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Lecture Notes on Normal Dilations

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product (x, y), and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$ equipped with the operator norm

$$\|A\| = \sup\{\|Ax\| : x \in \mathbf{C}^n,\ (x,x) = 1\}.$$

If $\mathcal{H}$ is n-dimensional, we identify $\mathcal{H}$ with $\mathbf{C}^n$ and $\mathcal{B}(\mathcal{H})$ with the algebra M_n of $n \times n$ complex matrices. The *numerical range* of an operator $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{(Ax, x) : x \in \mathcal{H},\ (x,x) = 1\}.$$

The spectrum of A is denoted by $\sigma(A)$.

We say that $A \in \mathcal{B}(\mathcal{H})$ has a *dilation* $\tilde{A} \in \mathcal{B}(\tilde{\mathcal{H}})$ if $A = V^*\tilde{A}V$ for some isometry $V : \mathcal{H} \to \tilde{\mathcal{H}}$; equivalently, $\tilde{A}$ is unitarily equivalent to a 2×2 operator-matrix of the form $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$. In such a case, A is called a *compression* of $\tilde{A}$.

Notably, normal dilations arises in structure theory as a sort of *non-commutative spectral decomposition* in terms of *a non-commutative resolution of the identity* . Namely, if A has a normal dilation $\tilde{A}$ with spectral projections $E(\cdot)$, and if $Q(S)$ is defined as the compression of $E(S)$ on the Hilbert space $\mathcal{H}$ for each Borel subset S of the the spectrum $\sigma(\tilde{A})$, then we get

$$I = \int dQ \quad \text{and} \quad A = \int \lambda dQ.$$

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In case that the normal dilation has a finite spectrum $\alpha_1, ...\alpha_n$, we can write the non-commutative resolution of identity in terms of finitely many positive operators Q_j, such that

$$I = \sum_{j=1}^{n} Q_j \quad \text{and} \quad A = \sum_{j=1}^{n} \alpha_j Q_j.$$

This lecture note is organized as follows. In Section 2, we examine normal dilations of finite spectra, with a detailed study of the underpinnings of the Mirman Theorem. In Section 3, we look into the structure of unitary dilations with particular concerns about constrained unitary dilations . Section 4 includes the structure theory of joint spectral circles in connection with simultaneous normal dilations for a pair of operators. In Section 5, we look in the recent result about higher-rank numerical ranges which can be regarded as compression values of operators.

2 Normal dilations with finite spectra

Let K be a convex compact subset of $\mathbf{C}$. It has been a major structure problem to determine whether any operator A can have a normal dilation $\tilde{A}$ whose spectrum is a subset of K. Obviously, we have numerical range inclusion $W(A) \subseteq W(\tilde{A}) \subseteq K$. It is natural to ask whether the inclusion $W(A) \subseteq K$ suffices to infer that A has a normal dilation with spectrum as a subset of K. It turns out the answer is true only for the case K is a triangle (covering the degenerate case when K is a line segment or a point).

If K is a line segment or a single point, then the condition $W(A) \subseteq K$ implies that A is a normal operator whose spectrum is a subset of K. Next theorem of Mirman ([14], see also [15], [6, Proposition 2.3])is a case of great significance in structure theory.

Theorem 2.1 (Mirman). *Let $A \in \mathcal{B}(\mathcal{H})$ and let $\gamma_1, \gamma_2, \gamma_3 \in \mathbf{C}$. The following conditions are equivalent.*

(a) *The numerical range $W(A)$ is included in the triangle with vertices $\gamma_1, \gamma_2, \gamma_3$.*

(b) *$A = V^*(B \otimes I)V$, where $B = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$, I is the identity operator on the Hilbert space $\mathcal{H}$, and $V : \mathcal{H} \to \mathbf{C}^3 \otimes \mathcal{H}$ is an operator satisfying $V^*V = I$.*

The Mirman Theorem is related to a numerical range in the shape of a triangle. The analogous statement for a square is invalid as shown in the following:

Example 2.2 ([6]. Let $A = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$ and let $B = \operatorname{diag}(1, -1, i, -i)$. Then $W(A) \subseteq W(B)$, where $W(A)$ is the circle centered at the origin with radius $\sqrt{2}/2$ and $W(B)$ is the square with four vertices $1, -1, i, -i$. However, A cannot be dilated to an operator of the form $B \otimes I$ as $\|A\| > \|B\|$.

