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PDE approach to utility maximization with partial information

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We consider a market model with one riskless security and a certain number of risky securities. The objective is to find an admissible self-financing investment strategy that maximizes the expected utility from terminal wealth at a given maturity and with a power utility function of the risk-averse type.

We assume that the dynamics of the risky assets are affected by "economic factors". We first consider the case where the economic factor evolve as a finite-state Markov process and then the one where it behaves as a conditionally Gaussian process. We allow these economic factors to be hidden, i.e. they may not be observed directly in both case. Information about these factors can therefore be obtained only by observing the prices of the risky assets.

Our problem is thus considered to be the type of a partially observed stochastic control problem. In the former case, we determine an equivalent complete observation control problem, where the new state is given by the pair (p_t, Y_t) consisting of the conditional state probability vector p_t for the hidden factor process and of the log-asset prices Y_t. In the latter the state variables are the conditional mean m_t, the conditional variance Π_t of the hidden factor X_t and the log prices. The equivalent complete observation control problems turn out to be of the types of risk sensitive stochastic control problems. By the method of Dynamic Programming (DP) we obtain nonlinear HJB equations. However, applying a transformation that is by now rather classical, the nonlinear HJB equations are transformed into linear ones. By means of a probabilistic representation as expectation of a suitable function of the underlying Markov process, we obtain unique viscosity solutions to the latter PDEs that induce unique viscosity solutions to the former. This probabilistic representation allows to obtain, on one hand, regularity results on the basis of classical results on expectations of functions of diffusion processes; on the other hand it allows to obtain a computational approach based on Monte Carlo simulation.

Portfolio optimization problems under partial information have been studied using two kinds of major methodologies, namely Dynamic Programming (DP) and the Martingale Methods (MM). As for MM, confer e.g. [5], [8], [9], [10], [16], [20], [22]. DP approach has benn used in [3], [18] and later in [13], [15] in relation to risk-sensitive stochastic control problems. Confer also [17] and [19].

1 Hidden Markov factor model

Let X_t be a finite state Markov chain whose state space $E = \{e_1, e_2, ..., e_k\}$ is assumed to be the set of the unit vectors in R^k.We assume the bond price S_t^0 is governed by the ordinary differential equation:

$$dS^0(t) = r(t, S_t)S^0(t)dt, \quad S^0(0) = s^0, \tag{1.1}$$

where $r(t, S)$ is a nonnegative bounded function on $[0, T] \times R_+^N$. The other secutrity prices S_t^i, $i = 1, 2, ..., N$, are assumed to satisfy

$$\begin{aligned} &dS^i(t) = S^i(t)\{a^i(t, X_t, S_t)dt + \textstyle\sum_{j=1}^N \sigma_j^i(t, S_t)dW_t^j\}, \\ &S^i(0) = s^i, \quad i = 1, ..., N, \end{aligned} \tag{1.2}$$

where $W_t = (W_t^j)_{j=1,..,N}$ is an N - dimensional standard Brownian motion process defined on $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ independent of X_t. Here we assume that $a^i(t, X, S)$ and $\sigma_j^i(t, S)$ are bounded and, for each t and X, locally Lipschitz continuous functions in S, σ is uniformly non degenerate, i.e. $z^*\sigma\sigma^*z \geq c|z|^2$, $\forall z \in R^N$, $\exists c > 0$.

Note that the markov chain X_t can be expressed by a martingale M_t of pure jump type such as

$$\begin{aligned} &dX_t = K(t)X_t dt + dM_t, \\ &X_0 = \xi, \end{aligned} \tag{1.3}$$

where $K(t)$ is a Q matrix of the Markov chain and ξ is a random variable taking its value on E.

Set

$$\mathcal{G}_t = \sigma(S(u); u \leq t).$$

and let us denote by h_t^i, $(i = 0, 1, ..., N)$ the portfolio proportion of the amount invested in the i-th security relative to the total wealth V_t that the investor possesses. It is defined as follows :

Definition 1.1 $(h^0(t), h(t)) \equiv (h^0(t), h^1(t), h^2(t), ..., h^N(t))^*$ *is said to be an invetment strategy if the following conditions are satisfied* *i) $h(t)$ is an R^N valued $\mathcal{G}_t$ - progressively measurable stochastic process such that*

$$\sum_{i=1}^{N} h^i(t) + h^0(t) = 1$$

ii) and that

$$P(\int_0^T |h(s)|^2 ds < \infty) = 1.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. When $(h^0(t), h(t)^*)_{0\leq t\leq T} \in \mathcal{H}(T)$ we will often write $h \in \mathcal{H}(T)$ for simplicity. For given $h \in \mathcal{H}(T)$ the wealth process $V_t = V_t(h)$ satisfies

