数理解析研究所講究録 1458 巻 2005 年 156-170

The family of inverses of operator monotone functions and operator inequalities

Mitsuru Uchiyama (内山　充)
Department of Mathematics (数学科)
Fukuoka University of Education (福岡教育大学)
Munakata, Fukuoka, 811-4192, Japan
e-mail uchiyama@fukuoka-edu.ac.jp

Abstract

Let $\mathcal{P}_+$ be the set of all non-negative operator monotone functions defined on $[0,\infty)$, and put $\mathcal{P}_+^{-1}=\{h : h^{-1}\in\mathcal{P}_+\}$. Then $\mathcal{P}_+\cdot\mathcal{P}_+^{-1}\subset\mathcal{P}_+^{-1}$ and $\mathcal{P}_+^{-1}\cdot\mathcal{P}_+^{-1}\subset\mathcal{P}_+^{-1}$. For a function $\tilde{h}(t)$ and a strictly increasing function h we write $\tilde{h}\preceq h$ if $\tilde{h}\circ h^{-1}$ is operator monotone. If $0\leqq\tilde{h}\preceq h$ and $0\leqq\tilde{g}\preceq g$ and if $h\in\mathcal{P}_+^{-1}$ and $g\in\mathcal{P}_+^{-1}\cup\mathcal{P}_+$, then $\tilde{h}\,\tilde{g}\preceq h\,g$. We will apply this result to polynomials and operator inequalities. Let $\{a_i\}_{i=1}^{n}$ and $\{b_i\}_{i=1}^{n}$ be non-increasing sequences, and put $u_+(t)=\prod_{i=1}^{n}(t-a_i)$ for $t\geqq a_1$ and $v_+(t)=\prod_{j=1}^{m}(t-b_j)$ for $t\geqq b_1$. Then $v_+\preceq u_+$ if $m\leqq n$ and $\sum_{i=1}^{k}b_i\leqq\sum_{i=1}^{k}a_i\quad(1\leqq k\leqq m)$: in particular, for a sequence $\{p_n\}_{n=0}^{\infty}$ of orthonormal polynomials, $(p_{n-1})_+\preceq(p_n)_+$. Suppose $0<r,p$ and $0<\alpha\leqq\frac{r}{p+r}$. Then $0\leqq A\leqq B$ implies $(e^{\frac{rA}{2}}A^se^{pA}e^{\frac{rA}{2}})^\alpha\leqq(e^{\frac{rA}{2}}B^se^{pB}e^{\frac{rA}{2}})^\alpha$ for $s=0$ or $1\leqq s\leqq 1+\frac{p}{r}$.

1 Introduction

Let A,B be bounded selfadjoint operators on a Hilbert space. A real-valued (Borel measurable) function $f(t)$ defined on a finite or infinite interval I in $\mathbf{R}$ is called an *operator monotone function* on I and denoted by $f\in P(I)$, provided $A\leqq B$ implies $f(A)\leqq f(B)$ for every pair A,B whose spectra lie in the interval I. When I is written as $[a,b)$ we simply write $P[a,b)$ instead of $P([a,b))$. It is easy to verify that if a sequence of functions in $P(I)$ converges to f pointwise on I then $f\in P(I)$ and that if $f(t)$ in $P(a,b)$ is continuous at a from the right then $f(t) \in P[a,b)$. It is well known that t^α $(0 < \alpha \leqq 1)$, $\log t$ and $\frac{t}{t+\lambda}$ $(\lambda > 0)$ are in $P(0,\infty)$. The following Löwner theorem [8](see also [4, 5]) is essential to the study of this area:

f is operator monotone on an open interval if and only if f has an analytic extension $f(z)$ to the open upper half plane Π_+ so that $f(z)$ maps Π_+ into itself, that is, $f(z)$ is a Pick function.

From this theorem it follows that $f(t)$ in $P(I)$ is constant or strictly increasing and that by Herglotz's theorem $f(t) \in P(0,\infty)$ can be expressed as:

$$f(t) = a + bt + \int_0^\infty \left(-\frac{1}{x+t} + \frac{x}{x^2+1} \right) d\nu(x),$$

where a, b are real constants with $b \geqq 0$ and $d\nu$ is a non-negative Borel measure on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{d\nu(x)}{x^2+1} < \infty.$$

Suppose $f \in P[0,\infty)$. Then we can rewrite the above expression as:

$$f(t) = f(0) + bt + \int_0^\infty \left(\frac{1}{x} - \frac{1}{x+t} \right) d\nu(x), \tag{1}$$

For further details on the operator monotone function we refer the reader to Chapter V of [4], [5] and [7].

Thanks to (1), we can see that if $f(t) \geqq 0$ is in $P[0,\infty)$, then the holomorphic extension $f(z)$ to Π_+ satisfies $\arg f(z) \leqq \arg z$. By making use of this property we can construct operator monotone functions as follows:
if $f(t)$ and $g(t)$ are both positive operator monotone on $(0,\infty)$, then so are $\frac{t}{f(t)}$, $f(t^\alpha)^{1/\alpha}$, $f(t^\alpha)g(t^{1-\alpha})$, $f(t)^\alpha g(t)^{1-\alpha}$ for $0 < \alpha < 1$ and $f(t)g(\frac{t}{f(t)})$; moreover, if f_i is positive operator monotone on $(0,\infty)$ for $1 \leqq i \leqq n$, then so is $(\prod_{i=1}^n f_i(t))^{1/n}$.

In contrast to this direct way, there is another way to construct an operator monotone function; indeed, it is the way to construct a function whose inverse is operator monotone:

(a) For the function $u(t)$ on $[a_1,\infty)$ defined by

$$u(t) = e^{ct} \prod_{i=1}^k (t - a_i)^{\gamma_i} \quad (c \geqq 0, a_1 > a_2 > \cdots > a_k, \quad \gamma_i > 0),$$

if $\gamma_1 \geqq 1$, then the inverse function $u^{-1}(s)$ is in $P[0,\infty)$ ([11, 12, 15]):

(b) If $0 \leqq f(t)$ is in $P[0, \infty)$, then so is the inverse function of $tf(t)$ ([2]).