Apparently, there is no easy structure theorem for general normal dilations of finite spectra. The following is sort of converse of the Mirman Theorem showing the best result along these lines.

Proposition 2.3. *Let K be a compact convex subset of $\mathbf{C}$ other than a triangle (or in the degenerate case as a line segment or a point). Then there exists a 2×2 matrix A with its numerical range $W(A) \subseteq K$, but A cannot be dilated to to normal operator $\tilde{A}$ with $\sigma(\tilde{A}) \subseteq K$.*

Idea of Proof. Let Γ be the circle (not necessarily to be centered at the origin) of smallest radius to surround the the compact convex set K. Then we can find an elliptical disk $\mathcal{E}$ included in K but there is no triangle Δ with three vertices on the circle Γ and Δ circumscribes $\mathcal{E}$. Specifically, suppose Γ is the circle centered at μ_0 of radius r; then $\mathcal{E}$ corresponds to a 2×2 matrix A such that $\|A - \mu_0 I\| > r$. Hence A cannot be dilated to any normal operator with its spectrum on the circle Γ. □

3 Unitary dilations

In [10], Halmos showed explicitly that each contraction $A \in \mathcal{B}(\mathcal{H})$ has a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of the form

$$U = \begin{pmatrix} A & \sqrt{1 - AA^*} \\ \sqrt{1 - A^*A} & -A^* \end{pmatrix}.$$

This result has generated a lot of research, including the far reaching Sz.-Nagy dilation theorem [17]: Each contraction $A \in \mathcal{B}(\mathcal{H})$ has a power unitary dilation; i.e., there is a unitary U satisfying

$$U^k = \begin{pmatrix} A^k & * \\ * & * \end{pmatrix}, \qquad k = 1, 2, \ldots.$$

In terms of 3×3 matrix-operator representation, it is possible to get a unitary dilation U in the form $\begin{pmatrix} * & * & * \\ O & A & * \\ O & O & * \end{pmatrix}$ which yields the power dilation $U^k = \begin{pmatrix} * & * & * \\ O & A^k & * \\ O & O & * \end{pmatrix}$ immediately.

In this section, we are particularly concerned about the structure of a contraction $A \in \mathcal{B}(\mathcal{H})$ subject to a constraint $A + A^* \le aI$ for some real a.

Theorem 3.1 (Choi and Li [7]). *Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction such that $A + A^* \le aI$ for some real number a. Then A has a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ satisfying $U + U^* \le aI$. In the case of $\mathcal{H}$ of dimension n, the matrix $U \in M_{2n}$ can be chosen such that its $2n$ eigenvalues are $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_n}$ with $2\cos\theta_j \le a$ for all j (i.e., non-real eigenvalues occur in conjugate pairs, real eigenvalues have even multiplicities).*

Obviously, the case $A + A^* \le aI$ with $a \ge 2$ is automatic while the case $A + A^* \le aI$ with $a < -2$ is vacuous.

Moreover, we can use Theorem 3.1 to get a spectral decomposition for non-normal constrained contractions in terms of "a non-commutative resolution of the identity". Here we state only the finite-dimensional case:

Corollary 3.2 (Choi and Li [7]). *Suppose $A \in M_n$ is a contraction satisfying $A + A^* \le aI_n$ for some real number a. Then there are n real numbers $\theta_1, \ldots, \theta_n \in [0, \pi]$ with $2\cos\theta_j \le a$ for all j, and positive semidefinite rank-1 matrices $Q_1, \ldots, Q_{2n} \in M_n$, such that*

$$I = \sum_{j=1}^{n} (Q_j + Q_{n+j}) \quad \text{and} \quad A = \sum_{j=1}^{n} \left(e^{i\theta_j} Q_j + e^{-i\theta_j} Q_{n+j}\right).$$

The constrained unitary dilation is particularly useful in the study of numerical ranges of operators. In particular, it can be used to affirm the conjecture of Halmos [11] about the closure of numerical range $\overline{W(A)}$.