$$\begin{aligned}
\frac{dV_t}{V_t} &= \textstyle\sum_{i=0}^{m} h^i(t)\frac{dS^i(t)}{S^i(t)} \\
&= h^0(t)r(t,S_t)dt + \textstyle\sum_{i=1}^{m} h^i(t)\{a^i(t,X_t,S_t)dt + \sum_{j=1}^{N}\sigma^i_j(t,S_t)dW^j_t\} \\
&= r(t,S_t)dt + h(t)^*(a(t,X_t,S_t) - r(t,S_t)\mathbf{1})dt + h(t)^*\sigma(t,S_t)dW_t, \\
V_0 &= v
\end{aligned}$$

under the assumption of the self-financing condition, where $\mathbf{1} = (1,1,...,1)^*$.

Our problem is the following. For a given constant $\mu < 1,\ \mu \neq 0$ maximize the expected (power) utility of terminal wealth up to the time horizon T, namely

$$J(v;h;T) = \frac{1}{\mu}E[e^{\mu \log V_T(h)}] = \frac{1}{\mu}E[V_T(h)^\mu], \quad \mu < 1,\ \mu \neq 0 \tag{1.4}$$

over the set $\mathcal{A}(T)$ of admissible startegies defined later . We consider here the maximization problem with partial information, since the economic factors X_t are in general not directly observable and so one has to select the strategies only on the basis of past information of the security prices.

2 Reduction to risk-sensitive stochastic control

Let us set

$$Y^i_t = \log S^i_t, \quad i = 0,1,2,...,N,$$

$Y_t = (Y^1_t, Y^2_t, ..., Y^N_t)^*$ and $\mathbf{e}^Y = (e^{Y^1}, ..., e^{Y^m})^*$. Then

$$dY^0_t = R(t, Y_t)dt$$

and

$$dY_t = \tilde{A}(t, X_t, Y_t)dt + \Sigma(t, Y_t)dW_t, \tag{2.1}$$

where

$$\bar{A}^i(t,x,y) = a^i(t,x,\mathbf{e}^y) - \tfrac{1}{2}(\sigma\sigma^*)^{ii}(t,\mathbf{e}^y),$$
$$\Sigma^i_j(t,y) = \sigma^i_j(t,\mathbf{e}^y), \quad R(t,y) = r(t,\mathbf{e}^y)$$

By Itô's formula we see that

$$dV_t^\mu = V_t^\mu\{-\mu\eta(t,X_t,Y_t,h_t)dt + \mu h_t^*\Sigma(t,Y_t)dW_t\}, \quad V_0 = v^\mu \tag{2.2}$$

and so,

$$V_t^\mu = v^\mu e^{-\mu\int_0^t \eta(s,X_s,Y_s,h_s)ds + \mu\int_0^t h_s^*\Sigma(s,Y_s)dW_s - \frac{\mu^2}{2}\int_0^t h_s^*\Sigma\Sigma^*(s,Y_s)h_s ds},$$

where

$$\eta(t,x,y,h) = \frac{1-\mu}{2}h^*\Sigma\Sigma^*(t,y)h - R(t,y) - h^*(A(t,x,y) - R(t,y)\mathbf{1}),$$
$$A^i(t,x,y) = a^i(t,x,\mathbf{e}^y).$$

Therefore our criterion can be written as

$$\frac{1}{\mu}E[V_T(h)^\mu] = \frac{v^\mu}{\mu}E[e^{-\mu\int_0^T \eta(s,X_s,Y_s,h_s)ds + \mu\int_0^T h_s^*\Sigma(s,Y_s)dW_s - \frac{\mu^2}{2}\int_0^T h_s^*\Sigma\Sigma^*(s,Y_s)h_s ds}].$$

Let us introduce a probability measure $\hat{P}$ on $(\Omega,\mathcal{F})$ defined by

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_T} = \rho_T,$$

where

$$\rho_T = e^{-\int_0^T \bar{A}(t,X_t,Y_t)^*(\Sigma\Sigma^*)^{-1}\Sigma(t,Y_t)dW_t - \frac{1}{2}\int_0^T \bar{A}^*(\Sigma\Sigma^*)^{-1}\bar{A}(t,X_t,Y_t)dt}. \tag{2.3}$$