Let $\mathcal{P}_+[a,b)$ be the set of all non-negative operator monotone functions defined on $[a,b)$ and $\mathcal{P}_+^{-1}[a,b)$ the set of increasing functions h defined on $[a,b)$ such that the range of h is $[0,\infty)$ and its inverse h^{-1} is operator monotone on $[0,\infty)$. Let g be a non-decreasing function and h a strictly increasing function. Then g is said to be *majorized* by h, in symbols $g \preceq h$, if $g \circ h^{-1}$ is operator monotone on the range of h, where $f \circ g$ means the composite function $f(g)$. In Section 2, we will show that $\mathcal{P}_+[a,b) \cdot \mathcal{P}_+^{-1}[a,b) \subset \mathcal{P}_+^{-1}[a,b)$ and $\mathcal{P}_+^{-1}[a,b) \cdot \mathcal{P}_+^{-1}[a,b) \subset \mathcal{P}_+^{-1}[a,b)$: the first relation includes (a) and (b) given above. Moreover, we will show that if $0 \leqq \tilde{h}_i \preceq h_i$ and $0 \leqq \tilde{f}_j \preceq f_j$ for $1 \leqq i \leqq m$, $1 \leqq j \leqq m$ and if the range of each f_j is $[0,\infty)$, then $\prod_{i=1}^m \tilde{h}_i \prod_{j=1}^n \tilde{f}_j(t) \preceq \prod_{i=1}^m h_i(t) \prod_{j=1}^n f_j(t)$. we will call this the *product theorem.*

Let us define real polynomials $u(t)$ and $v(t)$ by

$$u(t) = \prod_{i=1}^n (t - a_i), \quad v(t) = \prod_{j=1}^m (t - b_j),$$

where $a_i \geqq a_{i+1}$ and $b_j \geqq b_{i+1}$. The positive and increasing parts of $u(t)$ and $v(t)$ are denoted by $u_+(t)$ and $v_+(t)$ respectively, that is $u_+(t) := u(t)|_{[a_1,\infty)}$ and $v_+(t) := v(t)|_{[b_1,\infty)}$. In Section 3, we will show that $v_+ \preceq u_+$ if $m \leqq n$ and $\sum_{i=1}^k b_i \leqq \sum_{i=1}^k a_i$ $(k = 1, 2, \cdots, m)$. In particular, for a sequence $\{p_n\}_{n=0}^\infty$ of orthonormal polynomials with the positive leading coefficients we will show $(p_{n-1})_+ \preceq (p_n)_+$.

Let $h(t) \in \mathcal{P}_+^{-1}[0,\infty)$ and $f_i(t) \in \mathcal{P}_+[0,\infty)$ for $1 \leqq i \leqq n$, and put $g(t) = \prod_{i=1}^n f_i(t)$. Suppose $0 \leqq \tilde{h}(t) \preceq h(t)$. Then, in Section 4, we will show that for the function φ on $[0,\infty)$ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ for $0 \leqq t < \infty$

$$0 \leqq A \leqq B \Longrightarrow \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \geqq \varphi(g(A)^{\frac{1}{2}} h(A) g(A)^{\frac{1}{2}}) = g(A)\tilde{h}(A),$$

which yields a concrete operator inequality that extends the Furuta inequality.

2 $\mathcal{P}_+^{-1}(I)$ and Majorization

Our objectives are non-negative operator monotone functions. A function $f(t)$ that is operator monotone on $-\infty < t < \infty$ is affine, that is, of the form $\alpha t + \beta$. Therefore we confine our attention to operator monotone functions on a subinterval (a,b) of $(-\infty,\infty)$. We consider just the case $-\infty < a$; for, it seems that the case $-\infty < a$ is more useful than the case $b < \infty$. Then every increasing function defined on (a,b) has the right limit at a; accordingly we deal with functions defined on $[a,b)$. A "function" means a "continuous function" and "increasing" does "strictly increasing" throughout this paper.

Let us introduce the sets of increasing functions defined on a interval $[a,b)$, although we made a short mention of them in the first section:

Definition 1

$$\mathcal{P}_+[a,b) := \{f | f(t) \geqq 0, f(t) \in \mathcal{P}[a,b)\},$$
$$\mathcal{P}_+^{-1}[a,b) := \{h | h(t) \textit{is increasing with the range} [0,\infty), h^{-1} \in \mathcal{P}[0,\infty)\},$$

where h^{-1} stands for the inverse function of h.

It is clear that $\mathcal{P}_+[a,b)$ is a cone and that $(sh^{-1}+(1-s)k^{-1})^{-1} \in \mathcal{P}_+^{-1}[a,b)$ for $0 \leqq s \leqq 1$ if $h,k \in \mathcal{P}_+^{-1}[a,b)$. We note that $h(t) \in \mathcal{P}_+^{-1}[a,b)$ precisely if $h(t-c) \in \mathcal{P}_+^{-1}[a+c,b+c)$. We give simple examples: $1 \in \mathcal{P}_+[a,b)$ for any interval $[a,b)$, $t^\alpha \in \mathcal{P}_+[0,\infty)$ for $0<\alpha \leqq 1$, $t^\alpha \in \mathcal{P}_+^{-1}[0,\infty)$ for $\alpha \geqq 1$ and $(1+t)/(1-t) \in \mathcal{P}_+^{-1}[-1,1)$, because its inverse function $(t-1)/(t+1)$ belongs to $\mathcal{P}[0,\infty)$.