Theorem 3.3 (Chi and Li [7, Theorem 2.4]). *Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction. Then*

$$\overline{W(A)} = \bigcap\{\overline{W(U)} : U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } A\}.$$

(In the finite-dimensional case, the closure signs on the numerical ranges can be omitted.)

Idea of Proof. we consider any particular $\zeta \notin \overline{W(A)}$. Since $\overline{W(A)}$ is a compact convex set, there exists $\theta \in [0, 2\pi)$ and $\mu \in \mathbf{R}$ such that $e^{i\theta}\zeta + e^{-i\theta}\bar{\zeta} > \mu$, while $e^{i\theta}\overline{W(A)} = \overline{W(e^{i\theta}A)}$ is included in the closed half plane $\{z \in \mathbf{C} : z + \bar{z} \leq \mu\}$. By Theorem 3.1, there is a unitary dilation U of A such that $e^{i\theta}U + e^{-i\theta}U^* \leq \mu I_{2n}$. Hence $e^{i\theta}\zeta \notin \overline{W(e^{i\theta}U)}$ and $\zeta \notin \overline{W(U)}$. □

4 Joint spectral circles and unitary equivalence orbits

For $\mu \in \mathbf{C}$ and $r \geq 0$, we write $\Gamma(\mu; r) = \{z \in \mathbf{C} : |z - \mu| = r\}$ for the circle centered at μ with radius r. (When $r = 0$, the degenerated circle is the singleton $\{\mu\}$.) Notably, each single operator $A \in \mathcal{B}(\mathcal{H})$ is associated with a canonical circle $\Gamma(\mu_0; r)$ as follows:

Lemma 4.1 (Choi and Li [9]). *For each operator $A \in \mathcal{B}(\mathcal{H})$, there is a unique pair of $(\mu_0, r_0) \in \mathbf{C} \times [0, \infty)$ so that $r_0 = \|A - \mu_0 I\| \leq \|A - \mu I\|$ for every $\mu \in \mathbf{C}$.*

Proof. Assume that the inequality above is true for $\mu_0 = \mu_1$ and μ_{-1}. Then for $\tilde{\mu} = (\mu_1 + \mu_{-1})/2$, we have

$$2\|A - \tilde{\mu}I\|^2 \geq \|A - \mu_1 I\|^2 + \|A - \mu_{-1}I\|^2 \geq \|(A - \mu_1 I)^*(A - \mu_1 I) + (A - \mu_{-1}I)^*(A - \mu_{-1}I)\|$$

$$= \|2(A - \tilde{\mu}I)^*(A - \tilde{\mu}I) + \frac{|\mu_1 - \mu_{-1}|^2}{2} I\| = 2\|A - \tilde{\mu}I\|^2 + |\mu_1 - \mu_{-1}|^2/2;$$

it follows that $\mu_1 = \mu_{-1}$ as desired. □

Remark 4.2. If A is a normal operator, the canonical optimal circle $\Gamma(\mu_0; r_0)$ as determined in Lemma 4.1 is the circle, with minimum radius, enclosing the spectrum of A; i.e.,

$$r_0 = \min_{\mu \in \mathbf{C}} \max\{|\alpha - \mu| : \alpha \in \sigma(A)\} = \max\{|\alpha - \mu_0| : \alpha \in \sigma(A)\}.$$

In other words, the optimal circle is the unique circle Γ to enclose the whole spectrum $\sigma(A)$ subject to the additional condition:

(**)$\Gamma \cap \sigma(A)$ *is a nonempty set whose convex hull contains the center of* Γ.

Specifically, if A is a normal operator with a finite spectrum, then we will need only to consider finitely many circles Γ arising from any one of the following two types:

(1) Each pair of two points of $\sigma(A)$ determine the diameter of a circle Γ.

(2) Each acute-angle triangle with three vertices from $\sigma(A)$ determines a circle Γ passing through the vertices.

Among all circles of these two types, the optimal circle is the only circle to enclose the whole spectrum $\sigma(A)$.