Under this probability measure $\hat{P}$

$$\hat{W}_t = W_t + \int_0^t \Sigma^*(\Sigma\Sigma^*)^{-1}\bar{A}(s,X_s,Y_s)ds$$

is a Brownian motion process and the log price process Y_t can be rewritten as

$$dY_t = \Sigma(t,Y_t)d\hat{W}_t. \tag{2.4}$$

Under the new probability measure our criterion can be written as follows:

$$\frac{1}{\mu}E[V_T^\mu] = \frac{1}{\mu}v^\mu\hat{E}[e^{-\mu\int_0^T \eta(s,X_s,Y_s,h_s)ds + \mu\int_0^t h_s^*\Sigma(s,Y_s)dW_s - \frac{\mu^2}{2}\int_0^T h_s^*\Sigma\Sigma^*(s,Y_s)h_s ds}\rho_T^{-1}]$$
$$= \tfrac{1}{\mu}v^\mu\hat{E}[e^{-\mu\int_0^T \eta(s,X_s,Y_s,h_s)ds + \int_0^T Q(s,X_s,Y_s,h_s)^*dY_s - \frac{1}{2}\int_0^T Q^*\Sigma\Sigma^*Q(s,X_s,Y_s,h_s)ds}]$$

where
$$Q(t, X_t, Y_t, h_t) = (\Sigma\Sigma^*)^{-1}\bar{A}(t, X_t, Y_t) + \mu h_t.$$

Set
$$H_t = e^{-\mu\int_0^t \eta(s,X_s,Y_s,h_s)ds + \int_0^t Q(s,X_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^t Q^*\Sigma\Sigma^* Q(s,X_s,Y_s,h_s)ds}$$

and
$$q_t^i = \hat{E}[H_t X_t^i | \mathcal{G}_t],$$

where $X_t^i = \mathbf{1}_{\{e_i\}}(X_t)$. Then

$$\hat{E}[H_T|\mathcal{G}_T] = \sum_{i=1}^k \hat{E}[H_T X_T^i|\mathcal{G}_T] = \sum_{i=1}^k q_T^i$$

and we can see that q_t^i satisfies

$$(2.5)\qquad \begin{aligned} dq_t^i &= (K(t)q_t)^i dt - \mu\eta(t, e_i, Y_t, h_t)q_t^i dt + q_t^i Q^*(t, e_i, Y_t, h_t)dY_t, \\ q_0^i &= p_0^i \equiv P(\xi = e_i), \quad i = 1, 2, ..., k. \end{aligned}$$

(cf. [1],[4]) Furthermore we set

$$(2.6)\qquad \hat{H}_t = e^{-\mu\int_0^t \eta(s,p_s,Y_s,h_s)ds + \int_0^t Q(s,p_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^t Q^*\Sigma\Sigma^* Q(s,p_s,Y_s,h_s)ds},$$

where
$$p_t^i = P(X_t = e_i|\mathcal{G}_t), \quad i = 1, .., k,$$

and we employ the notation such that

$$f(s, p_s, y, h) := \sum_{i=1}^k f(s, e_i, y, h)p_s^i,$$

for a given function $f(s, x, y, h)$ on $[0, T] \times E \times R^N \times R^N$, while the defined function is the one on $[0, T] \times \Delta_{k-1} \times R^N \times R^N$ with the $k - 1$ dimensional simplex

$$\Delta_{k-1} = \{(d_1, d_2, .., d_k); d_1 + d_2 + ... + d_k = 1, \ \ 0 \le d_i \le 1, \ i = 1, .., k\}.$$

According to [21] it is known that p_t^i, $i = 1, 2, .., k$, satisfy the equation

$$dp_t^i = (K(t)p_t)^i dt + p_t^i[\bar{A}(t, e_i, Y_t)^* - \bar{A}(t, p_t, Y_t)^*][\Sigma\Sigma^*]^{-1}[dY_t - \bar{A}(t, p_t, Y_t)dt].$$

Namely,
(2.6)
$$dp_t = K(t)p_t + D(p_t)[\bar{A}(t, Y_t)^* - \mathbf{1}\bar{A}(t, p_t, Y_t)^*][\Sigma\Sigma^*]^{-1}[dY_t - \bar{A}(t, p_t, Y_t)],$$

where $\bar{A}(t, Y) := (\bar{A}^i(t, e_j, Y))$ is an $N \times k$ matrix and $D(p)$ is the diagonal matrix whose ii component is p^i.

Then we have

$$\begin{aligned} d(\hat{H}_t p_t^i) &= \hat{H}_t dp_t^i + p_t^i d\hat{H}_t + \langle \hat{H}, p^i \rangle_t \\ &= (K(t)\hat{H}_t p_t)^i dt - \mu\eta(t, e_i, Y_t, h_t)\hat{H}_t p_t^i dt + \hat{H}_t p_t^i Q^*(t, e_i, Y_t, h_t) dY_t. \end{aligned}$$

By comparing with (2.5), we see that $q_t^i = \hat{H}_t p_t^i$ and so,

$$\hat{E}[H_T|\mathcal{G}_T] = \sum_{i=1}^N q_T^i = \hat{H}_T.$$

Hence we have

Proposition 2.1

$$J(v; h; T) \equiv \frac{1}{\mu} E[V_T^\mu] = \frac{1}{\mu} v^\mu \hat{E}[H_T] = \frac{1}{\mu} v^\mu \hat{E}[\hat{H}_T].$$

