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Indecomposable representations of quivers on infinite dimensional Hilbert spaces

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We shall show that the theory of relative position of *several subspaces* of a Hilbert space is rich as subfactor theory.

We should study an *indecomposable* system of n subspaces in the sense that the system can not be isomorphic to a direct sum of two non-zero systems.

Many problems of linear algebra can be reduced to the classification of the systems of subpaces in a finite-dimensional vector space. In a finite-dimensional space, the classification of indecomposable systems of n subspaces for $n = 1, 2$ and 3 was simple. Jordan blocks give indecomposable systems of 4 subspaces. But there exist many other kinds of indecomposable systems of 4 subspaces. Therefore it was surprising that Gelfand and Ponomarev gave a complete classification of indecomposable systems of four subspaces in a finite-dimensional space over an algebraically closed field.

We study relative position of n subspaces in a separable infinite-dimensional Hilbert space.

Let H be a Hilbert space and $E_1, \ldots E_n$ be n subspaces in H. Then we say that $\mathcal{S} = (H; E_1, \ldots, E_n)$ is a system of n subspaces in H or a n-subspace system in H. A system $\mathcal{S}$ is called indecomposable if $\mathcal{S}$ can not be decomposed into a nontrivial direct sum.

For any bounded linear operator A on a Hilbert space K, we can associate a system $\mathcal{S}_A$ of four subspaces in $H = K \oplus K$ by

$$\mathcal{S}_A = (H; K \oplus 0, 0 \oplus K, \{(x, Ax); x \in K\}, \{(x, x); x \in K\}).$$

Two such systems $\mathcal{S}_A$ and $\mathcal{S}_B$ are isomorphic if and only if the two operators A and B are similar. The direct sum of such systems corresponds to the direct sum of the operators. In this sense the theory of operators is included into the theory of relative positions of four subspaces.

In particular on a finite dimesional space, Jordan blocks correspond to indecomposable systems. Moreover on an infinite dimensional Hilbert space, the above system $\mathcal{S}_A$ is indecomposable if and only if A is strongly irreducible, which is an infinite-dimensional analog of a Jordan block. Therefore there exist uncountably many indecomposable systems of four subspaces.

But it is rather difficult to know whether there exists another kind of indecomposable system of four subspaces. One of the main result of the paper[EW] is to give uncountably many, exotic, indecomposable systems of four subspaces on an infinite-dimensional separable Hilbert space.

Gelfand and Ponomarev introduced an integer valued invariant $\rho(\mathcal{S})$, called *defect*, for a system $\mathcal{S} = (H;E_1,E_2,E_3,E_4)$ of four subspaces by

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} \dim E_i - 2\dim H.$$

We extend the defect to a certain class of systems of four subspaces on an infinite dimesional Hilbert space using Fredholm index.

If a system $\mathcal{S} = (H;E_1,E_2,E_3,E_4)$ of four subspaces is finite-dimensional, then the defect $\rho(\mathcal{S})$ is an integer. Gelfand and Ponomarev showed that the possible value of defect $\rho(\mathcal{S})$ is exactly in $\{-2,-1,0,1,2\}$. We show that the set of values of defect for indecomposable systems of four subspaces in an infinite-dimesional Hilbert spaces is exactly $\{\frac{n}{3}; n \in \mathbb{Z}\}$.

In finite dimensional case, the classification of four subspaces is described as the classification of the representations of the extended Dynkin diagram $\widetilde{D_4}$. Recall that Gabriel listed Dynkin diagrams A_n, D_n, E_6, E_7, E_8 in his theory on finiteness of indecomposable representations of quivers. We will discuss on indecomposable representations of quivers on infinite-dimensinal Hilbert spaces . We shall show the following. Let Γ be a finite connected undirected graph. If there exist no infinite dimensional(Hilbert space) indecomposable representations of Γ with any orientations, then Γ is one of A_n, D_n, E_6, E_7, E_8.

Now we shall explain the framework of systems of subspaces.

Definition. Let H be a Hilbert space and $E_1,\dots,E_n$ be a closed subspaces in H. The ordered system $S = (H;E_1,\dots,E_n)$ is called an n-subspace system.

Definition. Let $S = (H;E_1,\dots,E_n)$ and $T = (K;F_1,\dots,F_n)$ be n-subspace systems. Then we say that S and T are ***isomorphic***($S \simeq T$) if there exists an invertible operator $\varphi \in B(H,K)$ such that $\varphi(E_i) = F_i (i = 1,\dots,n)$.