Definition 2 Let h be non-decreasing function on I and k an increasing function on J. Then we say that h is *majorized* by k, in symbols $h \preceq k$, if $J \subset I$ and $h \circ k^{-1}$ is operator monotone on $k(J)$.

We note that this definition coincides with the following:
for A,B whose spectra lie in J

$$k(A) \leqq k(B) \Longrightarrow h(A) \leqq h(B).$$

It is evident that $1 \preceq k$ for every increasing function $k(t)$.

We have the following properties:

(i) $k^\alpha \preceq k^\beta$ for any increasing function $k(t) \geqq 0$ and $0 < \alpha \leqq \beta$;

(ii) $g \preceq h, \quad h \preceq k \Longrightarrow g \preceq k$;

(iii) if τ is an increasing function whose range is the domain of k, then
$h \preceq k \Longleftrightarrow h \circ \tau \preceq k \circ \tau$;

(iv) $k \in \mathcal{P}_+^{-1}[a,b) \Longleftrightarrow t \preceq k, \ k([a,b)) = [0,\infty)$;

(v) if the range of k is $[0,\infty)$ and $h,k \geqq 0$, then
$h \preceq k \Longrightarrow h^2 \preceq k^2$;

(vi) $h \preceq k, \quad k \preceq h \Longleftrightarrow h = ck + d$ for real numbers $c > 0, d$.

(i),(ii),(iii) and (iv) are trivial. To see (v) suppose $\phi \in \mathcal{P}_+[0,\infty)$ satisfies $\phi(k(t)) = h(t)$ for $t \in J$ which is the domain of k. Then $\varphi(t) := \phi(\sqrt{t})^2 \in \mathcal{P}_+[0,\infty)$ and $\varphi(k^2) = h^2$; thereby $h^2 \preceq k^2$. To see (vi) assume $h \preceq k$, $k \preceq h$. Then there is ϕ on $k(J)$ such that ϕ and ϕ^{-1} are both operator monotone. Since an operator monotone function is increasing and concave, ϕ must be an increasing linear function. This implies $h(t) = ck(t) + d$ for $c > 0$. The converse is evident.

Lemma 2.1 *If $f(t)$, $g(t)$ and $h(t)$ are all in $\mathcal{P}_+[0,\infty)$, then $h(f(t))g(\frac{t}{f(t)})$ is in $\mathcal{P}_+[0,\infty)$ and so is $f(t)g(\frac{t}{f(t)})$ in particular.*

Proof. It is clear that $h(f(t))g(\frac{t}{f(t)})$ is well-defined on $(0,\infty)$ and has the analytic extension $h(f(z))g(\frac{z}{f(z)})$ to Π_+. Since $\arg f(z) \leqq \arg z$ for $z \in \Pi_+$, we have $\frac{z}{f(z)} \in \Pi_+$ for $z \in \Pi_+$. Hence

$$\arg g(\frac{z}{f(z)}) \leqq \arg \frac{z}{f(z)} < \pi, \quad \arg h(f(z)) \leqq \arg f(z) < \pi.$$

Thus $\arg h(f(z))g(\frac{z}{f(z)}) \leqq \arg z$, which implies $h(f(z))g(\frac{z}{f(z)})$ is a Pick function. Therefore, $h(f(t))g(\frac{t}{f(t)})$ belongs to $\mathcal{P}_+(0,\infty)$. By considering its continuous extension to $[0,\infty)$ we get $h(f(t))g(\frac{t}{f(t)}) \in \mathcal{P}_+[0,\infty)$; in particular, we get $f(t)g(\frac{t}{f(t)}) \in \mathcal{P}_+[0,\infty)$, although it has been well-known. $\square$.

$\tilde{h} \preceq h$, $\tilde{k} \preceq k$ does not necessarily imply $\tilde{h}\tilde{k} \preceq hk$ even if these functios are all non-negative. For instance, $1 \preceq t$, $t \preceq 1 + t^2$, where all functions are defined on $[0,\infty)$; but $t(1+t^2) \notin \mathcal{P}_+^{-1}[0,\infty)$ as proved in Example 2.1 of [15], that is to say, $t \preceq t(1+t^2)$ is false.

The following lemma is beneficial, so we name it.

Lemma 2.2 (Product lemma) *Let $h(t), g(t)$ be non-negative increasing functions defined on $[a,b)$, with $(hg)(a) = 0$, $(hg)(b-0) = \infty$, where $-\infty < a < b \leqq \infty$. Then for ψ_1, ψ_2 in $\mathcal{P}_+[0,\infty)$*

$$g \preceq hg \Longrightarrow h \preceq hg, \quad \psi_1(h)\psi_2(g) \preceq hg.$$

Proof. Define the functions ϕ_i on $[0,\infty)$ by

$$\phi_0(h(t)g(t)) = g(t), \quad \phi_1(h(t)g(t)) = h(t),$$
$$\phi_2(h(t)g(t)) = \psi_1(h(t))\psi_2(g(t)) \quad (a \leqq t < b).$$

We need to show that if ϕ_0 is operator monotone, then so are both ϕ_1 and ϕ_2. By putting $s = h(t)g(t)$, ϕ_1 and ϕ_2 can be expressed as

$$\phi_1(s) = \frac{s}{\phi_0(s)}, \quad \phi_2(s) = \psi_1(\frac{s}{\phi_0(s)})\psi_2(\phi_0(s)).$$

As we mentioned in the first section, $\phi_1 \in \mathcal{P}_+[0,\infty)$, and $\phi_2 \in \mathcal{P}_+[0,\infty)$ follows from Lemma2.1. $\square$

We note that if $\psi_2(t) = t$ in the above lemma , then the last relation implies $\psi_1(h)\, g \preceq h\, g$.