Now our problem is reduced to the risk-sensitive stochastic control problem with full information. Indeed, introduce a new probability measure $\tilde{P}$ with the density

$$\begin{aligned} \zeta_T &= e^{\int_0^T Q(s,p_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^T Q^*\Sigma\Sigma^* Q(s,p_s,Y_s,h_s)ds} \\ &= e^{\int_0^T Q(s,p_s,Y_s,h_s)^*\Sigma(s,Y_s)d\hat{W}_s - \frac{1}{2}\int_0^T Q^*\Sigma\Sigma^* Q(s,p_s,Y_s,h_s)ds} \end{aligned}$$

defined by

$$\left.\frac{d\tilde{P}}{d\hat{P}}\right|_{\mathcal{G}_T} = \zeta_T.$$

Then under the probability measure $\tilde{P}$

$$\tilde{W}_t = \int_0^t (\Sigma\Sigma^*)^{-\frac{1}{2}}(s, Y_s) dY_s - \int_0^t \Sigma(s, Y_s)^* Q(s, p_s, Y_s, h_s) ds$$

is a standard $\mathcal{G}_t$ - Brownian motion process. Then

$$\frac{1}{\mu} v^\mu \hat{E}[\hat{H}_T] = \frac{1}{\mu} v^\mu \tilde{E}[\exp\{-\mu \int_0^T \eta(s, p_s, Y_s, h_s) ds\}]$$

and so our problem is maximizing the risk-sensitive criterion

$$\frac{1}{\mu} v^\mu \tilde{E}[\exp\{-\mu \int_0^T \eta(s, p_s, Y_s, h_s) ds\}], \tag{2.8}$$

subject to (p_t, Y_t) on $\Delta_{k-1} \times R^N$ governed by :

$$\begin{aligned} dp_t &= D(p_t)[\bar{A}(t, Y_t)^* - \mathbf{1}\bar{A}(t, p_t, Y_t)^*][\Sigma\Sigma^*]^{-1}\Sigma(t, Y_t) d\tilde{W}_t \\ &\quad + \{K(t)p_t + \mu D(p_t)[\bar{A}(t, Y_t)^* - \mathbf{1}\bar{A}(t, p_t, Y_t)^*]h_t\} dt. \end{aligned} \tag{2.9}$$

and

$$dY_t = \Sigma(t, Y_t) d\tilde{W}_t + \{\bar{A}(t, p_t, Y_t) + \mu\Sigma\Sigma^*(t, Y_t) h_t\} dt \tag{2.10}$$

defined on the filtered probability space $(\Omega, \mathcal{F}, \tilde{P}; \mathcal{G}_t)$.

3 HJB equation

Now we consider the HJB equation of the risk-sensitive stochastic control problem (2.8)-(2.10) to find an optimal strategy for the problem maximizing the expected power utiltity (1.4) at time maturity T with partial information. Set $Z := (p, Y)^* \in \Delta_{k-1} \times R^N$ and

$$\beta(s, Z) := (K(s)p, \bar{A}(s, p, Y))^*$$

$$\alpha(s, Z) := (D(p)[\bar{A}^*(s, Y) - \mathbf{1}\bar{A}^*(s, p, Y)](\Sigma\Sigma^*)^{-1}\Sigma(s, Y), \Sigma(s, Y))^*$$

$$\beta_\mu(s, Z, h) := \beta(s, Z) + \mu\alpha(s, Z)\Sigma^*(s, Y)h$$

$$(3.1) \qquad \begin{cases} dZ_s &= \beta_\mu(s, Z_s; h_s)ds + \alpha(s, Z_s)d\tilde{W}_s, \quad s \in [t, T] \\ Z_t &= z \end{cases}$$

Let us define the value function of our risk-sensitive stochastic control problem starting at t up to time horizon T:

$$(3.2) \qquad w(t, z) = \sup_{\mathcal{A}(t,T)} \log \tilde{E}[e^{-\mu \int_t^T \eta(s, Z_s, h_s)ds}]$$

Then we have

$$(3.3) \qquad \sup_{h \in \mathcal{A}(0,T)} J(v; h; T) = \frac{1}{\mu} v^\mu e^{w(0, z_0)}, \quad z_0 = (p_0, \log S_0)$$