Definition. Let $S = (H;E_1,\dots,E_n)$ and $T = (K;F_1,\dots,F_n)$ be n-subspace systems. Define the direct sum $S \oplus T$ by $S \oplus T = (H \oplus K; E_1 \oplus F_1,\dots,E_n \oplus F_n)$.

Definition. Define $S = 0$ by $H = 0$.

Definition. We say that S is indecomposable if, for n -subspace systems S_1 and S_2, $S \simeq S_1 \oplus S_2$, then $S_1 \simeq 0$ or $S_2 \simeq 0$.

In this framework, for $S, T \in B(K)$ we have that $S \simeq T$(similar) if and only if $S_S \simeq S_T$(isomorphic).

In this framework, Gelfand Ponomarev completely classified indecomposable n-subspace systems ($n = 1,2,3,4$) in a finite dimensional vector spaces.

The extension of Gelfand Ponomarev results (1970) to the infinite dimensional case is

considered in [EW]M.Enomoto,Y.Watatani,Relative position of four subspaces in a Hilbert space, Advanced Math.,in press.

Now we shall describe the Gelfand Ponomarev results .

[1]the Gelfand Ponomarev results

For $n = 1$, that is, one subspace case. Then the underlyng space is only one dimensional space. Thus we have $H = \mathbb{C}, E_1 = \mathbb{C}$ or 0.

For $n = 2$, that is, two subspaces case. Then the underlyng space is only one dimensional space. Thus we have $H = \mathbb{C}, E_i = \mathbb{C}$ or 0. $(i = 1,2)$.

For $n = 3$, that is, three subspaces case. Then the case in which the underlyng space is a one dimensional space occurs. That is, we have $H = \mathbb{C}, E_i = \mathbb{C}$ or 0. $(i = 1,2,3)$.

But the other case occurs. That is, $H = \mathbb{C}^2, E_1 = \mathbb{C}\binom{1}{0}, E_2 = \mathbb{C}\binom{0}{1}, E_3 = \mathbb{C}\binom{1}{1}$.

For $n = 4$, that is, four subspaces case. In this case they have many interesing cases as follows. The classification is carried by the invariant

defect$\rho(S) = \sum_{i=1}^{4} \dim(E_i) - 2\dim H$.

They showed thatthe set of the possible values of defects is $\{0, \pm 1, \pm 2\}$.

At first, we mention the case(A) ,the dimension of the whole space is even.

(A)$\dim H = 2k$ is even for some integer $k \geq 0$

Let H be a space with a basis $\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$.

(1)$(H; E_1, E_2, E_3, E_4)$ with $\rho(S) = -1$

$$\begin{aligned} H &= [e_1, \ldots, e_k, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [(e_2 + f_1), \ldots, (e_k + f_{k-1})], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k)]. \end{aligned}$$

(2)$(H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 1$

$$\begin{aligned} H &= [e_1, \ldots, e_k, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [e_1, (e_2 + f_1), \ldots, (e_k + f_{k-1}), f_k], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k)]. \end{aligned}$$

(3)$(H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 0$

$$\begin{aligned} H &= [e_1, \ldots, e_k, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [e_1, (e_2 + f_1), \ldots, (e_k + f_{k-1})], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k)]. \end{aligned}$$

(4)$(H; E_1, E_2, E_3, E_4)$ with $\rho(S) = 0$

$$\begin{aligned} H &= [e_1, \ldots, e_k, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [(e_1 + \lambda f_1), (e_2 + f_1 + \lambda f_2), \ldots, (e_k + f_{k-1} + \lambda f_k)], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k)]. \end{aligned}$$

At first, we mention the case(B) ,the dimension of the whole space is odd.

(B) $\dim H = 2k + 1$ is odd for some integer $k \geq 0$.

Let H be a space with a basis $\{e_1, \ldots, e_k, e_{k+1}, f_1, \ldots, f_k\}$.

(5)$(H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = -1$

$$\begin{aligned} H &= [e_1, \ldots, e_k, e_{k+1}, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k, e_{k+1}],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [(e_2 + f_1), \ldots, (e_{k+1} + f_k)], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k)]. \end{aligned}$$

(6)$(H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 1$

$$\begin{aligned} H &= [e_1, \ldots, e_k, e_{k+1}, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k, e_{k+1}],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [e_1, (e_2 + f_1), \ldots, (e_{k+1} + f_k)], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k), e_{k+1}]. \end{aligned}$$

