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非等方曲率流の自己相似曲線の非一意性

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Abstract In this paper, we consider self-similar solutions for an anisotropic curvature flow equation in the plane. For some (nonsymmetric) interfacial energy, we show that there exists a self-similar curve which is not a local minimizer of the entropy under the area constraint. As its result, we obtain non-uniqueness of self-similar solutions for the anisotropic flow.

1 Introduction

This paper is concerned with strictly convex simple closed smooth curves shrinking self-similarly by the anisotropic curvature flow equation

(1.1) $$v=-\gamma(\theta)\kappa,$$

where γ is a positive continuous function on S^1 and θ, v and κ are the outward normal angle, velocity and curvature of a smooth closed curve in the plane, respectively.

If h is a positive smooth function on S^1 with $h_{\theta\theta}+h>0$ and $\phi(x,y):=\max_{\theta\in S^1}\frac{x\cos\theta+y\sin\theta}{h(\theta)}$ for $(x,y)\in\mathbf{R}^2$, then the motion of smooth closed curves in $\mathbf{R}^2$ governed by (1.1) with $\gamma:=h(h_{\theta\theta}+h)$ is driven from the gradient flow for the anisotropic interfacial energy

$$L:=\oint h(\theta)ds=\int_0^{l_0}h(\theta(s))ds$$

with respect to the metric $\sqrt{\oint\phi(u)^2h(\theta)ds}$, where l_0, s and u denote the arc length of a curve, the arc length parameter and a velocity field, respectively. See, e.g., Proposition 1.6 and Remark 1.8 of [6]. Further, the smooth convex curve

$$(x,y)(\theta,t)\ =\ \sqrt{-2t}\,(h(\theta)\cos\theta-h_\theta(\theta)\sin\theta,h(\theta)\sin\theta+h_\theta(\theta)\cos\theta)$$

with the support function

$$\max_{\tau\in S^1}(x(\tau,t)\cos\theta+y(\tau,t)\sin\theta)$$

$$=x(\theta,t)\cos\theta+y(\theta,t)\sin\theta\;=\;\sqrt{-2t}\,h(\theta)$$

from $(0,0)\in\mathbf{R}^2$ is a self-similar shrinking solution to (1.1) in $t\in(-\infty,0)$. We note that h is also the support function from $(0,0)\in\mathbf{R}^2$ of the convex set $\{(x,y)\in\mathbf{R}^2\,|\,\phi(x,y)\le 1\}$.

For rescaled solutions of (1.1), Gage and Li [7] have proved the following sub-convergence theorem to self-similar solutions:

Theorem 1 *Let $\gamma\in C^2(S^1)$ be positive. If the initial curve is simple closed and strictly convex, then the solution to (1.1) remains so and shrinks to a single point $(x_0,y_0)\in\mathbf{R}^2$ at a time $T\in\mathbf{R}$. Further, for any sequence $t_k\in(-\infty,T)$ with $\lim_{k\to\infty}t_k=T$, there exist a subsequence $t_{k'}$ and a positive function $\tilde\sigma\in C^4(S^1)$ such that the rescaled family $\tilde h(\theta,t)$ of the support function $h(\theta,t)$ of the solution from the extinction point $(x_0,y_0)\in\mathbf{R}^2$ with the enclosed area $\frac12\oint\tilde h(\tilde h_{\theta\theta}+\tilde h)\,d\theta=\frac12\oint\gamma\,d\theta$ satisfies*

$$\lim_{k'\to\infty}\|\tilde h(\theta,t_{k'})-\tilde\sigma(\theta)\|_{C^2(S^1)}=0$$

and $\tilde\sigma(\tilde\sigma_{\theta\theta}+\tilde\sigma)\equiv\gamma$ holds.