The HJB equation of $w(t, z)$ can be written as

(3.4)
$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}[\alpha\alpha^* D^2 w] + \frac{1}{2}(\nabla w)^*\alpha\alpha^*\nabla w \\ \quad + \sup_h[\beta_\mu(t, z, h)^*\nabla w + \mu\gamma^*(t, z)h - \frac{1}{2}\mu(1-\mu)h^*\Sigma\Sigma^* h] + \mu R(t, z) = 0 \\ w(T, z) = 0 \end{cases}$$

where

$$\gamma(t, z) = A(t, p, Y) - R(t, z)\mathbf{1}.$$

It is easy to see that (3.4) could be written as

$$(3.5) \qquad \begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}[\alpha\alpha^* D^2 w] + \frac{1}{2(1-\mu)}(\nabla w)^*\alpha\alpha^*\nabla w + \Phi^*\nabla w + \Psi = 0 \\ w(T, z) = 0, \end{cases}$$

where

$$\Phi(t, z) := \beta(t, z) + \frac{\mu}{1-\mu}\alpha(t, z)\Sigma^{-1}(t, z)\gamma^*(t, z)$$

$$\Psi(t,z) = \mu R(t,z) + \frac{\mu}{2(1-\mu)}\gamma^*(t,z)(\Sigma\Sigma^*)^{-1}(t,z)\gamma(t,z).$$

Here we employ the transformation used in [6], [7]. Set

$$v(t,z) = e^{\frac{1}{1-\mu}w(t,z)}$$

Then we have

$$(3.6) \qquad \begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\alpha\alpha^* D^2 v] + \Phi^*(t,z)\nabla v + \frac{\Psi(t,z)}{1-\mu}v = 0 \\ v(T,z) = 1. \end{cases}$$

It can be easily seen that $v(t,z)$ is a viscosity solution for (3.6) if and only if $w = (1-\mu)\log v$ is a viscosity solution for (3.5). Furthermore the solution of (3.6) has an expression such that

$$(3.7) \qquad v(t,z) = E_{t,z}[\exp\{\frac{1}{1-\mu}\int_t^T \Psi(s,\bar{Z}_s)ds\}],$$

where

$$(3.8) \qquad \begin{cases} d\bar{Z}_s &= \Phi(s,\bar{Z}_s)ds + \alpha(s,\bar{Z}_s)d\tilde{W}_s \\ \bar{Z}_t &= z, \end{cases}$$

provided that the stochastic differential equation (3.8) has a unique solution and (3.7) has a finite value. Under the assumptions in section 1 (3.8) has a unique solution and $v(t,x)$ defined by the formula (3.7) turns out to be a viscosity solution of (3.6). Thus we have the following proposition.

Proposition 3.1 *Under the assumptions in section 1 equation (3.5) has a unique viscosity solution w and it is expressed as $w(t,z) = (1-\mu)\log v$, where v is the function defined by (3.7).*

Under stronger assumptions on r, a^i, b^i_j such that they are C^2 functions with derivatives of polynomial growth we have by Theorem 5.5 in [2] that $\bar{v}(t,z)$, and therefore also $\bar{w}(t,z)$, are of class C^2 and with derivatives of polynomial growth. The formal Bellman equation (3.5) becomes thus an equation having a classical solution. In that case set

$$h(t,z) := (\Sigma\Sigma^*)^{-1}(t,z)[\Sigma(t,z)\alpha^*(t,z)\frac{\nabla v(t,z)}{v(t,z)}]$$

and solve

$$dZ_s = \beta_\mu(s,Z_s;h(s,Z_s))ds + \alpha(s,Z_s)d\tilde{W}_s, \quad s\in[0,T], \quad Z_0 = z_0.$$

Then $h(t,Z_t)$ turns out to be an optimal strategy.

4 Conditionally Gaussian case

Let us consider the same problem as above for a market model with the economic factor X_t governed by the stochastic differential equation

$$(4.1) \qquad dX_t = B(t, X_t, Y_t)dt + \Lambda(t, X_t, Y_t)dW_t, \quad X_0 = x \in R^n,$$

such that the logprice of the securities is determined by

$$dY_t^0 = r(t, X_t, Y_t)dt,$$

$$(4.2) \qquad dY_t = A(t, X_t, Y_t)dt + \Sigma(t, Y_t)dW_t,$$

where W_t is an $N+n$ dimensional Brownian motion process. Here $A(t,x,y)$, $B(t,x,y)$ and $\Lambda(t,x,y)$ are continuous on $[0,T] \times R^n \times R^N$, locally Lipshitz and at most linear growth with respect to y, and $r(t,x,y)$ is a nonnegative bounded function. Our criterion is defined as before. Namely

$$J(v;h;T) := \frac{1}{\mu}E[V_T^\mu] = \frac{1}{\mu}v^\mu E[e^{-\mu\int_0^T \eta(t,X_t,Y_t,h_t)dt + \mu\int_0^T h_t^*\Sigma(t,Y_t)dW_t - \frac{\mu^2}{2}h_t^*\Sigma\Sigma^*(t,Y_t)h_t dt}].$$