To see the full significance of the spectral circles, we need the optimal normal dilations. Recall that every contraction in $\mathcal{B}(\mathcal{H})$ has a unitary dilation. Applying affine transformations, we see that if $A \in \mathcal{B}(\mathcal{H})$, $\mu \in \mathbf{C}$ and $r \geq 0$ satisfy $\|A - \mu I\| \leq r$, then A has a normal dilation $\tilde{A}$ such that $\sigma(\tilde{A}) \subseteq \Gamma(\mu; r)$.

Proposition 4.3 (Choi and Li [9, Section 3.2]). *Suppose $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} & \sup\{\|A - U^*AU\| : U \text{ is unitary }\} \\ = \; & \min \sup\{\|\tilde{A} - \tilde{U}^*\tilde{A}\tilde{U}\| : \tilde{U} \text{ is unitary in } \mathcal{B}(\mathcal{H}\oplus\mathcal{H})\}, \end{aligned}$$

where min *is taken over all possible normal dilations $\tilde{A}$ of A acting on the larger Hilbert space $\mathcal{H}\oplus\mathcal{H}$. Moreover, let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| \leq \|A - \mu I\| \quad \text{for every } \mu \in \mathbf{C},$$

and let $r_0 = \|A - \mu_0 I\|$. Then each normal dilation $\tilde{A}$ of A so that $\sigma(\tilde{A}) \subseteq \Gamma(\mu_0; r_0)$ satisfies

$$2r_0 = \sup\{\|A - U^*AU\| : U \text{ is unitary}\} = \sup\{\|\tilde{A} - \tilde{U}^*\tilde{A}\tilde{U}\| : \tilde{U} \text{ is unitary}\}.$$

Without referring to normal dilations, we can re-state Prop. 4.3 as follows:

Theorem 4.4 (Choi and Li [9, Section 3.2]). *Let $A \in B(H)$, and let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| \leq \|A - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Set $r_0 = \|A - \mu_0 I\|$. Then

$$2r_0 = \sup\{\|A - U^*AU\| : U \text{ unitary}\}$$

and

$$\|f(A) + U^*g(A)U\| \leq \max_{z\in\Gamma(\mu_0;r_0)} |f(z)| + \max_{z\in\Gamma(\mu_0;r_0)} |g(z)|$$

for each unitary U and each pair of polynomials $f(z)$ and $g(z)$.

Note that the equality $2r_0 = \sup\{\|A - U^*AU\| : U \text{ unitary}\}$ can be viewed as the optimal case of the last inequality with $f(z) = z - \mu_0$ and $g(z) = \mu_0 - z$.

We can further extend the above discussion to two operators $A, B \in \mathcal{B}(\mathcal{H})$ and obtain the following theorem concerning their joint spectral circles in connection with the distance between their unitary similarity orbits.

Theorem 4.5 (Choi and Li [9, Theorem 3.3]). *Let $A, B \in \mathcal{B}(\mathcal{H})$, and let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \leq \|A - \mu I\| + \|B - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Set $r_1 = \|A - \mu_0 I\|$ and $r_2 = \|B - \mu_0 I\|$. Then

$$\sup\{\|A - U^*BU\| : U \text{ unitary}\} = r_1 + r_2 \tag{4.1}$$

and

$$\|f(A)+U^*g(B)U\| \leq \max_{z\in\Gamma(\mu_0;r_1)} |f(z)| + \max_{z\in\Gamma(\mu_0;r_2)} |g(z)| \tag{4.2}$$

for each unitary U and each pair of polynomials $f(z)$ and $g(z)$.

Note that (4.1) can be viewed as the equality case of (4.2) for $f(z)=z-\mu_0$ and $g(z)=\mu_0-z$.

The next proposition gives a description for the set of complex numbers μ_0 in the statement of Theorem 4.5.

Proposition 4.6 (Choi and Li [9, Prop.3.4]). *Let $A, B \in \mathcal{B}(\mathcal{H})$, and let $S(A,B)$ be the set of complex numbers μ_0 satisfying*

$$\|A-\mu_0 I\| + \|B-\mu_0 I\| \leq \|A-\mu I\| + \|B-\mu I\| \quad \textit{for all } \mu \in \mathbf{C}.$$

Then $S(A,B)$ is a compact convex set which is either a singleton or a line segment.