Remark 2.1 In the proof of Lemma2.2, it is crucial that ψ_1 and ψ_2 are in $\mathcal{P}_+[0,\infty)$. For instance, put $h(t) = g(t) = t$ for $0 \leqq t < \infty$. Then clearly $g \preceq h\, g$. Consider the functions $\psi_1(t) = \psi_2(t) = t-1$ in $\mathcal{P}_+[1,\infty) \cap \mathcal{P}[0,\infty)$. But the function ϕ_2 defined by $\phi_2(t^2) = \psi_1(t)\psi_2(t) = (t-1)^2$ is not operator monotone even on $[1,\infty)$ (see Section 1 of [11]).

Lemma 2.3 *Suppose $a \geqq 0$ and $h(t) \in \mathcal{P}_+^{-1}[a,b)$. Then*

$$t \cdot h(t) \in \mathcal{P}_+^{-1}[a,b), \quad \text{i.e., } t \preceq th(t).$$

Moreover,

$$h(t) \preceq th(t), \quad t^2 \preceq th(t).$$

The lemma below is a generalization of the statement (b) in Introduction.

Lemma 2.4 *Suppose $a \geqq 0$. Then*

$$\mathcal{P}_+[a,b) \cdot \mathcal{P}_+^{-1}[a,b) \subset \mathcal{P}_+^{-1}[a,b).$$

Moreover, for $f(t) \in \mathcal{P}_+[a,b)$, $h(t) \in \mathcal{P}_+^{-1}[a,b)$ and $\psi(t), \phi(t) \in \mathcal{P}_+[0,\infty)$

$$h \preceq h\, f, \quad f \preceq h\, f, \quad \psi(h)\, \phi(f) \preceq h\, f.$$

Lemma 2.5 *Let $f(t)$ be in $\mathcal{P}_+[0,\infty)$ and $h(t)$ in $\mathcal{P}_+^{-1}[f(0), f(\infty))$. Then*

$$t \cdot (h \circ f)(t) \in \mathcal{P}_+^{-1}[0,\infty), \quad t \cdot f \preceq t \cdot (h \circ f).$$

Lemma 2.6 *Suppose $a \geqq 0$. Then*

$$\mathcal{P}_+^{-1}[a,b) \cdot \mathcal{P}_+^{-1}[a,b) \subset \mathcal{P}_+^{-1}[a,b).$$

Moreover, for $h(t), k(t) \in \mathcal{P}_+^{-1}[a,b)$ and for $\psi_1(t), \psi_2(t) \in \mathcal{P}_+[0,\infty)$

$$t \preceq h\, k, \quad k \preceq h\, k, \quad \psi_1(h)\, \psi_2(k) \preceq h\, k.$$

Lemma 2.7 *Suppose $a \geqq 0$. Let $h(t) \in \mathcal{P}_+^{-1}[a,b)$ and $f_j(t) \in \mathcal{P}_+[a,b)$ for $1 \leqq j \leqq n$. Put $g(t) = \prod_{j=1}^n f_j(t)$. Then $h(t)g(t) \in \mathcal{P}_+^{-1}[a,b)$. Moreover, for $\psi_1, \psi_2 \in \mathcal{P}_+[0,\infty)$*

$$h \preceq hg, \quad g \preceq hg, \quad \psi_1(h)\psi_2(g) \preceq hg.$$

Remark 2.2 If $\psi_1(t) = h^{-1}(t) - c$ for $c \leqq a$, $\psi_1(t) = t$ and $\psi_1(t) = 1$, then $\psi_1(h) = t - c$, $\psi_1(h) = h$ and $\psi_1(h) = 1$, respectively. Similarly, $\psi_2(g)$ may be g or 1.

So far we have assumed $a \geqq 0$ concerning the domain $[a,b)$, but now let us consider the case $a < 0$. For a function $h(t)$ and a real number c, we put $h_c(t) = h(t+c)$. It is evident that $h(t) \in \mathcal{P}_+[a,b)$ if and only if $h_c(t) \in \mathcal{P}_+[a-c,b-c)$, and $h(t) \in \mathcal{P}_+^{-1}[a,b)$ if and only if $h_c(t) \in \mathcal{P}_+^{-1}[a-c,b-c)$, because $h^{-1}(s) = h_c^{-1}(s)+c$ for $0 \leqq s < \infty$. By the property (iii), $k(t) \preceq h(t)$ if and only if $k_c(t) \preceq h_c(t)$. Moreover, the function ϕ satisfying $\phi(h(t)) = k(t)$ for $a \leqq t < b$ also satisfies $\phi(h_c(t)) = k_c(t)$ for $a - c \leqq t < b - c$. Thus we can see Lemma2.4, Lemma2.6 and Lemma2.7 hold even for $a < 0$. We note that $\tilde{h}(t) = t - c$ if $\tilde{h}_c(t) = t$. Thus by Lemma2.7, we get

Theorem 2.8 (Product theorem) *Suppose $-\infty < a < b \leqq \infty$. Then*

$$\mathcal{P}_+[a,b) \cdot \mathcal{P}_+^{-1}[a,b) \subset \mathcal{P}_+^{-1}[a,b), \quad \mathcal{P}_+^{-1}[a,b) \cdot \mathcal{P}_+^{-1}[a,b) \subset \mathcal{P}_+^{-1}[a,b).$$

Further, let $h_i(t) \in \mathcal{P}_+^{-1}[a,b)$ for $1 \leqq i \leqq m$, and let $g_j(t)$ be a finite product of functions in $\mathcal{P}_+[a,b)$ for $1 \leqq j \leqq n$. Then for $\psi_i, \phi_j \in \mathcal{P}_+[0,\infty)$

$$\prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t) \in \mathcal{P}_+^{-1}[a,b), \quad \prod_{i=1}^m \psi_i(h_i) \prod_{j=1}^n \phi_j(g_j) \preceq \prod_{i=1}^m h_i \prod_{j=1}^n g_j.$$

Remark 2.3 By Remark2.2 $\psi_i(h_i)$ may be $t - c_i$ for $c_i \leqq a$, h_i or 1, and $\phi_j(g_j)$ may be g_j or 1. Hence, for $0 \leqq m_1 + m_2 \leqq m$, $\quad 0 \leqq n_1 \leqq n$

$$\prod_{i=1}^{m_1}(t - c_i) \prod_{i=1}^{m_2} h_i(t) \prod_{j=1}^{n_1} g_j(t) \preceq \prod_{i=1}^m h_i \prod_{j=1}^n g_j,$$

where we put $\prod_{j=1}^{n_1} g_j(t) = 1$ if $n_1 = 0$.