Dohmen, Giga and Mizoguchi [5] extended the existence result of a self-similar solution to $\gamma\in C(S^1)$. That is: *For any positive function $\gamma\in C(S^1)$, there exists a positive function $\sigma\in C^2(S^1)$ such that $\sigma(\sigma_{\theta\theta}+\sigma)\equiv\gamma$.* On the other hand, the uniqueness was proved by Gage [6] for π-periodic $\gamma\in C^\infty(\mathbf{R})$ and Dohmen and Giga [4] (see also Giga [8, §3.2]) π-periodic $\gamma\in C(\mathbf{R})$. Precisely, it was shown that: *For any positive π-periodic function $\gamma\in C(\mathbf{R})$, there uniquely exists a positive function $\sigma\in C^2(S^1)$ such that $\sigma(\sigma_{\theta\theta}+\sigma)\equiv\gamma$.* See, e.g., [1], [3] and [8] for related results. The uniqueness problem for 2π-periodic γ (not π-periodic, i.e., not symmetric) had been left open ([8]). In this paper, we give a negative answer to this problem. In fact, we prove that the uniqueness of the self-similar solutions dose not necessarily hold for the anisotropic curve shortening flow (1.1) with 2π-periodic γ unless it is π-periodic.

Theorem 2 *There exist two different positive functions σ and $\tilde{\sigma} \in C^\infty(S^1)$ such that $\sigma(\sigma_{\theta\theta} + \sigma) \equiv \tilde{\sigma}(\tilde{\sigma}_{\theta\theta} + \tilde{\sigma}) > 0$.*

In Introduction of [1], Andrews had pointed out that for the more general flow $v = -\gamma\kappa^\alpha$ with any $\alpha > 0$, there exists a positive function $\gamma \in C^\infty(S^1)$ such that self-similar solutions are not unique. However, its proof may not have been published yet.

After this work was completed, the author was informed of the way of the proof by Andrews [2], which has also proved Theorem 2. It has been sketched in [8, p. 152]. Our proof might differ completely and seem more direct and simple.

2 The proof of Theorem 2

The following is the main technical result in this paper and proved in Section 3.

Theorem 3 *There exist functions σ and $w \in C^\infty(S^1)$ such that $\sigma > 0$, $\sigma_{\theta\theta} + \sigma > 0$, $\oint \frac{\sigma(w_{\theta\theta}+w)^2}{\sigma_{\theta\theta}+\sigma} d\theta + \oint w(w_{\theta\theta} + w) d\theta < 0$ and $\oint \sigma(w_{\theta\theta} + w) d\theta = 0$.*

Since $\oint h(h_{\theta\theta} + h) d\theta = (1+a)^2 \oint \sigma(\sigma_{\theta\theta} + \sigma) d\theta + 2(1+a)b \oint \sigma(w_{\theta\theta} + w) d\theta + b^2 \oint w(w_{\theta\theta} + w) d\theta$ holds for the function $h = (1+a)\sigma + bw$ on S^1 with constants a and b, we can easily see the following lemma by using the implicit function theorem or the formula of the solutions to a quadratic equation.

Lemma 4 *Let σ and $w \in C^2(S^1)$. If $\oint \sigma(\sigma_{\theta\theta} + \sigma) d\theta \neq 0$, then there exist δ_1 and $\delta_2 > 0$ such that the following holds:*

There uniquely exists a function c from $(-\delta_1, \delta_1)$ to $(-\delta_2, \delta_2)$ such that the functions $\bar{h}^s \in C^2(S^1)$ defined by

$$\bar{h}^s(\theta) := (1 + c(s))\sigma(\theta) + sw(\theta) \tag{2.1}$$

satisfy

$$\oint \bar{h}^s(\bar{h}^s_{\theta\theta} + \bar{h}^s) d\theta = \oint \sigma(\sigma_{\theta\theta} + \sigma) d\theta \tag{2.2}$$

for all $s \in (-\delta_1, \delta_1)$. Further, c is a real-analytic function.

By Theorem 3 and Lemma 4, we show the following proposition.

Proposition 5 *There exist functions σ and $w \in C^\infty(S^1)$ such that $\sigma > 0$, $\sigma_{\theta\theta} + \sigma > 0$ and*

$$\oint \sigma(\sigma_{\theta\theta} + \sigma) \log \frac{\bar{h}^s_{\theta\theta} + \bar{h}^s}{\sigma_{\theta\theta} + \sigma} d\theta > 0 \tag{2.3}$$

holds for all sufficiently small $s > 0$, where $\bar{h}^s \in C^\infty(S^1)$ is defined by (2.1).