Introduce a probability measure $\hat{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_T} = \bar{\rho}_T, .$$

where

$$(4.3) \qquad \bar{\rho}_T = e^{-\int_0^T A(t,X_t,Y_t)^*(\Sigma\Sigma^*)^{-1}\Sigma(t,Y_t)dW_t - \frac{1}{2}\int_0^T A^*(\Sigma\Sigma^*)^{-1}A(t,X_t,Y_t)dt}.$$

Then we have

$$\begin{aligned} J(v;h;T) &= \frac{1}{\mu}v^\mu \hat{E}[\bar{\rho}_T^{-1}e^{-\mu\int_0^T \eta(t,X_t,Y_t,h_t)dt + \mu\int_0^T h_t^*\Sigma(t,Y_t)dW_t - \frac{\mu^2}{2}h_t^*\Sigma\Sigma^*(t,Y_t)h_t dt}] \\ &\equiv \frac{1}{\mu}v^\mu \hat{E}[e^{-\mu\int_0^T \eta(t,X_t,Y_t,h_t)dt}\Psi_T], \end{aligned}$$

where

$$\Psi_t = e^{\int_0^t Q(s,X_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^t Q^*\Sigma\Sigma^* Q(s,X_s,Y_s,h_s)ds}$$

and

$$Q(s,x,y,h) = (\Sigma\Sigma^*)^{-1}A(t,x,y) + \mu h.$$

Set

$$q(t)(\varphi(t)) = q^h(t)(\varphi(t)) := \hat{E}[e^{-\mu\int_0^t \eta(s,X_s,Y_s,h_s)ds}\Psi_t\varphi(t,X_t,Y_t)|\mathcal{G}_t].$$

Then

$$\frac{1}{\mu}E[V_T^\mu] = \frac{1}{\mu}v^\mu \hat{E}[q^h(T)(1)]$$

and we can see that $q(t)$ satisfies the Modified Zakai equation

$$q(t)(\varphi(t)) = q(0)(\varphi(0)) + \int_0^t q(s)(\frac{\partial \varphi}{\partial s} + L\varphi - \mu\eta_s(\cdot)\varphi$$

$$+\mu h_s^*(\Sigma\Lambda^* D_x\varphi + \Sigma\Sigma^* D_y\varphi))ds$$

$$+\int_0^t q(s)(\varphi Q_s^*\Sigma\Sigma^* + (D_x\varphi)^*\Lambda\Sigma^* + (D_y\varphi)^*\Sigma\Sigma^*)(\Sigma\Sigma^*)^{-1}dY_s,$$

where

$$Q_s = Q(s,\cdot,Y_s,h_s), \quad \eta_s = \eta(s,\cdot,Y_s,h_s)$$

(cf. [13]). We consider a rather specific case such that

$$\begin{aligned} A(t,x,y) &= A_0(t,y) + A_1(t,y)x, \\ B(t,x,y) &= B_0(t,y) + B_1(t,y)x, \qquad \Lambda(t,x,y) = \Lambda(t,y). \end{aligned}$$

Namely X_t is assumed to be a conditionally Gaussian diffusion process with conditional mean vector m_t and conditinal variance Π_t:

$$E[e^{\sqrt{-1}\theta^* X_t}|\mathcal{G}_t] = \exp(\sqrt{-1}\theta^* m_t - \frac{1}{2}\theta^*\Pi_t\theta), \quad \theta \in R^n,$$

and m_t and Π_t are known to satisfy

(4.4)

$$\begin{cases} dm_t &= B(t,m_t,Y_t)dt + (\Pi A_1^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(dY_t - A(t,m_t,Y_t)dt) \\ m_0 &= x := E[X_0] \end{cases}$$

$$(4.5) \quad \begin{cases} \dot{\Pi}_t &= B_1(t,Y_t)\Pi_t + \Pi_t B_1(t,Y_t)^* + \Lambda\Lambda^*(t,Y_t) \\ & \qquad -(\Lambda\Sigma^* + \Pi_t A_1^*)(\Sigma\Sigma^*)^{-1}(\Lambda\Sigma^* + \Pi_t A_1)^*(t,Y_t) \\ \Pi_0 &= E[(X_0 - x)(X_0 - x)^*]. \end{cases}$$

In a similar way to [13] we can see that the solution of the modified Zakai equation $q(t)$ has a representation such that

$$q(t)(\varphi(t)) = \Gamma_t \int \varphi(t, m_t + \Pi_t^{\frac{1}{2}} z, Y_t)\frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}|z|^2}dz,$$

where

$$\Gamma_t = e^{\int_0^t Q(s,m_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^t Q^*\Sigma\Sigma^* Q(s,m_s,Y_s,h_s)ds - \mu\int_0^t \eta(s,m_s,Y_s,h_s)ds}$$