Remark 4.7. In the case of normal operators A and B, the evaluation of $S(A,B)$ is much related to the geometrical positions of $\sigma(A)$ and $\sigma(B)$. In particular, writing $A=A_1+iA_2$, we can estimate the norm of A by means of the joint optimal spectral circles for the pair (A_1, iA_2). The following may be the most important result in the structure theory of operator norm computation.

Theorem 4.8 (Choi and Li [8, Theorem 2.1]). *Suppose A and B are self-adjoint operators subject to $a_1 I \leq A \leq a_2 I$ and $b_1 I \leq B \leq b_2 I$. Assume further that $a_2 \geq |a_1|$ and $b_2 \geq |b_1|$.*

(i) *If $a_1 b_2 + a_2 b_1 \geq 0$, then*

$$\|A+iB\| \leq |a_2+ib_2| = \sqrt{a_2^2+b_2^2}.$$

(ii) *If $a_1 b_2 + a_2 b_1 \leq 0$, then*

$$\|A+iB\| \leq \tau+\tau',$$

where

$$\tau = |a_1-z_0| = |a_2-z_0| = \frac{1}{2}\sqrt{(a_1-a_2)^2+(b_1+b_2)^2}$$

and

$$\tau' = |ib_1-z_0| = |ib_2-z_0| = \frac{1}{2}\sqrt{(a_1+a_2)^2+(b_1-b_2)^2}$$

with $z_0 = \{(a_1+a_2)+i(b_1+b_2)\}/2$.

(iii) *The bounds in* (i) *and* (ii) *are sharp in the following sense: If $\{a_1,a_2\}\subseteq\sigma(A)$ and $\{b_1,b_2\}\subseteq\sigma(B)$, then there exists a unitary U such that $\|A+iU^*BU\|$ attains the upper bound.*

The case of the summation of two unitary operators is also worthy of special mention:

Theorem 4.9 (Choi and Li [8, Theorem 3.2]). *Let U and V be unitary operators. If their spectra $\sigma(U)$ and $\sigma(V)$ can be separated by a straight line, then*

$$\|U+V\| \leq \max\{|u+v| : u\in\sigma(U) \textit{ and } v\in\sigma(V)\}.$$

Otherwise, $||U+V|| \leq 2$*. Moreover, the inequalities are sharp as* $\sup\{||U+W^*VW|| : W \text{ is unitary}\}$ *is equal to the right-hand side in each case.*

Next, we replace the unitary V by $-V$ to get another statement from a different angle of view:

Corollary 4.10 (Choi and Li [8, Corollary 3.5]). *Suppose U and V are unitary operators. Then there is a sharp inequality:*

$$||U - V|| \leq \max\{|u - v| : u \in \sigma(U) \text{ and } v \in \sigma(V)\},$$

if the right hand side $< \sqrt{2}$ *and*

$$||U - V|| \leq 2, \text{ otherwise.}$$

Remark 4.11. The inequality in Corollary 4.10 is sharp in the following sense: Let n be a fixed integer larger than 2 and let $r \in [0, 2]$ and let $\Phi(r) = max\{||U - V||\}$ where U and V run through all pairs of $n \times n$ unitary matrices satisfying $|u - v| \leq r$ for $u \in \sigma(U)$ and $v \in \sigma(V)$. Then

$$\Phi(r) = r \text{ if } r \in [0, \sqrt{2})$$

and

$$\Phi(r) = 2 \text{ if } r \in [\sqrt{2}, 2].$$

Thus we get a quantitative description about the norm change with respect to the spectral variation. Actually, it is a folklore fact that $\Phi(r)$ is continuous at $r = 0$. Namely, if all eigenvalues of a unitary matrix U is near to a single complex number, then U is near to a scaler matrix. It may be surprising that $\Phi(r)$ remains to be continuous when r is not too small and then there is an astonishing jump discontinuity only at $r = \sqrt{2}$.