(7)$(H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 0$

$$\begin{aligned} H &= [e_1, \ldots, e_k, e_{k+1}, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k, e_{k+1}],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [e_1, (e_2 + f_1), \ldots, (e_{k+1} + f_k)], \\ E_4 &= [(e_1 + f_1), \ldots, (e_k + f_k)]. \end{aligned}$$

(8)$(H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = -2$

$$\begin{aligned} H &= [e_1, \ldots, e_k, e_{k+1}, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k],\ E_2 = [f_1, \ldots, f_k], \\ E_3 &= [(e_2 + f_1), \ldots, (e_{k+1} + f_k)], \\ E_4 &= [(e_1 + f_2), \ldots, (e_{k-1} + f_k), (e_k + e_{k+1})]. \end{aligned}$$

(9)$(H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 2$

$$\begin{aligned} H &= [e_1, \ldots, e_k, e_{k+1}, f_1, \ldots, f_k], \\ E_1 &= [e_1, \ldots, e_k, e_{k+1}],\ E_2 = [f_1, \ldots, f_k, e_{k+1}], \\ E_3 &= [e_1, (e_2 + f_1), \ldots, (e_{k+1} + f_k)], \\ E_4 &= [f_1, (e_1 + f_2), \ldots, (e_{k-1} + f_k), (e_k + e_{k+1})]. \end{aligned}$$

[2]Some of our results[EW].

We consider the Gelfand Ponomarev results in an infinite dimensional Hilbert space setting.

At first we can construct an uncountable family of indecomposable systems of four subspaces.

Example.(an uncountable family of indecomposable systems of four subspaces) Let $K = \ell^2(\mathbb{N})$ and $H = K \oplus K$. Consider a unilateral shift $S : K \to K$. For a parameter $\alpha \in \mathbb{C}$, let $E_1 = K \oplus 0, E_2 = 0 \oplus K, E_3 = \{(x, (S + \alpha I)x) | x \in K\}$ and $E_4 = \{(x, x) | x \in K\}$. Then the system $\mathcal{S}_\alpha = (H; E_1, E_2, E_3, E_4)$ of four subspaces are indecomposable. If $\alpha \neq \beta$, then $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are not isomorphic, because the spectra $\sigma(S + \alpha) \neq \sigma(S + \beta)$ and $S + \alpha I$ and $S + \beta I$ are not similar. Thus we can easily construct an uncountable family $(\mathcal{S}_\alpha)_\alpha, (\alpha \in \mathbb{C})$of indecomposable systems of four subspaces.

This example extended to the unbounded case as follows.

Definition.(closed operator systems) We say that a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is a *closed operator system* if there exist Hilbert spaces K_1, K_2 and closed operators $T : K_1 \supset D(T) \to K_2$, $S : K_2 \supset D(S) \to K_1$ such that $H = K_1 \oplus K_2$ and $E_1 = K_1 \oplus 0$,

$$E_2 = 0 \oplus K_2,\ E_3 = \{(x, Tx); x \in D(T)\}, E_4 = \{(Sy, y); y \in D(S)\}.$$

We can show exotic example which does not come from closed operator systems.

Exotic examples. Let $L = \ell^2(\mathbb{N})$ with a standard basis $\{e_1, e_2, \ldots\}$. Put $K = L \oplus L$ and $H = K \oplus K = L \oplus L \oplus L \oplus L$. Consider a unilateral shift $S : L \to L$ by $Se_n = e_{n+1}$ for $n = 1, 2, \ldots$. For a fixed paramater $\gamma \in \mathbb{C}$ with $|\gamma| \geq 1$, we consider an operator

$$T_\gamma = \begin{pmatrix} \gamma S^* & I \\ 0 & S \end{pmatrix} \in B(K) = B(L \oplus L).$$

Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$,
$E_3 = \{(x, T_\gamma x) \in K \oplus K; x \in K\} + \mathbb{C}(0,0,0,e_1)$ =graph$T_\gamma + \mathbb{C}(0,0,0,e_1)$, and
$E_4 = \{(x,x) \in K \oplus K; x \in K\}$. Consider a system $\mathcal{S}_\gamma = (H; E_1, E_2, E_3, E_4)$. We shall show that $\mathcal{S}_\gamma$ is indecomposable. If $|\gamma| > 1$, then $\mathcal{S}_\gamma$ is not isomorphic to any closed operator systems under any permutation.

We shall extend their notion of defect for a certain class of systems relating with Fredholm index.

Definition Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. For any distinct $i,j = 1,2,3,4$, define an adding operator$A_{ij} : E_i \oplus E_j \ni (x,y) \to x + y \in H$. Then $\mathrm{Ker}A_{ij} = \{(x,-x) \in E_i \oplus E_j; x \in E_i \cap E_j\}$ and $\mathrm{Im}A_{ij} = E_i + E_j$.

