend the result above to any subspace of Toeplitz operators we need an extra property, property $\mathbb{A}_{1/2}(1)$, of $\mathcal{T}(\mathbb{D})$, for definition and the proof see [14].

Corollary 3.11. *Every w^*-closed subspace consisting of Toeplitz operators is 2-hyperreflexive with constant at most 5.*

Compering the above results the following problem arises:

Problem 3.12. *Which reflexive subspaces of $\mathcal{T}(\mathbb{D})$ are hyperreflexive?*

It is worth to add that, in the context of Theorem 3.5, the Bergman shift was investigated in [8].

4. Toeplitz operators on the bidisc and the unit ball

We can also consider Hardy spaces $H^2(\mathbb{D}^2)$, $H^\infty(\mathbb{D}^2)$ on the bidisc $\mathbb{D}^2$ and the projection $P_{H^2(\mathbb{D}^2)}\colon L^2(\mathbb{T}^2) \to H^2(\mathbb{D}^2)$. For $\varphi \in L^\infty(\mathbb{T}^2)$ we define a Toeplitz operator with a symbol φ as

$$T_\varphi f = P_{H^2(\mathbb{D}^2)}(\varphi f) \text{ for } f \in H^2(\mathbb{D}^2).$$

Then the multiplication operators by independent variables can be written as

$$(T_{z_i} f)(z_1, z_2) = z_i f(z_1, z_2) \text{ for } f \in H^2(\mathbb{D}^2),\ i = 1, 2.$$

The space of all Toeplitz operators we denote by $\mathcal{T}(\mathbb{D}^2)$ and the algebra of analytic Toeplitz operators by $\mathcal{A}(\mathbb{D}^2) = \{T_\varphi : \varphi \in H^\infty(\mathbb{D}^2)\}$, which is equal to $\mathcal{W}(T_{z_1}, T_{z_2})$.

Similarly like in one variable case we have the following characterization (see [19, Proposition 3.3])

$$\mathcal{T}(\mathbb{D}^2) = \{A \in B(H^2(\mathbb{D}^2)) : A = T_{z_i}^* A T_{z_i},\ i = 1, 2\}. \tag{7}$$

Theorem 3.1 and Proposition 3.3 can be generalized to bidisc situation.

Theorem 4.1. [17, 19]

(1) *The algebra $\mathcal{A}(\mathbb{D}^2) = \mathcal{W}(T_{z_1}, T_{z_2})$ is reflexive.*
(2) *The subspace $\mathcal{T}(\mathbb{D}^2)$ is transitive.*

The proof of reflexivity of $\mathcal{A}(\mathbb{D}^2)$ is similar to the disc case. Note that the set $\{k_{\lambda_1,\lambda_2} = (1 - \bar{\lambda}_1 z_1)^{-1}(1 - \bar{\lambda}_2 z_2)^{-1} : \lambda_1, \lambda_2 \in \mathbb{D}\}$ is dense in $H^2(\mathbb{D}^2)$ and that the functions k_{λ_1,λ_2} are joint eigenvectors for T_{z_i}, $i = 1, 2$.

Thus the following problem arise:

Problem 4.2. *How to characterize reflexive subspaces of $\mathcal{T}(\mathbb{D}^2)$?*

In the contexts of hyperreflexivity of $\mathcal{A}(\mathbb{D})$ (Theorem 3.9), we can ask about hyperreflexivity of $\mathcal{A}(\mathbb{D}^2)$. As we have noticed, one of the tools for the proof of hyperreflexivity of $\mathcal{A}(\mathbb{D})$ was the Nehari's theorem.

In [5], [11] it was shown that the Nehari's theorem can not be extended to bidisc case. Thus we have the following problem.

Problem 4.3. *Is $\mathcal{A}(\mathbb{D}^2)$ hyperreflexive?*

The characterizations (5) was the main tool to prove Proposition 3.4 thus (7) allows us to see that

Theorem 4.4. *The space of all Toeplitz operators on the bidisc $\mathcal{T}(\mathbb{D}^2)$ is 2–reflexive.*

The main tool to prove 2–hyperreflexivity of the space of Toeplitz operators on the disc was the Arveson's projection. In [20] the existence of the similar projection was shown for a bidisc using two-variable Banach limit. Hence the 2–hyperreflexivity can be shown.