Proof. The proof is easy and omitted here.

Now, we prove Theorem 2 by using Theorem 1 and Proposition 5.
Proof of Theorem 2. First, let σ and $w \in C^\infty(S^1)$ be the functions given in Proposition 5. Then, $\sigma > 0$ and $\sigma_{\theta\theta} + \sigma > 0$ hold. Further, we can take small $s > 0$ such that (2.2), (2.3), $\bar{h}^s > 0$ and $\bar{h}^s_{\theta\theta} + \bar{h}^s > 0$ hold. Set a positive function

$$\gamma := \sigma(\sigma_{\theta\theta} + \sigma) \in C^\infty(S^1).$$

Let us consider the solution to (1.1) with the initial curve

$$(x, y)(\theta, 0) := (\bar{h}^s(\theta) \cos\theta - \bar{h}^s_\theta(\theta) \sin\theta, \bar{h}^s(\theta) \sin\theta + \bar{h}^s_\theta(\theta) \cos\theta),$$

whose support function from $(0, 0) \in \mathbf{R}^2$ is $\bar{h}^s(\theta)$. Then, by Theorem 1, there exist a sequence $t_k \geq 0$ and a positive function $\tilde{\sigma} \in C^2(S^1)$ with

$$\gamma = \tilde{\sigma}(\tilde{\sigma}_{\theta\theta} + \tilde{\sigma}) \tag{2.4}$$

such that the rescaled family $\tilde{h}(\theta, t)$ of its support function $h(\theta, t)$ from its extinction point $(x_0, y_0) \in \mathbf{R}^2$ with $\oint \tilde{h}(\tilde{h}_{\theta\theta} + \tilde{h})\, d\theta = \oint \gamma\, d\theta$ satisfies

$$\lim_{k\to\infty} \|\tilde{h}(\theta, t_k) - \tilde{\sigma}(\theta)\|_{C^2(S^1)} = 0. \tag{2.5}$$

Because $h \geq 0$ holds by (1.1), so is $\tilde{h}$. Hence, from (2.5), we see $\tilde{\sigma} \geq 0$. So, from (2.4) and $\gamma > 0$, we obtain $\tilde{\sigma} > 0$. Moreover, by $\gamma \in C^\infty(S^1)$, we can also see $\tilde{\sigma} \in C^\infty(S^1)$. Therefore, in order to complete this proof, it suffices to prove that σ and $\tilde{\sigma}$ are different.

We have $(x_0 \cos\theta + y_0 \sin\theta) + h(\theta, 0) = \bar{h}^s(\theta)$. Hence, by (2.2), we see $\oint h(\theta, 0)(h_{\theta\theta}(\theta, 0) + h(\theta, 0))\, d\theta = \oint \gamma(\theta)\, d\theta$ and $\tilde{h}(\theta, 0) = h(\theta, 0)$. Therefore, $\tilde{h}(\theta, 0) = \bar{h}^s(\theta) - (x_0 \cos\theta + y_0 \sin\theta)$ also holds. From this, we have

$$\oint \gamma(\theta) \log \frac{\gamma(\theta)}{\tilde{h}_{\theta\theta}(\theta, 0) + \tilde{h}(\theta, 0)} d\theta = \oint \gamma(\theta) \log \frac{\gamma(\theta)}{\bar{h}^s_{\theta\theta}(\theta) + \bar{h}^s(\theta)} d\theta.$$

On the other hand, as it is known that the entropy $\oint \gamma(\theta)\log\frac{\gamma(\theta)}{\tilde{h}_{\theta\theta}(\theta,t)+\tilde{h}(\theta,t)}d\theta$ is nonincreasing in Corollary 5.7 of [7], we also have

$$\oint \gamma(\theta)\log\frac{\gamma(\theta)}{\tilde{h}_{\theta\theta}(\theta,t_k)+\tilde{h}(\theta,t_k)}d\theta \leq \oint \gamma(\theta)\log\frac{\gamma(\theta)}{\bar{h}^s_{\theta\theta}(\theta)+\bar{h}^s(\theta)}d\theta.$$