We say $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is a *Fredholm* system if A_{ij} is a Fredholm operator for any $i,j = 1,2,3,4$ with $i \neq j$.

Definition We say $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is a *quasi-Fredholm* system if $E_i \cap E_j$ and $(E_i + E_j)^\perp$ are finite-dimensional for any $i \neq j$. In the case we define the *defect* $\rho(\mathcal{S})$ of $\mathcal{S}$ by

$$\rho(\mathcal{S}) := \frac{1}{3}\sum_{1 \leq i < j \leq 4}(\dim(E_i \cap E_j) - \dim(E_i + E_j)^\perp))$$

which coincides with the Gelfand-Ponomarev original defect if H is finite-dimensional.

Example(a fractional value). Let S be a unilateral shift on $K = \ell^2(\mathbb{N})$. Then the operator system $\mathcal{S}_S$ is an indecomposable. It is not a Fredholm system but a quasi-Fredholm system and $\rho(\mathcal{S}_S) = -\frac{1}{3}$. The operator system $\mathcal{S}_{S+\frac{1}{2}I}$ is a Fredholm system and $\rho(\mathcal{S}_{S+\frac{1}{2}I}) = -\frac{2}{3}$. Moreover $(\mathcal{S}_{T+\alpha I})_\alpha$ ($\alpha \in \mathbb{C}$)is uncountable family of indecomposable , quasi-Fredholm systems. Fredholm systems among them and their defect are given by

$$\rho(\mathcal{S}_{S+\alpha I}) = \begin{cases} -\frac{2}{3}, & (|\alpha|< 1 \text{ and } |\alpha - 1|< 1) \\ -\frac{1}{3}, & (|\alpha|< 1 \text{ and } |\alpha - 1|> 1) \text{ or } (|\alpha|> 1 \text{ and } |\alpha - 1|< 1) \\ 0, & (|\alpha|> 1 \text{ and } |\alpha - 1|> 1). \end{cases}$$

Example. For $\gamma \in \mathbb{C}$ with $|\gamma|\geq 1$, let $\mathcal{S}_\gamma = (H;E_1,E_2,E_3,E_4)$ be an exotic system of four subspaces in Theorem as above. Then $\mathcal{S}_\gamma$ is a quasi-Fredholm system and

$$\rho(\mathcal{S}_\gamma) = 1.$$

Theorem. The set of the possible values of the defect of indecomposable systems of four subspaces is exactly $\mathbb{Z}/3$.

Theorem. For any $n \in \mathbb{Z}$ there exist uncountable family of indecomposable systems $\mathcal{S}$ of four subspaces with the same defect $\rho(\mathcal{S}) = \frac{n}{3}$.

About exotic indecomposable systems of four subspaces, we have the following:

Theorem. There exists uncountable family of exotic indecomposable systems $\mathcal{S}$ of four subspaces with the defect $\frac{2n+1}{3}(n \in \mathbb{N})$.

Next we can generalize Coxeter functors in an infinite dimensional Hilbert space. case.

Definition.(Coxeter functor Φ^+) Let $\mathcal{S} = (H;E_1,\dots,E_n)$ be a system of n subspaces in a Hilbert space H. Let $R := \oplus_{i=1}^n E_i$ and

$$\tau : R \ni x = (x_1,\dots,x_n) \mapsto \tau(x) = \sum_{i=1}^n x_i \in H.$$

Define $\mathcal{S}^+ = (H^+;E_1^+,\dots,E_n^+)$ by

$$H^+ := \mathrm{Ker}\tau \text{ and } E_k^+ := \{(x_1,\dots,x_n) \in H^+; x_k = 0\}.$$

Let $\mathcal{T} = (K;F_1,\dots,F_n)$ be another system of n subspaces in a Hilbert space K and $\varphi : \mathcal{S} \to \mathcal{T}$ be a homomorphism. Since $\varphi : H \to K$ is a bounded linear operator with $\varphi(E_i) \subset F_i$, we can define a bounded linear operator $\varphi^+ : H^+ \to K^+$ by $\varphi^+(x_1,\dots,x_n) = (\varphi(x_1),\dots,\varphi(x_n))$. Since $\varphi^+(E_i^+) \subset F_i^+$, φ^+ define a homomorphism $\varphi^+ : \mathcal{S}^+ \to \mathcal{T}^+$. Thus we can introduce a covariant functor $\Phi^+ : \mathcal{S}ys^n \to \mathcal{S}ys^n$ by

$$\Phi^+(\mathcal{S}) = \mathcal{S}^+ \text{ and } \Phi^+(\varphi) = \varphi^+.$$