Set

$$\check{W}_t = \int_0^t (\Sigma\Sigma^*)^{-\frac{1}{2}}(s,Y_s)dY_s.$$

Then, under the probability measure $\hat{P}$, $\check{W}_t$ is a standard Brownian motion process and m_t and Y_t satisfy

$$\begin{aligned} dm_t &= B(t,m_t,Y_t)dt - (\Pi A_1^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A(t,m_t,Y_t)dt \\ &\quad +(\Pi_t A_1^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-\frac{1}{2}}d\check{W}_t \\ m_0 &= x \end{aligned} \tag{4.6}$$

$$dY_t = (\Sigma\Sigma^*)^{\frac{1}{2}}d\check{W}_t. \tag{4.7}$$

Thus we are reduced to considering the stochastic control problem with full information maximizing the criterion:

$$J = \frac{1}{\mu}v^\mu \hat{E}[e^{\int_0^T Q(s,m_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^T Q^*\Sigma\Sigma^* Q(s,m_s,Y_s,h_s)ds - \mu\int_0^T \eta(s,m_s,Y_s,h_s)ds}].$$

Here the state dymnamics is $n+N+n\times n$ dimensional process (m_t, Y_t, Π_t) governed by equations (4.6),(4.7) and (4.5).

We confine ourselves to a more specific case such that

$$A_1(t,y) = A_1(t), \quad \Sigma(t,y) = \Sigma(t), \qquad B_1(t,y) = B_1(t), \quad \Lambda(t,y) = \Lambda(t).$$

Let us introduce a probability measure $\bar{P}$ defined by

$$\left.\frac{d\bar{P}}{d\hat{P}}\right|_{\mathcal{G}_T} = e^{\int_0^T Q(s,m_s,Y_s,h_s)^* dY_s - \frac{1}{2}\int_0^T Q^*\Sigma\Sigma^* Q(s,m_s,Y_s,h_s)ds}.$$

Then under the probability measure $\bar{P}$

$$\bar{W}_t = \check{W}_t - \int_0^t (\Sigma\Sigma^*)^{\frac{1}{2}}Q(s,m_s,Y_s,h_s)ds$$

is a standard Brownian motion process and the criterion can be rewritten as

$$J = \frac{1}{\mu}v^\mu \bar{E}[e^{-\mu\int_0^T \eta(s,m_s,Y_s,h_s)ds}].$$

In the present case (4.5) is an ordinary differential equation not dependent on the observation Y_t and itself solved. Therefore our state dynamics is considered to be (m_t, Y_t) governed by

$$\begin{aligned} dY_t &= (A(t,m_t,Y_t) + \mu\Sigma\Sigma^* h_t)dt + (\Sigma\Sigma^*)^{\frac{1}{2}} d\bar{W}_t \\ dm_t &= \{B(t,m_t,Y_t) + \mu(\Pi_t A_1^* + \Lambda\Sigma^*)h_t\}dt + (\Pi_t A_1^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-\frac{1}{2}} d\bar{W}_t. \end{aligned}$$

Now we have arrived at similar situation to the problem considered in section 2. and 3.

Set

$$z = (y,m)^*, \quad \beta(t,z) = (A(t,m,y), B(t,m,y))^*$$

$$\zeta(t) = (\Sigma\Sigma^*(t), (\Pi_t A_1^* + \Lambda\Sigma^*)(t))^*, \quad \alpha(t) = ((\Sigma\Sigma^*)^{\frac{1}{2}}, (\Pi_t A_1^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-\frac{1}{2}})^*.$$

Then, controlled process is described as:

$$dZ_s = (\beta(s,Z_s) + \mu\zeta(s)h_s)ds + \alpha(s)d\bar{W}_s, \quad Z_t = z. \tag{4.8}$$

Introduce the value function on time interval $[t,T]$:

$$w(t,z) = \sup_{h.} \log \bar{E}[e^{-\mu\int_t^T \eta(s,Z_s,h_s)ds}]. \tag{4.9}$$

Then,

$$\sup_h J(v,h;T) = \frac{1}{\mu} v^\mu e^{w(0,z_0)}, \quad z_0 := (\log S_0, x). \tag{4.10}$$

The HJB equation of the value function is the following.