Theorem 4.5. [20] *The space of all Toeplitz operators on the bidisc $\mathcal{T}(\mathbb{D}^2)$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}(\mathbb{D}^2)) \leqslant 2$.*

The two–variable generalization of the disc $\mathbb{D}$ is not only bidisc $\mathbb{D}^2$, but also the two dimensional unit ball $\mathbb{B}^2$.

Let $H^2(\mathbb{B}^2)$ and $H^\infty(\mathbb{B}^2)$ be the Hardy spaces on the ball $\mathbb{B}^2$. Let us denote by $P_{H^2(\mathbb{B}^2)}$ a projection from $L^2(\partial\mathbb{B}^2)$ onto $H^2(\mathbb{B}^2)$. Similarly as for the disc case we can define a Toeplitz operator. For $\varphi \in L^\infty(\partial\mathbb{B}^2)$ we define an operator $(T_\varphi f) = P_{H^2(\mathbb{B}^2)}(\varphi f)$ for $f \in H^2(\mathbb{B}^2)$. We keep notation $\mathcal{A}(\mathbb{B}^2)$ for analytic and $\mathcal{T}(\mathbb{B}^2)$ for all Toeplitz operators in the ball $\mathbb{B}^2$.

It is known that

Theorem 4.6. [19, 18]

(1) *The algebra $\mathcal{A}(\mathbb{B}^2)$ is reflexive.*
(2) *The subspace $\mathcal{T}(\mathbb{B}^2)$ is transitive.*

Moreover, we get by [6] and [10, Corollary 2.13 and 2.14] the following

Theorem 4.7. *The algebra $\mathcal{A}(\mathbb{B}^2)$ is hyperreflexive.*

Hence we can state the following problem.

Problem 4.8. *How to characterize the reflexive subspaces of $\mathcal{T}(\mathbb{B}^2)$?*

5. 2–hyperreflexivity of generalized Toeplitz operators

The idea of generalized Toeplitz operators comes from replacing in the characterization (5) the backward shift T_z^* by any contraction. Precisely, for given contractions $S, T \in B(\mathcal{H})$, an operator $X \in B(\mathcal{H})$ is called a *generalized Toeplitz operator with respect to S and T* if $X = SXT^*$. This type of operators was investigated in [21]. The space of all such operators we denote by $\mathcal{T}(S,T) = \{X \in B(\mathcal{H}) : X = SXT^*\}$. Note that this characterization implies w^*–closeness of $\mathcal{T}(S,T)$.

In [18] this idea was extended to two variables. Having in mind characterization (7) of Toeplitz operators on the torus we can set the following definition. For given pairs of commuting contractions S_1, S_2 and $T_1, T_2 \in B(\mathcal{H})$, an operator $X \in B(\mathcal{H})$ is called a *generalized Toeplitz operator with respect to pairs S_1, S_2 and T_1, T_2* if $X = S_1XT_1^*$ and $X = S_2XT_2^*$. The space of all such operators we denote by $\mathcal{T}(S_1, S_2; T_1, T_2) = \{X \in B(\mathcal{H}) : X = S_1XT_1^*, X = S_2XT_2^*\}$. This space is also w^*–closed.

The reflexive behavior of the space of generalized Toeplitz operators $\mathcal{T}(S,T)$ depends on contractions S, T (the same we can tell about two variables case). For example, if the underling Hilbert space is the Hardy space on the unit circle and $S = T = T_z^*$ then $\mathcal{T}(T_1^*, T_2^*) = \mathcal{T}(\mathbb{T})$ is transitive. On the other hand, the space $\mathcal{T}(S,T)$ might be even reflexive. For example if $S = T = I_H$ then $\mathcal{T}(I_H, I_H) = B(\mathcal{H})$ which is reflexive.