Definition.(Coxeter functor Φ^-) Let $\mathcal{S} = (H;E_1,\dots,E_n)$ be a system of n subspaces in a Hilbert space H. Let $e_i^\perp \in B(H)$ be the projection onto $E_i^\perp \subset H$. Let $Q := \oplus_{i=1}^n E_i^\perp$ and

$$\mu : H \ni x \mapsto \mu(x) = (e_1^\perp x,\dots,e_n^\perp x) \in Q.$$

Then $\mu^* : Q \to H$ is given by $\mu^*(y_1,\dots,y_n) = \sum_{i=1}^n y_i$. Define $H^- :=\mathrm{Ker}\mu^* \subset Q$. Put $q_- : Q \to H^-$ is the canonical projection. Define $\mathcal{S}^- = (H^-;E_1^-,\dots,E_n^-)$ by

$$E_i^- := \overline{q_-(0 \oplus E_i^\perp \oplus 0)} \subset H^-.$$

Our definition of $\mathcal{S}^- = (H^-;E_1^-,\dots,E_n^-)$ coincides with the original one by Gelfand and

Ponomarev up to isomorphism in the case of finite-dimensional spaces.

Define $\Phi^-(\mathcal{S}) := \mathcal{S}^- = (H^-; E_1^-, \ldots, E_n^-)$. Then there is a relation between $\mathcal{S}^+$ and $\mathcal{S}^-$.

Theorem. Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Then we have

$$\Phi^-(\mathcal{S}) = \Phi^\perp \Phi^+ \Phi^\perp(\mathcal{S}).$$

Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H and $\mathcal{T} = (K; F_1, \ldots, F_n)$ be another system of n subspaces in a Hilbert space K. Let $\varphi : \mathcal{S} \to \mathcal{T}$ be a homomorphism, i.e., $\varphi : H \to K$ is a bounded linear operator with $\varphi(E_i) \subset F_i$. Define $\varphi^- : \Phi^-(\mathcal{S}) \to \Phi^-(\mathcal{T})$ by

$$\varphi^- := \Phi^\perp \Phi^+ \Phi^\perp(\varphi).$$

Thus we can introduce a covariant functor $\Phi^- : \mathcal{S}ys^n \to \mathcal{S}ys^n$ by

$$\Phi^-(\mathcal{S}) = \mathcal{S}^- \text{ and } \Phi^-(\varphi) = \varphi^-.$$

Definition. Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H.

Then $\mathcal{S}$ is said to be *reduced from above* if for any $k = 1, \ldots, n$, $\sum_{i \neq k} E_i = H$. Similarly $\mathcal{S}$ is said to be *reduced from below* if for any $k = 1, \ldots, n$, $\sum_{i \neq k} E_i^\perp = H$.

Example.(1) Any bounded operator system is reduced from above and reduced from

below. (2)The exotic examples are reduced from above and reduced from below.

Theorem. (duality) Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from above. Then we have

$$\Phi^- \Phi^+(\mathcal{S}) \cong \mathcal{S}.$$

Similarly we have the follwoing:

Theorem. (duality) Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from below. Then we have

$$\Phi^+ \Phi^-(\mathcal{S}) \cong \mathcal{S}.$$

Theorem. Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from above and $\mathcal{S}^+ = \Phi^+(\mathcal{S})$ is reduced from below. If $\mathcal{S}$ is indecomposable, then $\Phi^+(\mathcal{S})$ is also indecomposable.

Example. Let $\mathcal{S}_\gamma = (H; E_1, E_2, E_3, E_4)$ be an exotic example. Then $\mathcal{S}_\gamma$ is reduced from

above and $\Phi^+(\mathcal{S}_\gamma)$ is reduced from below. Since $\mathcal{S}_\gamma$ is indecomposable, $\Phi^+(\mathcal{S}_\gamma)$ is also indecomposable.

Similarly we have the following:

Theorem. Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from below and $\mathcal{S}^- = \Phi^-(\mathcal{S})$ is reduced from above. If $\mathcal{S}$ is indecomposable, then $\Phi^-(\mathcal{S})$ is also indecomposable.

We shall show that the Coxeter functors Φ^+ and Φ^- preserve the defect under certain conditions.

Theorem. Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Suppose that $\mathcal{S}$ is reduced from above. If $\mathcal{S}$ is a quasi-Fredholm system, then $\Phi^+(\mathcal{S})$ is also a quasi-Fredholm system and $\rho(\Phi^+(\mathcal{S})) = \rho(\mathcal{S})$.