In the proof of the following theorem, we will use the elementary result that if $h_n(t)$ $(n = 1, 2, \cdots)$ is continuous and increasing on a finite closed interval I and if $\{h_n\}$ converges pointwise to a continuous function h on I, then $\{h_n\}$ converges uniformly to h on I.

Theorem 2.9 *Suppose $h_n(t) \in \mathcal{P}_+^{-1}[a,b)$ $(n = 1, 2, \cdots)$. If $\{h_n\}$ converges pointwise to a continuous function h on $[a,b)$ and if $h(t) > 0$ for $t > a$ and $h(b-0) = \infty$, then*

$$h \in \mathcal{P}_+^{-1}[a,b).$$

Moreover, let $\tilde{h}_n$ $(n = 1, 2, \cdots)$ be increasing functions on $[a,b)$, and let $\{\tilde{h}_n\}$ converge pointwise to a continuous function $\tilde{h}$ on $[a,b)$. Then

$$\tilde{h}_n \preceq h_n \ (n = 1, 2, \cdots) \Longrightarrow \tilde{h} \preceq h.$$

In the above theorem, we assumed the continuity of h for the sake of simplicity, though this can be derived by an elementary argument.

Corollary 2.10 *Let $h \in \mathcal{P}_+^{-1}[a,b)$, and let g_n, for each n, be a finite product of functions in $\mathcal{P}_+[a,b)$. Suppose $\{g_n\}$ converges pointwise to g on $[a,b)$ such that g is increasing and continuous. Then, for $\psi_1, \psi_2 \in \mathcal{P}_+[0,\infty)$*

$$\psi_1(h)\psi_2(g) \preceq h\,g \in \mathcal{P}_+^{-1}[a,b).$$

Proof. By the product theorem $h\,g_n \in \mathcal{P}_+^{-1}[a,b)$ and $h \preceq h\,g_n$. Thus by Theorem2.9 $h\,g \in \mathcal{P}_+^{-1}[a,b)$ and $h \preceq h\,g$. By the product lemma we get $\psi_1(h)\psi_2(g) \preceq h\,g$. □

3 Polynomials

The aim of this section is to apply the product theorem to real polynomials. (a) in the first section has been shown by analytic extension method in [11]. But its proof was not so easy. To begin with, let us give a simple proof of (a) by using the product theorem:

Another proof of (a). Since $\gamma_1 \geqq 1$, $(t-a_1)^{\gamma_1} \in \mathcal{P}_+^{-1}[a_1,\infty)$. For each γ_i $(2 \leqq i \leqq k)$ take a large natural number n to ensure that $n - 1 < \gamma_i \leqq n$; then $(t-a_i)^{\gamma_i/n} \in \mathcal{P}_+[a_1,\infty)$. Thus by the product theorem $h(t) := \prod_{i=1}^{k}(t-a_i)^{\gamma_i} \in \mathcal{P}_+^{-1}[a_1,\infty)$. Besides, $(1 + \frac{c}{n}t) \in \mathcal{P}_+[a_1,\infty)$ for n so that $-\frac{n}{c} \leqq a_1$

and $g_n(t) := (1 + \frac{c}{n}t)^n$ converges to e^{ct} for every t, which is increasing and continuous. By Corollary2.10 $h(t)e^{ct} \in \mathcal{P}_+^{-1}[a_1, \infty)$. $\square$

The same argument as the above proof leads us to the following:

Proposition 3.1 *Let $h \in \mathcal{P}_+^{-1}[a, b)$ for $-\infty < a < b \leqq \infty$. Then for $c \geqq 0$ and $\psi_1, \psi_2 \in \mathcal{P}_+[0, \infty)$,*

$$h\, e^{ct} \in \mathcal{P}_+^{-1}[a, b), \quad \psi_1(h)\psi_2(e^{ct}) \preceq h\, e^{ct}.$$

For non-increasing sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, we consider the positive and increasing functions $u(t)$ and $v(t)$ defined by

$$u(t) = \prod_{i=1}^{n}(t - a_i) \quad (t \geqq a_1), \quad v(t) = \prod_{i=1}^{m}(t - b_i) \quad (t \geqq b_1). \tag{2}$$

Lemma 3.2 *Suppose $v \preceq u$ for u and v given in (2). Then $m \leqq n$.*

Proof. Since $v \circ u^{-1}(s)$ is concave and non-negative on $0 \leqq s < \infty$, $v(u^{-1}(s))/s$ is decreasing. Therefore $v(t)/u(t)$ is decreasing on $a_1 \leqq t < \infty$. This implies $m \leqq n$. $\square$

Lemma 3.3 *Let u and v be polynomials defined by (2). Then*

$$m \leqq n, \quad b_i \leqq a_i \ (1 \leqq i \leqq m) \Longrightarrow v \preceq u.$$

Proof. Consider $t - a_i$ and $t - b_i$ as functions on $[a_1, \infty)$ and $[b_1, \infty)$ respectively. It is evident that $(t - a_1) \in \mathcal{P}_+^{-1}[a_1, \infty)$ and $(t - a_i) \in \mathcal{P}_+[a_1, \infty)$ for every i. Since $\psi(t) := t + (a_i - b_i) \in \mathcal{P}_+[0, \infty)$ and $\psi(t - a_i) = t - b_i$ for $t \geqq a_1$, $(t - b_i) \preceq (t - a_i)$ for every i. Hence the product theorem yields $v(t) \preceq u(t)$. $\square$

The following theorem indicates that the "majorization between sequenses" leads us to the "majorization between functions" introduced in the second section.