Hence, by (2.4) and (2.5), $\oint \gamma\log\tilde{\sigma}\,d\theta \leq \oint\gamma\log\frac{\gamma}{\bar{h}^s_{\theta\theta}+\bar{h}^s}\,d\theta$ holds. From this and (2.3), we see $\oint\gamma\log\tilde{\sigma}\,d\theta < \oint\gamma\log\sigma\,d\theta$ and $\tilde{\sigma}\not\equiv\sigma$. q.e.d.

3 The proof of Theorem 3

In the following lemma, it might be essential that $\sigma\in C^2(S^1)$ was not π-periodic. Our proof deduces it from eigenvalue problems.

Lemma 6 *For any $K\geq 2$, there exists $\sigma\in C^\infty(S^1)$ such that $\sigma(\theta)>0$ and $\sigma_{\theta\theta}(\theta)+\sigma(\theta)>0$ hold for all $\theta\in S^1$ and*

$$\sigma_{\theta\theta}(\theta)+\sigma(\theta)=K\sigma(\theta) \tag{3.1}$$

holds for all $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$.

Proof. We first take $\theta_0\in(\frac{\pi}{2},\pi)$ such that $\tan(\pi-\theta_0)>\sqrt{K-1}$. Then, we also take $\varepsilon\in(0,1)$ such that

$$\sqrt{1-\varepsilon}\tan(\sqrt{1-\varepsilon}(\pi-\theta_0))>\sqrt{K-1} \tag{3.2}$$

and a positive function $f\in C^\infty(S^1)$ such that $f(\theta)\equiv K$ for $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $f(\theta)\equiv\varepsilon$ for $\theta\in S^1\setminus(-\theta_0,\theta_0)$ and $\varepsilon\leq f(\theta)\leq K$ for $\theta\in S^1$. Set a constant

$$a_0 := \frac{e^{\sqrt{K-1}\theta_0}+e^{-\sqrt{K-1}\theta_0}}{\cos(\sqrt{1-\varepsilon}(\pi-\theta_0))}>0,$$

and define a positive function $\underline{\sigma}\in C(S^1)$ by

$$\underline{\sigma}(\theta) := \begin{cases} e^{\sqrt{K-1}\theta}+e^{-\sqrt{K-1}\theta}, & \text{for } \theta\in[-\theta_0,\theta_0],\\ a_0\cos(\sqrt{1-\varepsilon}(\pi-\theta)), & \text{for } \theta\in(\theta_0,-\theta_0+2\pi).\end{cases}$$

Then, we see $\underline{\sigma}_{\theta\theta}+\underline{\sigma}=K\underline{\sigma}\geq f\underline{\sigma}$ in $(-\theta_0,\theta_0)$ and $\underline{\sigma}_{\theta\theta}+\underline{\sigma}=\varepsilon\underline{\sigma}=f\underline{\sigma}$ in $S^1\setminus[-\theta_0,\theta_0]$. Further, from (3.2), we also see $(D^+\underline{\sigma})(\theta_0)=-(D^-\underline{\sigma})(-\theta_0)=$