Finally, we establish the full generalization of Proposition 4.3 in a two-variable version. Suppose $\tilde{A}$ and $\tilde{B}$ are normal dilations of A and B. We have

$$\sup\{\|U^*AU - V^*BV\| : U, V \text{ unitary}\} \leq \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U}, \tilde{V} \text{ unitary}\};$$

i.e., the distance between the unitary orbits of A and B is not larger than that of their normal dilations. Nevertheless, the following theorem shows that there always exist appropriate normal dilations whose unitary orbits are not farther apart.

Proposition 4.12 (Choi and Li [9, Prop. 3.5]). *Suppose $A, B \in B(\mathcal{H})$. Then*

$$\begin{aligned} & \sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\ = \ & \min \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U} \text{ and } \tilde{V} \text{ are unitaries}\}, \end{aligned}$$

where min *is taken over all possible normal dilations $\tilde{A}$ and $\tilde{B}$ of A and B on the common Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Moreover, let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \leq \|A - \mu I\| + \|B - \mu I\| \quad \text{for every } \mu \in \mathbf{C},$$

$r_1 = \|A - \mu_0 I\|$*, and* $r_2 = \|B - \mu_0 I\|$*. Then the set*

$$\mathcal{C} = \{(\tilde{A}, \tilde{B}) : \tilde{A} \text{ and } \tilde{B} \text{ are normal dilations of } A \text{ and } B \text{ on a common}$$

Hilbert space $\mathcal{H} \oplus \mathcal{H}$ with $\sigma(\tilde{A}) \subseteq \Gamma(\mu_0; r_1)$ and $\sigma(\tilde{B}) \subseteq \Gamma(\mu_0; r_2)\}$

is non-empty, and every pair $(\tilde{A}, \tilde{B}) \in \mathcal{C}$ satisfies

$$r_1 + r_2 = \sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\}$$
$$= \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U} \text{ and } \tilde{V} \text{ are unitaries in } \mathcal{B}(\mathcal{H} \oplus \mathcal{H})\}.$$

5 Higher-rank numerical ranges

In this section, we initiate the study of higher-rank versions of the standard numerical range for matrices. A primary motivation arises through the basic problem of error correction in quantum computing. Specifically, the development of theoretical and ultimately experimental techniques to overcome the errors associated with quantum operations is central to continued advances in quantum computing. (See, [5].)

For each positive integer k, the *rank-k numerical range* of an operator T is a subset of the complex plane defined by

$$\Lambda_k(T) = \{\lambda \in \mathbf{C} : PTP = \lambda P \text{ for some rank}-k \text{ projection } P\}.$$

Actually, the elements of $\Lambda_k(T)$ are sort of "compression-values" for T, since $\lambda \in \Lambda_k(T)$ if and only if the $k \times k$ scalar matrix λI_k is the compression of T to a k-dimensional subspace. This means that T is unitarily equivalent to a 2×2 block matrix of the form
$T = \begin{pmatrix} \lambda I_k & A \\ B & C \end{pmatrix}$. Equivalently, T is a "dilation" of the scalar matrix λI_k, or, $T - \lambda I$ maps a k-dimensional subspace into its orthogonal complement. Note that, in particular, when N is an $n \times n$ normal matrix with eigenvalues $\{\alpha_j : j = 1, ...n\}$ (including multiplicity), then a complex number $\lambda \in \Lambda_k(N)$ iff there exist $k \times k$ rank-1 positive semi-definite matrices Q_j, such that

$$I_k = \sum_{j=1}^{n} Q_j \quad \text{and} \quad \lambda I_k = \sum_{j=1}^{n} \alpha_j Q_j.$$

The following set inclusions may be readily verified for any operator T:

(i) $W(T) = \Lambda_1(T) \supseteq \Lambda_2(T) \supseteq \ldots \supseteq \Lambda_N(T)$.

(ii) $\Lambda_k(T) = \Lambda_k(W^*TW)$ for all unitary W.

(iii) $\Lambda_k(\alpha T + \beta I) = \alpha\Lambda_k(T) + \beta \quad \forall \beta \in \mathbf{C}$ and nonzero $\alpha \in \mathbf{C}$.