$$\begin{aligned} &\tfrac{\partial w}{\partial t} + \tfrac{1}{2}\mathrm{tr}[\alpha\alpha^*(t)D^2 w] + \beta(t,z)^* Dw + \tfrac{1}{2}\nabla w \alpha\alpha^* \nabla w \\ &\qquad + \sup_h\{\mu h_t^* \zeta(t)^* Dw - \mu\eta(s,z,h)\} = 0 \\ &w(T,z) = 0, \end{aligned} \tag{4.11}$$

which can be rewritten as

(4.12)

$$\begin{aligned} &\tfrac{\partial w}{\partial t} + \tfrac{1}{2}\mathrm{tr}[\alpha\alpha^*(t)D^2 w] + \beta(t,z)^* Dw \\ &\quad + \tfrac{\mu}{1-\mu}\gamma(t,z)^*(\Sigma\Sigma^*)^{-1}\zeta(t)^* Dw + \tfrac{1}{2(1-\mu)}(\nabla w)^*\alpha\alpha^*(\nabla w) + \mu r(t,z) \\ &\quad + \tfrac{\mu}{2(1-\mu)}\gamma(t,z)^*(\Sigma\Sigma^*)^{-1}\gamma(t,z) = 0 \\ &w(T,z) = 0, \end{aligned}$$

where $\gamma(t,z) = \bar{A}(t,z) - r(t,z)\mathbf{1}$. Noting that

$$\zeta(t)(\Sigma\Sigma^*)^{-1}\zeta(t)^* = \alpha\alpha(t)^*$$

setting $v(t,z) = e^{\frac{1}{1-\mu}w(t,z)}$ as before we obtain the equation for $v(t,z)$:

$$(4.13)\quad \begin{aligned} &\tfrac{\partial v}{\partial t} + \tfrac{1}{2}\mathrm{tr}[\alpha\alpha^*(t)D^2 v] + \beta(t,z)^* Dv \\ &+\tfrac{\mu}{1-\mu}\gamma(t,z)^*(\Sigma\Sigma^*)^{-1}\zeta(t)Dv \\ &+\{\tfrac{\mu}{2(1-\mu)^2}\gamma(t,z)^*(\Sigma\Sigma^*)^{-1}\gamma(t,z) + \mu r(t,z)\}v = 0 \\ &v(T,z) = 1. \end{aligned}$$

This is nothing but a linear equation and the solution $v(t,z)$ has the Feynman-Kac representation:

$$(4.14)\qquad v(t,z) = \bar{E}_z[e^{\frac{1}{1-\mu}\int_t^T \Psi(s,\tilde{Z}_s)ds}],$$

where

$$\Psi(t,z) = \frac{\mu}{2(1-\mu)}\gamma(t,z)^*(\Sigma\Sigma^*)^{-1}\gamma(t,z) + \mu r(t,z)$$

and

$$d\tilde{Z}_s = \{\beta(s,\tilde{Z}_s) + \frac{\mu}{1-\mu}\zeta(s)(\Sigma\Sigma^*)^{-1}\gamma(s,\tilde{Z}_s)\}ds + \alpha(s)d\bar{W}_s, \quad \tilde{Z}_t = z.$$

Now we are in a similar situation to the previous section. Namely, $w(t,x)$ is a viscosity solution to (4.12) if and only if $v(t,z) = e^{\frac{1}{1-\mu}w(t,z)}$ is a viscosity solution to (4.13). Since under the assumptions in the present section (4.13) has a unique viscosity solution expressed by (4.14) we see that (4.12) has a unique viscosity solution. Moreover under some regularity assumptions on the coefficients of (4.13) we can see that $v(t,x)$ in (4.14) is sufficiently smooth and the solution turns out to be a classical solution. In that case, by setting

$$\begin{aligned} h(t,z) &= \tfrac{1}{1-\mu}(\Sigma\Sigma^*)^{-1}\{\gamma(t,z) + \zeta(t)^*\nabla w(t,z)\} \\ &= \tfrac{1}{1-\mu}(\Sigma\Sigma^*)^{-1}\{\gamma(t,z) + \tfrac{(1-\mu)\zeta(t)^*}{v(t,z)}\nabla v(t,z)\} \end{aligned}$$

and solving stochastic differential equation

$$\begin{aligned} d\hat{Z}_t &= \{\beta(t,\hat{Z}_t) + \mu\zeta(t)h(t,\hat{Z}_t)\}dt + \alpha(t)d\bar{W}_t, \\ \hat{Z}_0 &= (Y_0, m_0)^* = (\log S(0), x)^* \end{aligned}$$

we can obtain an optimal strategy

$$\hat{h}_t = h(t, \hat{Z}_t).$$

The optimal value is

$$\hat{J} = \frac{1}{\mu} v^{\mu} \bar{E}[e^{-\mu \int_0^T \eta(s, \hat{Z}_s, \hat{h}_s)}].$$

Finally we note that if $r(t,x,y) = r(t)$, then the present problem is exactly same as the one studied in [13] and the solution $w(t,x)$ of (4.12) is obtained by solving two kinds of Riccati equations as was seen in [13].

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