Theorem 3.4 *Let $u(t)$ and $v(t)$ be polynomials defined by (2). Then*

$$m \leqq n, \quad \sum_{i=1}^{k} b_i \leqq \sum_{i=1}^{k} a_i \ (1 \leqq k \leqq m) \Longrightarrow v \preceq u.$$

We do not know yet if the converse of Theorem3.4 holds.

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials with the positive leading coefficient. It is known that each p_n has n simple zeros $a_1 > a_2 > \cdots > a_n$ and there is a zero b_i of p_{n-1} in (a_{i+1}, a_i). This means $b_i < a_i$ for $i = 1, 2, \cdots, n-1$. Thus, by Lemma3.3 we have

Corollary 3.5 [15] *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials with the positive leading coefficient.Denote the restricted part of p_n to $[a, \infty)$ abusively by p_n, where a is the maximum zero of p_n. Then*

$$p_{n-1} \preceq p_n.$$

Now we give a bit of results related to characteristic polynomials of matrices. Let A be a $n \times n$ matrix with singular values $s_1 \geqq s_2 \geqq \cdots \geqq s_n$. Then

$$\|A\|_k := \sum_{i=1}^{k} s_i$$

is called *k-norm* of A. It is well-known that $\|A\|_k \leqq \|B\|_k$ for $k = 1, 2, \cdots, n$, if and only if $|||A||| \leqq |||B|||$ for every unitarily invariant norm. By using Theorem3.4 we can easily verify the following:

Corollary 3.6 *Let A, B be $n \times n$ non-negative matrices and p_A, p_B their characteristic polynomials. Then*

$$\|A\|_k \leqq \|B\|_k \ (1 \leqq k \leqq n) \Longrightarrow p_A \preceq p_B.$$

We finally deal with a general real polynomial $w(t)$ with imaginary zeros.

Theorem 3.7 *Let $u(t)$ be the polynomial defined in* (2) *and $w(t)$ the polynomial with imaginary zeros defined by*

$$w(t) = \prod_{j=1}^{m}(t - \alpha_j) \quad (\Re\alpha_1 \leqq t < \infty),$$

where $\Re\alpha_1 \geqq \Re\alpha_2 \geqq \cdots \geqq \Re\alpha_m$. Then

$$m \leqq n, \quad \sum_{j=1}^{k} \Re\alpha_j \leqq \sum_{j=1}^{k} a_j \ (1 \leqq k \leqq m) \Longrightarrow w \preceq u.$$

4 Operator Inequalities

In this section we apply the product theorem to operator inequalities. Our interest is inqualities concerning non-negative operators. So we only deal with functions defined on $[0,\infty)$. Let us recall that for $\phi(t)\in\mathcal{P}_+[0,\infty)$

$$X^*X\leqq 1\Longrightarrow \phi(X^*AX)\geqq X^*\phi(A)X \quad (A\geqq 0),$$
$$X^*X\geqq 1\Longrightarrow \phi(X^*AX)\leqq X^*\phi(A)X \quad (A\geqq 0).$$

The first inequality is called the Hansen-Pedersen inequality [7], from which the second one follows (cf.[13]).

Lemma 4.1 *Let $\phi(t)$ and $f(t)$ be in $\mathcal{P}_+[0,\infty)$. Suppose $h(t)$ and $\tilde{h}(t)$ are non-negative functions on $[0,\infty)$. If $\phi(h(t)f(t))=\tilde{h}(t)f(t)$, then*

$$0\leqq A\leqq B\Rightarrow \begin{cases}\phi(f(A)^{\frac{1}{2}}h(B)f(A)^{\frac{1}{2}})\geqq f(A)^{\frac{1}{2}}\tilde{h}(B)f(A)^{\frac{1}{2}},\\ \phi(f(B)^{\frac{1}{2}}h(A)f(B)^{\frac{1}{2}})\leqq f(B)^{\frac{1}{2}}\tilde{h}(A)f(B)^{\frac{1}{2}}.\end{cases}$$

Proposition 4.2 *Let $h(t)\in\mathcal{P}_+^{-1}[0,\infty)$ and $f_i(t)\in\mathcal{P}_+[0,\infty)$ for $i=1,2,\cdots$. For each natural number n put $g_n(t)=\prod_{i=1}^n f_i(t)$ and define the function ϕ_n on $[0,\infty)$ by*

$$\phi_n(h(t)g_n(t))=tg_n(t) \quad (0\leqq t<\infty). \tag{3}$$

Then

$$0\leqq A\leqq B\Rightarrow \begin{cases}\phi_n(g_n(A)^{\frac{1}{2}}h(B)g_n(A)^{\frac{1}{2}})\geqq Ag_n(A)\\ \phi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}})\leqq Bg_n(B)\end{cases} \tag{4}$$

Proof. We will show only the first inequality of (4) since the second one can be similarly shown. The product theorem says $\phi_n\in\mathcal{P}_+[0,\infty)$ for every n. By Lemma4.1

$$\phi_1(g_1(A)^{\frac{1}{2}}h(B)g_1(A)^{\frac{1}{2}})\geqq g_1(A)^{\frac{1}{2}}Bg_1(A)^{\frac{1}{2}}\geqq Ag_1(A).$$