Theorem. Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Suppose that $\mathcal{S}$ is reduced from below. If $\mathcal{S}$ is a quasi-Fredholm system, then $\Phi^{-}(\mathcal{S})$ is also a quasi-Fredholm system and $\rho(\Phi^{-}(\mathcal{S})) = \rho(\mathcal{S})$.

[3] Representations of quivers on infinite dimensional Hilbert spaces

Next we consider to extend 4-sbspace systems to general finite connected directed graph.

We consider representation of quivers(finite directed graphs) on Hilbert spaces.

Gabriel showed that quivers which have only finite numbers of indecomposable representations are ADE. We consider this Gabriel theorem in the infinite dimensional setting.

In order to do this, we need some definitions.

(1)General Definition

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. The set V represents the set of vertices of Γ and the set of E represents the set of arrows of Γ. For $\ell \in E$, $s(\ell)$ represents the starting point of ℓ and $r(\ell)$ represents the end point of ℓ.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that a Hilbert space representation (H, f) of Γ is a pair of a family $H = (H_v)_{v \in V}$ of Hilbert spaces H_v and a family of $f = (f_\ell)_{\ell \in E}$ of bounded linear operators f_ℓ such that $f_\ell : H_{s(\ell)} \to H_{r(\ell)}$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver. Let (H, f) and (K, g) be representations of Γ. We put

$Hom((H,f),(K,g) := \{T = (T_x)_{x \in V};\ T_x \in B(H_x, K_x),\ T_{r(e)} f_e = g_e T_{s(e)} (\forall e \in E)\}$.

If $(H, f) = (K, g)$ holds, we denote $End((H,f)) = Hom((H,f),(H,f))$. Put $Idem((H,f)) = \{T \in End((H,f)); T \textit{ is idempotent.}\}$

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver.Let (H, f) and (W, g) be Hilbert space representations of Γ. We say that (H, f) and (W, g) are isomorphic $(H, f) \simeq (W, g)$ if there exists a family $\varphi = (\varphi_v)_{v \in V}$ of bounded invertible linear operators $\varphi_v \in B(H_v, K_v)$ such that $\varphi_{r(\ell)} f_\ell = g_\ell \varphi_{s(\ell)} (\forall \ell \in E)$.

Definition. Let $\Gamma = (V, E, s, r)$ be a finite quiver.Let (H, f) and (W, g) be Hilbert space representations of Γ. Define the direct sum $(H, f) \oplus (W, g)$.

Definition. Let(K, g) be representations of Γ. Then we put $(K, g) = 0$ as

$K_x = 0 (\forall x \in V)$.

Definition. Let(H, f) be representations of Γ. Then (H, f) is indecomposable if $(H, f) \cong (K, g) \oplus (K', g')$ then $(K, g) \cong 0$ or $(K', g') \cong 0$.

Proposition. Let (H, f)be a representation of Γ. Then (H, f) is indecomposable if and only if $Idem(H,f) = \{0,1\} = \{(0_x)_x, (1_x)_x\}$.

As a fundamental tool, we can construct reflection functors on representations of quivers.

(2)A definition of reflection functors

Definition. Let $\Gamma = (V,E,s,r)$ be a finite quiver. We say that a vertex $\beta \in V$ is a sink if $\beta \neq s(\ell)(\forall \ell \in E)$. We put $\Gamma^\beta = \{\ell \in E; r(\ell) = \beta\}$. Then we can construct a new quiver $\sigma^+_\beta(\Gamma)$ as follows. Define the set of vertices of $\sigma^+_\beta(\Gamma) := V$, For $\ell \in E$, we put the arrow $\overline{\ell}$ which has the opposite direction of ℓ ,therefore
$s(\overline{\ell}) = r(\ell), r(\overline{\ell}) = s(\ell)$. Define $\overline{\Gamma^\beta} = \{\overline{\ell}; \ell \in \Gamma^\beta\}$. Define the set of arrows of $\sigma^+_\beta(\Gamma) := \overline{\Gamma^\beta} \cup (E \setminus \Gamma^\beta)$.

Let (H,f) be a Hilbert space representation of a finite quiver $\Gamma = (V,E,s,r)$.

We construct a new representation (W,g) of $\sigma^+_\beta(\Gamma)$ using by (H,f). Define $h_\beta : \oplus_{\ell\in\Gamma^\beta} H_{s(\ell)} \to H_\beta$, by $h_\beta((x_{s(\ell)})_{\ell\in\Gamma^\beta}) = \sum_{\ell\in\Gamma^\beta} f_\ell(x_{s(\ell)})$.