(iv) $\Lambda_k(T) = \overline{\Lambda_k(T^*)} = \Lambda_k(T^{transpose})$.

(v) $\Lambda_k(T) \subseteq \Lambda_k(\operatorname{Re} T) + i\,\Lambda_k(\operatorname{Im} T)$.

(vi) $\Lambda_{k_1+k_2}(T_1 \oplus T_2) \supseteq \Lambda_{k_1}(T_1) \cap \Lambda_{k_2}(T_2)$.

(vii) $\Lambda_k(T_1 \oplus T_2) \supseteq \{t\lambda_1 + (1-t)\lambda_2 : t \in [0,1], \lambda_j \in \Lambda_k(T_j) \text{ for } j = 1, 2\}$.

It is rather easy to describe the numerical ranges of very high ranks as follows:

Proposition 5.1 ([4, Proposition 2.2]). *Let T be an $n \times n$ matrix and suppose that $2k > n$. Then the rank-k numerical range $\Lambda_k(T)$ is an empty set or a singleton set. If $\Lambda_k(T) = \{\lambda_0\}$ is a singleton set with $2k > n$, then λ_0 is an eigenvalue of geometric multiplicity at least $2k - n$. In particular, $\Lambda_n(T)$ is non-empty if and only if T is a scalar matrix.*

In the normal case, this leads to more detailed information for large values of k.

Corollary 5.2 ([4]). *Let T be an $n \times n$ normal matrix and suppose that $2k > n$. Then the rank-k numerical range $\Lambda_k(T)$ is an empty set or a singleton set. In fact, the case $\Lambda_k(T) = \{\lambda_0\}$ occurs if and only if there is a $(2n-2k) \times (2n-2k)$ matrix T_0 such that*

$$T = \lambda_0 I_{2k-n} \oplus T_0,$$

and λ_0 belongs to $\Lambda_{n-k}(T_0)$.

It is a simple matter to derive a general description of $\Lambda_k(T)$ in the Hermitian case for arbitrary k as follows:

Theorem 5.3 (Choi, Kribs, and Zyczkowski [4, Theorem 2.4]). *Let A be an $n \times n$ Hermitian matrix with eigenvalues (counting multiplicities) given by $a_n \geq \ldots \geq a_2 \geq a_1$. Then $\Lambda_k(A) = [a_k, a_{n+1-k}]$.*

Note that in the theorem above, the interval $[a, b]$ is regarded as an empty set if $a > b$, while $[a, a]$ is the singleton set $\{a\}$.

We finish this section by discussing the case of normal matrices. First note that item (v) above and Theorem 5.3 give a crude containment result for $\Lambda_k(T)$ for arbitrary T. Indeed, $\Lambda_k(T)$ is a subset of the rectangular region in the complex plane $\{\alpha + i\beta : \alpha \in \Lambda_k(\mathrm{Re}(T)), \beta \in \Lambda_k(\mathrm{Im}(T))\}$. We can do better to obtain a more refined containment in the normal case. The following result follows from the proof of the previous theorem.

Corollary 5.4 ([4]). *Let N be an $n \times n$ normal matrix and let k be a fixed positive integer. Then the k-th numerical range*

$$\Lambda_k(N) \subseteq \cap_\Gamma(\mathrm{co}\,\Gamma),$$

where co *stands for the convex hull and Γ runs through all $(n+1-k)$-point subsets (counting multiplicities) of the spectrum of N.*

Most likely, the inclusion sign of Corollary 5.4 can be changed to equality, but we don't have any proof at present.

Conjecture 5.5. *If N is an $n \times n$ normal matrix, then its rank-k numerical range $\Lambda_k(N)$ coincides with the intersection of the convex hulls $\mathrm{co}\,\Gamma$, where Γ runs through all $(N+1-k)$-point subsets (counting multiplicities) of the spectrum of N.*

In conjunction with Conjecture 5.5, the following is outstanding:

Conjecture 5.6. *For any operator T, the rank-k numerical range $\Lambda_k(T)$ is a convex set.*

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