Assume (4) holds for n, that is

$$\phi_n(g_n(A)^{\frac{1}{2}}h(B)g_n(A)^{\frac{1}{2}})\geqq Ag_n(A),$$

and denote the larger side (or the smaller side) of this inequality by K (or H). The function ψ_n defined by $\psi_n(tg_n(t))=f_{n+1}(t)$ is in $\mathcal{P}_+[0,\infty)$, because $f_{n+1}(t)\preceq t\preceq tg_n(t)$. Putting $s=tg_n(t)$, we have

$$\phi_{n+1}(\phi_n^{-1}(s)\psi_n(s))=s\psi_n(s).$$

Applying Lemma4.1 to this equality and the inequality $H \leqq K$, we get

$$\phi_{n+1}(\psi_n(H)^{\frac{1}{2}}\phi_n^{-1}(K)\psi_n(H)^{\frac{1}{2}}) \geqq \psi_n(H)^{\frac{1}{2}}K\psi_n(H)^{\frac{1}{2}} \geqq H\psi_n(H).$$

This yields

$$\phi_{n+1}(g_{n+1}(A)^{\frac{1}{2}}h(B)g_{n+1}(A)^{\frac{1}{2}}) \geqq A\,g_n(A)\,f_{n+1}(A) = A\,g_{n+1}(A),$$

because $\psi_n(H) = \psi_n(Ag_n(A)) = f_{n+1}(A)$ and $\phi_n^{-1}(K) = g_n(A)^{\frac{1}{2}}h(B)g_n(A)^{\frac{1}{2}}$. Thus we have obtained the first required inequality of (4). □

Remark 4.1 (4) is the generalization of the Furuta inequality [6](also see [9, 14]):

$$0 \leqq A \leqq B \Longrightarrow \begin{cases} (A^{r/2}B^pA^{r/2})^{\frac{1+r}{p+r}} & \geqq (A^{r/2}A^pA^{r/2})^{\frac{1+r}{p+r}}, \\ (B^{r/2}A^sB^{r/2})^{\frac{1+r}{p+r}} & \leqq (B^{r/2}B^pB^{r/2})^{\frac{1+r}{p+r}}, \end{cases}$$

where $r > 0$, $p \geqq 1$. In fact, let us substitute t^p for $h(t)$ and t^r for $g_n(t)$ in (3), where n is taken as $n-1 < r \leqq n$. Since the function ϕ_n satisfies

$$\phi_n(t) = t^{\frac{1+r}{p+r}},$$

(4) deduces the above inequalities.

Proposition 4.3 *Let $h(t) \in \mathcal{P}_+^{-1}[0,\infty)$, and let $\tilde{h}(t)$ be a non-negative function on $[0,\infty)$ such that*

$$\tilde{h} \preceq h.$$

Let $f_i(t) \in \mathcal{P}_+[0,\infty)$ for $i = 1, 2, \cdots$, and put $g_n(t) = \prod_{i=1}^n f_i(t)$. Then for the function φ_n defined by $\varphi_n(h(t)g_n(t)) = \tilde{h}(t)g_n(t)$

$$0 \leqq A \leqq B \Rightarrow \begin{cases} \varphi_n(g_n(A)^{\frac{1}{2}}h(B)g_n(A)^{\frac{1}{2}}) \geqq g_n(A)^{\frac{1}{2}}\tilde{h}(B)g_n(A)^{\frac{1}{2}}, \\ \varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \leqq g_n(B)^{\frac{1}{2}}\tilde{h}(A)g_n(B)^{\frac{1}{2}}. \end{cases} \tag{5}$$

Theorem 4.4 *Let $h(t) \in \mathcal{P}_+^{-1}[0,\infty)$, and let $\tilde{h}(t)$ be a non-negative function on $[0,\infty)$ such that*

$$\tilde{h} \preceq h.$$

Let $g_n(t)$ be a finite product of functions in $\mathcal{P}_+[0,\infty)$ for $i = 1, 2, \cdots$. Suppose $\{g_n\}$ converges pointwise to g on $[0,\infty)$ such that g is increasing and continuous. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$

$$0 \leqq A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases} \tag{6}$$

Furthermore, if $\tilde{h} \in \mathcal{P}_+[0,\infty)$, then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq \tilde{h}(A)g(A), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq \tilde{h}(B)g(B). \end{cases} \tag{7}$$

Now we apply Theorem4.4 to power functions.

Proposition 4.5 *Let $h(t) \in \mathcal{P}_+^{-1}[0,\infty)$, and let g be a pointwise limit of $\{g_n\}$, where $g_n(t)$ is a finite product of functions in $\mathcal{P}_+[0,\infty)$. If*

$$0 < \alpha < 1, \quad h(t)^\alpha g(t)^{\alpha-1} \preceq h(t),$$

then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^\alpha \geqq g(A)^{\frac{1}{2}}h(B)^\alpha g(B)^{\alpha-1}g(A)^{\frac{1}{2}}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^\alpha \leqq g(B)^{\frac{1}{2}}h(A)^\alpha g(A)^{\alpha-1}g(B)^{\frac{1}{2}}. \end{cases} \tag{8}$$

Furthermore, if

$$h(t)^\alpha g(t)^{\alpha-1} \in \mathcal{P}_+[0,\infty),$$

then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^\alpha \geqq (h(A)g(A))^\alpha, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^\alpha \leqq (h(B)g(B))^\alpha. \end{cases} \tag{9}$$

Proof. Put $\tilde{h}(t) = h(t)^\alpha g(t)^{\alpha-1}$. Then the assumption means $\tilde{h}(t) \preceq h(t)$. For φ on $[0,\infty)$ defined by

$$\varphi(h(t)g(t)) = \tilde{h}(t)g(t),$$

by Theorem4.4, we have

$$\varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}.$$

Since $\varphi(h(t)g(t)) = \tilde{h}(t)g(t) = (h(t)g(t))^\alpha$, or $\varphi(s) = s^\alpha$ for $0 \leqq s < \infty$, the above inequality coincides with the first inequality of (8). The rest can be shown in the same fasion. $\square$

It seems that Proposition4.5 has numerous applications.