Put $W_\beta := \ker h_\beta \subset \oplus_{\ell\in\Gamma^\beta} H_{s(\ell)}$. Let $i_\beta : W_\beta \to \oplus_{\ell\in\Gamma^\beta} H_{s(\ell)}$ be the inclusion map.
Let $Proj_e : \oplus_{\ell\in\Gamma^\beta} H_{s(\ell)} \to H_{s(e)}$ be the projection map.
Then ,for $\overline{e} \in \overline{\Gamma^\beta}$, we put $g_{\overline{e}} = (Proj_e) i_\beta$, $g_\ell = f_\ell (\ell \notin \Gamma^\beta), W_v = H_v (v \neq \beta)$.
From this, we can define a new representation (W,g) of $\sigma^+_\beta(\Gamma)$.

Definition. Let $\Gamma = (V,E,s,r)$ be a finite quiver. We say that a vertex $\alpha \in V$ is a source if $\alpha \neq r(\ell)(\forall \ell \in E)$. We put $\Gamma^\alpha = \{\ell \in E; s(\ell) = \alpha\}$.
Then we can construct a new quiver $\sigma^-_\alpha(\Gamma)$ as follows.
Put $\overline{\Gamma^\alpha} := \{\overline{\ell}; \ell \in \Gamma^\alpha\}$.
The set of vertices of $\sigma^-_\alpha(\Gamma) := V$.
The set of arrows of $\sigma^-_\alpha(\Gamma) := \overline{\Gamma^\alpha} \cup (E \setminus \Gamma^\alpha)$.

Let (H,f) be a Hilbert space reoresentation of a finite quiver $\Gamma = (V,E,s,r)$.
We construct a new representation (W,g) of $\sigma^-_\alpha(\Gamma)$ using by (H,f).
Define $\widetilde{h_\alpha} : H_\alpha \to \oplus_{\ell\in\Gamma^\alpha} H_{r(\ell)}$, by $\widetilde{h_\alpha}(x) := (f_\ell(x))_{\ell\in\Gamma^\alpha}$.
Put $W_\alpha := (\mathrm{Im}\widetilde{h_\alpha})^\perp \subset \oplus_{\ell\in\Gamma^\alpha} H_{r(\ell)}$. For $e \in \Gamma^\alpha$,let $j_e : H_{r(e)} \to \oplus_{\ell\in\Gamma^\alpha} H_{r(\ell)}$be the inclusion map and $Proj_{W_\alpha} : \oplus_{\ell\in\Gamma^\alpha} H_{r(\ell)} \to W_\alpha$ be the projection.
We put $g_{\overline{e}} = (Proj_{W_\alpha}) j_e$, $g_\ell = f_\ell(\ell \notin \Gamma^\alpha), W_v = H_v(v \neq \alpha)$.
Thus we can get a new representation (W,g) of $\sigma^-_\alpha(\Gamma)$.

(3)Decomposition Theorem、Duality Theorem、Indecomposability Theorem

Theorem (Decomposition)
(1)Let $\Gamma = (V,E,s,r)$ be a finite quiver.
Let $\beta \in V$ be a sink. Let (H,f) be a Hilbert space representation of Γ.
(Assumption) the set $\sum_{\ell\in\Gamma^{\beta}} \mathrm{Im}(f_\ell)$ is a closed set.
Then, there exists a Hilbert space representation $(\widetilde{H},\widetilde{f})$ of Γ such that
$(H,f) \simeq \sigma_\beta^- \sigma_\beta^+ (H,f) \oplus (\widetilde{H},\widetilde{f})$.

(2)Let $\Gamma = (V,E,s,r)$ be a finite quiver.
Let $\alpha \in V$ be a source. Let (H,f) be a Hilbert space representation of Γ.
(Assumption) the set $\sum_{\ell\in\Gamma^{\alpha}} \mathrm{Im}(f_\ell)^*$ is a closed set.

Then, there exists a Hilbert space representation $(\widetilde{H},\widetilde{f})$ of Γ such that
$(H,f) \simeq \sigma_\alpha^+ \sigma_\alpha^- (H,f) \oplus (\widetilde{H},\widetilde{f})$.

Definition. Let $\Gamma = (V,E,s,r)$ be a finite quiver.
Let $\beta \in V$ be a sink. Let (H,f) be a Hilbert space representation of Γ.
Then we say that (H,f) is full at β if $\sum_{\ell\in\Gamma^{\beta}} \mathrm{Im}(f_\ell) = H_\beta$.