$a_0\sqrt{1-\varepsilon}\sin(\sqrt{1-\varepsilon}(\pi-\theta_0)) > \sqrt{K-1}(e^{\sqrt{K-1}\theta_0}+e^{-\sqrt{K-1}\theta_0}) > \sqrt{K-1}$ $(e^{\sqrt{K-1}\theta_0}-e^{-\sqrt{K-1}\theta_0}) = (D^-\underline{\sigma})(\theta_0) = -(D^+\underline{\sigma})(-\theta_0)$. Therefore, because $\underline{\sigma} \in C(S^1)$ is a positive sub-solution to $\sigma_{\theta\theta}+(1-f)\sigma = 0$ in S^1, the first eigenvalue of $\sigma_{\theta\theta}+(1-f)\sigma = \lambda\sigma$ in S^1 is nonnegative. On the other hand, because the first eigenvalue of $\sigma_{\theta\theta}+(1-K)\sigma = \lambda\sigma$ in S^1 is $1-K \in (-\infty, 0]$, there exists $\rho_0 \in [0,1]$ such that 0 is the first eigenvalue of $\sigma_{\theta\theta}+(1-g)\sigma = \lambda\sigma$ in S^1 with $g := (1-\rho_0)K + \rho_0 f \in C^\infty(S^1)$. So, we have a positive function $\sigma \in C^\infty(S^1)$ such that $\sigma_{\theta\theta}+\sigma = g\sigma$. Because of $g > 0$ in S^1 and $g \equiv K$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, this completes the proof. q.e.d.

Next, we show the following.

Lemma 7 *There exists a constant $K \geq 2$ such that the following holds:*
For any linear function $J[\cdot]$ from $C^\infty(S^1)$ to $\mathbf{R}$, there exists $w \in C^\infty(S^1)$ such that $J[w] = 0$, $K^{-1}\oint(w_{\theta\theta}+w)^2 d\theta + \oint w(w_{\theta\theta}+w)d\theta < 0$ and the support of w is contained in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Proof. We take a cut-off function $\chi \in C^\infty(S^1)$ such that $\chi(\theta) \equiv 1$ for $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, $\chi(\theta) \equiv 0$ for $\theta \in S^1 \setminus (-\frac{\pi}{2}, \frac{\pi}{2})$ and $0 \leq \chi(\theta) \leq 1$ for $\theta \in S^1$. For $(a,b) \in \mathbf{R}^2$ with $a^2+b^2 = 1$, we define $g^{a,b} \in C^\infty(S^1)$ by

$$g^{a,b}(\theta) := \chi(\theta)(a\sin 4\theta - b\cos 4\theta).$$

Then, we define a constant

$$K := 2 + \frac{1}{3\pi}\max_{a^2+b^2=1}\oint(g^{a,b}_{\theta\theta}+g^{a,b})^2 d\theta. \tag{3.3}$$

For any linear functional J on $C^\infty(S^1)$, there exists $(a_0, b_0) \in \mathbf{R}^2$ with $a_0^2+b_0^2 = 1$ such that $J[g^{a_0,b_0}] = 0$. So, we set $w := g^{a_0,b_0} \in C^\infty(S^1)$ and see that $J[w] = 0$ and the support of w is contained in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, $\oint w(w_{\theta\theta}+w)d\theta = \oint(w^2 - w_\theta^2)d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(w^2 - w_\theta^2)d\theta \leq \pi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w_\theta^2 d\theta \leq \pi - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} w_\theta^2 d\theta = \pi - 16\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(a_0\cos 4\theta + b_0\sin 4\theta)^2 d\theta = -3\pi$ holds. Hence, from (3.3), we obtain $\oint(w_{\theta\theta}+w)^2 d\theta + K\oint w(w_{\theta\theta}+w)d\theta \leq -6\pi$. q.e.d.

Now, we prove Theorem 3 from Lemmas 6 and 7.
Proof of Theorem 3. The proof is easy and omitted here.

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REFERENCES

[1] B. Andrews, *Calc. Var.*, 7 (1998), 315-371.
[2] B. Andrews, *in preparation.*
[3] K.-S. Chou, X.-P. Zhu, *Indiana Univ. Math. J.*, 48 (1999), 139-154.
[4] C. Dohmen, Y. Giga, *Proc. Japan Acad. A*, 70 (1994), 252-255.
[5] C. Dohmen, Y. Giga, N. Mizoguchi, *Calc. Var.*, 4 (1996), 103-119.
[6] M. E. Gage, *Duke Math. J.*, 72 (1993), 441-466.
[7] M. E. Gage, Y. Li, *Duke Math. J.*, 75 (1994), 79-98.
[8] Y. Giga, *Sugaku Expositions*, 16 (2003), 135-152.

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