Corollary 4.6 *Let a_i, s_i $(i = 1, \cdots, n)$ and r be real numbers such that $0 = a_0 < a_i$, $0 \leqq s_i$ and $0 < r$. Put $s = \sum_{i=0}^n s_i$. If $0 < s \leqq 1$, $0 < \alpha \leqq 1$, or if*

$$1 \leqq s_0,\ r(s - s_0 - 1) \leqq s_0,\ 0 < \alpha \leqq \frac{1+r}{s+r},$$

then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} \prod_{i=0}^n (B + a_i)^{s_i} A^{\frac{r}{2}})^\alpha \geqq (A^{\frac{r}{2}} \prod_{i=0}^n (A + a_i)^{s_i} A^{\frac{r}{2}})^\alpha, \\ (B^{\frac{r}{2}} \prod_{i=0}^n (B + a_i)^{s_i} B^{\frac{r}{2}})^\alpha \geqq (B^{\frac{r}{2}} \prod_{i=0}^n (A + a_i)^{s_i} B^{\frac{r}{2}})^\alpha. \end{cases}$$

We take notice that the Furuta inequality is just the case of $s_0 = p, s_i = 0$ for $i \geqq 1$ in the above inequalities.

Corollary 4.7 *Let g be a pointwise limit of $\{g_n\}$, where $g_n(t)$ is a finite product of functions in $\mathcal{P}_+[0, \infty)$. If $0 < r$, $0 < \alpha \leqq \frac{r}{s+r}$, then*

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (g(A)^{\frac{r}{2}} g(B)^s g(A)^{\frac{r}{2}})^\alpha \geqq (g(A)^{\frac{r}{2}} g(A)^s g(A)^{\frac{r}{2}})^\alpha, \\ (g(B)^{\frac{r}{2}} g(A)^s g(B)^{\frac{r}{2}})^\alpha \leqq (g(B)^{\frac{r}{2}} g(B)^s g(B)^{\frac{r}{2}})^\alpha. \end{cases} \tag{10}$$

The case of $g(t) = e^t$ in (9) has been shown in [3](cf. [10]).

Example 4.1 Let $h(t) = \prod_{i=0}^n (t + a_i)^{s_i}$, where $a_i \geq 0, s_i \geqq 0$. If $s \leqq r$, then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} |(h(B)e^B)^s (h(A)e^A)^r| \geqq (h(A)e^A)^{(s+r)}, \\ |(h(A)e^A)^s (h(B)e^B)^r| \leqq (h(B)e^B)^{(s+r)}, \end{cases}$$

where $|X| := (X^* x)^{1/2}$.

Indeed, consider $g(t)$ in the preceding corollary as $\prod_{i=0}^n (t + a_i)^{s_i} e^t$ with $a_i \geq 0, s_i \geqq 0$, and substitute $2r$ for r and $2s$ for s; since $1/2 \leqq r/(s+r)$ if $s \leqq r$, by (10) we get the above inequalities.

Corollary 4.8 *Let $h(t) = \prod_{i=0}^n (t+a_i)^{s_i} e^{st}$, where $a_0 = 0, a_i > 0, s_0 \geqq 1, s_i \geqq 0$, and put $s = \sum_{i=0}^n s_i$. If $0 < r$, $r(s-1) \leqq s$, $0 < \alpha \leqq \frac{r}{s+r}$, then*

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (e^{\frac{r}{2}A} h(B) e^{\frac{r}{2}A})^\alpha \geqq (e^{\frac{r}{2}A} h(A) e^{\frac{r}{2}A})^\alpha, \\ (e^{\frac{r}{2}B} h(B) e^{\frac{r}{2}B})^\alpha \geqq (e^{\frac{r}{2}B} h(A) e^{\frac{r}{2}B})^\alpha. \end{cases}$$

Acknowledgement The author express his thanks to Professor T. Ando for reading the original manuscript and giving him valuable comments.

References

[1] T. Ando, On some operator inequalities, Math. Ann. **279** (1987), 157–159.

[2] T. Ando, Comparison of norms $|||f(A)-f(b)|||$ and $|||f(|A-B|)|||$, Math. Z. **197**(1988),403–409.

[3] T. Ando, Math. Z

[4] R. Bhatia, Matrix Analysis, Springer, 1996.

[5] W. Donoghue, Monotone matrix functions and analytic continuation, Springer, 1974.

[6] T. Furuta, $A \geqq B \geqq 0$ assures $(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$ for $r \geqq 0, p \geqq 0, q \geqq 1$ with $(1+2r)q \geqq p+2r$, Proc. Amer. Math. Soc., **101** (1987), 85–88.

[7] F. Hansen, G. K. Pedersen, Jensen's inequality for operators and Löwner's theorem, Math. Ann., **258** (1982), 229–241.

[8] K. Löwner, Über monotone matrixfunktionen, Math. Z., **38** (1934), 177–216.

[9] K. Tanahashi, Best possibility of the Furuta inequality, Proc. A. M. S., **124** (1996), 141–146.

[10] M. Uchiyama, Some exponential operator inequalities, Math Inequal. Appl., **2**(1999),469–472.

[11] M. Uchiyama, Operator monotone functions which are defined implicitly and operator inequalities, J. Funct. Anal., **175** (2000), 330–347.

[12] M. Uchiyama, M. Hasumi, On some operator monotone functions, Integral Equations Operator Theory, **42** (2002), 243–251.

[13] M. Uchiyama, Mixed matrix (operator) inequalities, Linear A. A.,**341** (2002)249–257.

[14] M. Uchiyama, Criteria for monotonicity of operator means, J. Math. Soc. Japan, **55**(2003)197–207.

[15] M. Uchiyama, Inverse functions of polynomials and orthogonal polynomials as operator monotone functions, Transaction of Amer. Math. Soc. **355**(2003) 4111–4123