Definition. Let $\Gamma = (V,E,s,r)$ be a finite quiver.
Let $\alpha \in V$ be a source. Let (H,f) be a Hilbert space representation of Γ.

Then we say that (H,f) is co-full at α if $\sum_{\ell\in\Gamma^{\alpha}} \mathrm{Im}(f_\ell)^* = H_\alpha$.

Then we have a duality between σ_β^+ and σ_α^-.
Theorem (Duality)
(1) Let $\Gamma = (V,E,s,r)$ be a finite quiver.
Let $\beta \in V$ be a sink. Let (H,f) be a Hilbert space representation of Γ.
(Assumption) (H,f) is full at β.
Then
$(H,f) \simeq \sigma_\beta^- \sigma_\beta^+ (H,f)$.

(2)(H,f) is full at β.
Let $\alpha \in V$ be a source. Let (H,f) be a Hilbert space representation of Γ.
(Assumption) (H,f) is co-full at α.
Then $(H,f) \simeq \sigma_\alpha^+ \sigma_\alpha^- (H,f)$.

Theorem (Indecomposability)
(1) Let $\Gamma = (V,E,s,r)$ be a finite quiver.

Let $\beta \in V$ be a sink. Let (H,f) be a Hilbert space representation of Γ which is

indecomposable and $\dim H_\beta \geq 2$.
(Assumption) $\sum_{\ell \in \Gamma^\beta} \mathrm{Im}(f_\ell)$ is a closed set.
Then
$\sigma_\beta^+(H,f)$ is indecomposable.
(2)

Let $\Gamma = (V,E,s,r)$ be a finite quiver. Let $\alpha \in V$ be a source.

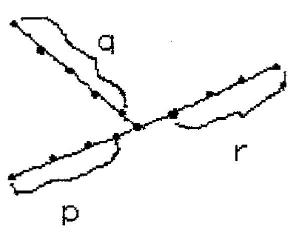
Let (H,f) be a Hilbert space representation of Γ which is indecomposable and $\dim H_\beta \geq 2$.
(Assumption) $\sum_{\ell \in \Gamma^\alpha} \mathrm{Im}(f_\ell)^*$ is a closed set.
Then
$\sigma_\alpha^-(H,f)$ is indecomposable.

We have now the following results about the indecomposable representations of finite quivers on infinite dimensional Hilbert spaces.

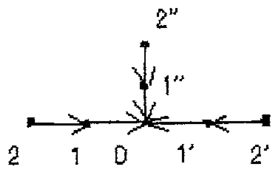
Theorem.
Let Γ be a finite connected undirected graph. If there exist no infinite dimensional(Hilbert space) indecomposable representations of Γ with any orientations, then Γ is one of
A_n, D_n, E_6, E_7, E_8.

Proposition.
There exist indecomposable representation of $\widetilde{A_0}, \widetilde{A_n} (n \geq 1), \widetilde{D_n} (n \geq 4)$ on an infinite dimensional Hilbert space.

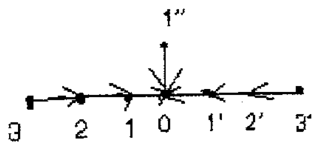
Hence we need to investigate the special trees $T_{p,q,r}$.



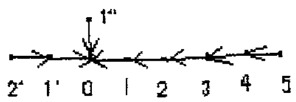
In order to do this, we only need to check $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$.



$\widetilde{E_6}$



$\widetilde{E_7}$



$\widetilde{E_8}$.

We give an indecomposable representation (H,f) of $\widetilde{E_6}$ on an infinite dimensional Hilbert space.

We take an infinite dimensional Hilbert space K and a unilateral shift on K.
We put
$H_0 = K \oplus K \oplus K$,
$H_1 = K \oplus 0 \oplus K$,
$H_2 = 0 \oplus 0 \oplus K$,
$H_{1'} = K \oplus K \oplus 0$,
$H_{2'} = 0 \oplus K \oplus 0$,
$H_{1''} = \{(x,x,x); x \in K\} + \{(y,Sy,0); y \in K\}$
$H_{2''} = \{(x,x,x); x \in K\}$.
We also give inclusion maps along the each arrows.

This gives an indecomposable representation of $\widetilde{E_6}$ on an infinite dimensional Hilbert space.

Similarly we can give indecomposable representations of $\widetilde{E_7}, \widetilde{E_8}$ on an infinite dimensional Hilbert space.

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