In [20] the linear projection $\pi\colon B(\mathcal{H}) \to \mathcal{T}(S,T)$ and projection $\pi\colon B(\mathcal{H}) \to \mathcal{T}(S_1, S_2; T_1, T_2)$ with the similar properties as Arveson's projection were constructed. Hence we can estimate the reflexive behavior by

Theorem 5.1. [20] *Let $S, T \in B(\mathcal{H})$ be contractions. Then $\mathcal{T}(S,T)$ is 2–hyperreflexive.*

We have also a two variables version of Theorem 5.1.

Theorem 5.2. [20] *Let $S_1, S_2 \in B(\mathcal{H})$ and $T_1, T_2 \in B(\mathcal{H})$ be two pairs of commuting contractions. Then $\mathcal{T}(S_1, S_2; T_1, T_2)$ is 2–hyperreflexive.*

References

[1] W. T. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), 208–233.

[2] E. A. Azoff, *On finite rank operators and preannihilators*, Mem. Amer. Math. Soc., No. 357, 1986.

[3] E. A. Azoff and M. Ptak, *A dichotomy for linear spaces of Toeplitz operators*, J. Funct. Anal. **156** (1998), 411-428.

[4] E. A. Azoff and H. A. Shehada, *Literal embeddings of linear spaces of operators*, Indiana Univ. Math. J. **42**, (1993), 571–589.

[5] M. Bakonyi and D. Timotin, *On a conjecture of Cotlar and Sadosky on multidimensional Hankel operators*, C. R. Acad. Sci. Paris, Serie I **325** (1997), 1071–1075.

[6] H. Bercovici, *Hyper-Reflexivity and the Factorization of Linear Functionals*, J. Funct. Anal. **158** (1998), 242–252.

[7] J. B. Conway, *A course in operator theory*, American Mathematical Society, Providence, 2000.

[8] J. B. Conway and M. Ptak, *The harmonic functional calculus and hyperreflexivity*, Pacific J. Math., **204** (2002), 19–29.

[9] K. Davidson, *The distance to the analytic Toeplitz operators*, Illinois J. Math. **31** (1987), 265–273.

[10] J. Eschmeier, *On C_{00}–representations of $H^\infty(G)$ with dominating Harte spectrum*, Banach Algebras '97 (Blaubeuren), 135–151, de Gruyter, Berlin 1998.

[11] S. H. Ferguson, *The Nehari problem for the hardy space on the torus*, J. Operator Theory **40** (1998), 309–321.

[12] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, 1967.

[13] K. Kliś, M. Ptak, *Quasinormal operators are hyperreflixive*, Banach Center Publication Institute Mathematics, Polish Academy of Sciences, to appear.

[14] K. Kliś, M. Ptak, *k–hypereflexive subspaces*, Huston J. Math., to appear.

[15] J. Kraus and D. Larson, *Reflexivity and distance formulae*, Proc. London Math. Soc. **53** (1986), 340–356.

[16] W. Loginov and V. Shulman, *Hereditary and intermediate reflexivity of W^* algebras*, Math. USSR-Izv. **9** (1975), 1189–1201.

[17] M. Ptak, *On the reflexivity of the pairs of isometries and of tensor products of some operator algebras*, Studia Math., **83** (1986), 47–53.

[18] M. Ptak, *Reflexivity of Multiplication Operators in Certain Domains in $\mathbb{C}^N$*, Bull. Acad. Pol. Sci., Ser. Math. **37** (1989), 217–220.

[19] M. Ptak, *On the reflexivity of multigenerator algebras*, Dissertationes Mathematicae, CCCLXXVIII, (1998), pp 68.

[20] M. Ptak, *The projection on the space of Toeplitz operators*, Ann. Polon. Math., to appear.

[21] V. Pták, P. Vrbová, *Opeartors of Toeplitz and Hankel type*, Acta Sci. Math. **52** (1988), 117–140.

[22] D. Sarason, *Invariant subspaces and unstarred operator algebras*, Pacific J. Math. **17** (1966), 511–517.

MAREK PTAK, INSTITUTE OF MATHEMATICS, UNIVERSITY OF AGRICULTURE, AL.MICKIEWICZA 24/28, 30-059 KRAKÓW, POLAND

E-mail address: rmptak@cyf-kr.edu.pl